String-Steilkurs 2008: Introduction to String Theory

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von Herrn S. Theisen auf dem String-Steilkurs 2008
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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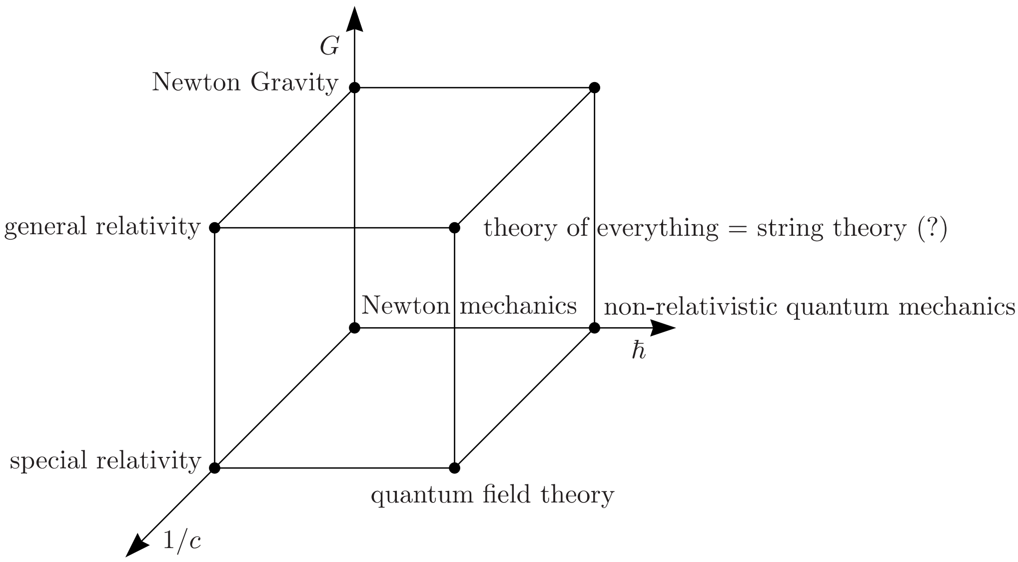
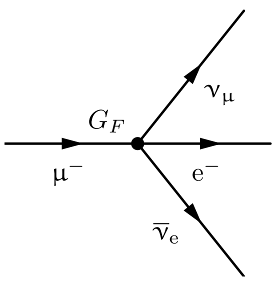
Introduction to string theory

1.1 The cGh-cube

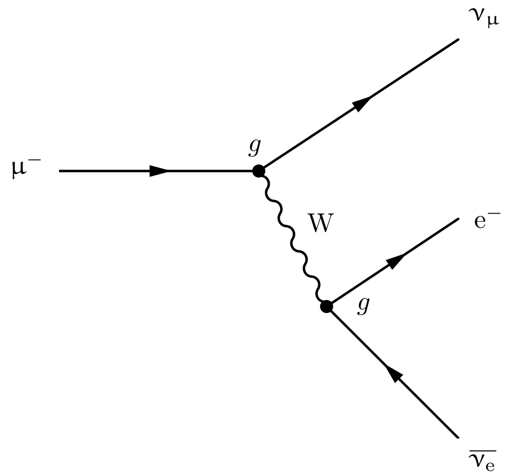


1.2 Quantization of gravity

Why can one not quantize gravity? For answering this question, let us first look at the weak interaction theory (Fermi theory):



This theory is sick; unitarity is violated at high energies. That is why one has to introduce the W-boson:



$$G_F \sim \frac{g^2}{M_W^2}$$

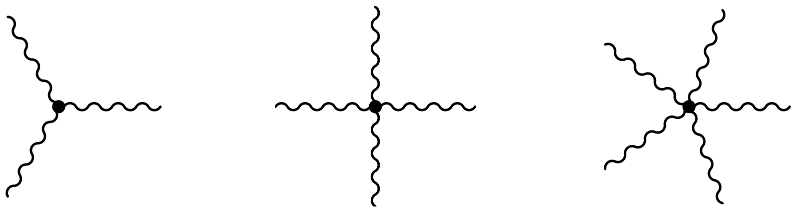
Then the problems with renormalizability are resolved; it is a renormalizable quantum field theory. Now, let us look at the action of gravity:

$$S = \frac{1}{16\pi G_N} \int \mathrm{d}^4x \, \sqrt{g} R \, . \tag{1.1}$$

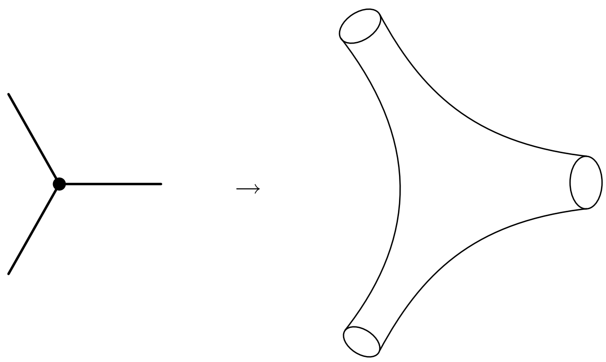
If one expands the action around the background $\eta_{\mu\nu}$, whereas

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \, , \quad \kappa^2 \sim G_N \, , \tag{1.2}$$

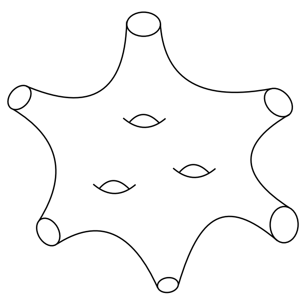
one ends up with the following vertices (to first order):



The problem with this theory is, that it is not renormalizable. Since one has an infinite number of interaction vertices the theory has no predictive power. The solution to pull apart vertices does not work. However, one can "pull apart" the world line of the graviton.



Hence, one replaces world lines of particles by world sheets of strings.



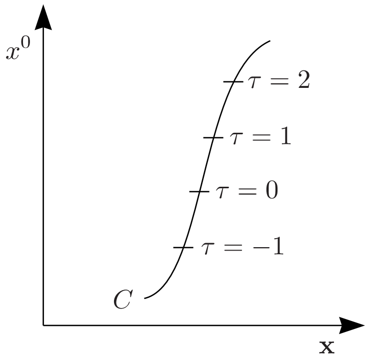
Such interactions can all be pulled apart to the above interaction.

Chapter 2

The bosonic string

2.1 The classical bosonic string

We want to have a look on the world-line of a relativistic point particle:



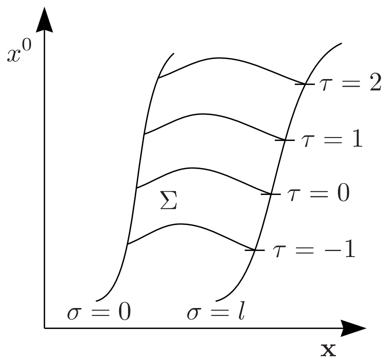
τ is the parametrization of the world-line. Recall that the action is given by

$$S = -mc \int_{s_1}^{s_2} \mathrm{d}s = -mc \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \sqrt{-\eta_{\mu\nu}\dot{x}^\mu \dot{x}^\nu}\,, \tag{2.1}$$

X is a map from the world line to space-time.

$$X : C \to M^{1,D-1},\ \tau \mapsto x^\mu(\tau)\,. \tag{2.2}$$

The action for a relativistic string is called Nambu-Goto action.



Now, X is a map of the world sheet to space-time:

$$X : \Sigma \to M,\ (\sigma, \tau) \mapsto x^\mu(\sigma, \tau)\,. \tag{2.3}$$

The action is given by:

$$S_{\mathrm{NG}} = \text{“area of } \Sigma\text{”} = \int \mathrm{d}\sigma\, \mathrm{d}\tau\, \mathscr{L}_{\mathrm{NG}}\,, \tag{2.4}$$

with

$$\mathscr{L}_{\rm NG} = -\frac{1}{2\pi\alpha'}\sqrt{(\dot{X}^\mu X'_\mu)^2 - (\dot{X})^2(X')^2} = -\frac{1}{2\pi\alpha^2}\sqrt{-\det(h_{ab})}\,, \tag{2.5}$$

where dots denote derivatives with respect to τ and dashes derivatives with respect to σ.

$$h_{ab} = \frac{\partial}{\partial\sigma^a}X^\mu\frac{\partial}{\partial\sigma^b}X^\nu\eta_{\mu\nu}\,, \tag{2.6}$$

is the induced metric on Σ with $\sigma^a = (\sigma,\tau)$.

Comments

- α' is a constant of dimension (length)^2.
- It holds that $\alpha' = l_S^2$, where l_S is the only dimension-full parameter in string theory.
- generalization: $(M,\eta) \to (M,G)$, $h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)$

Consider the Polyakov-action (area of the world sheet measured with the induced metric):

$$S_P = -\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\sqrt{-\gamma}\gamma^{ab}h_{ab}\,, \quad h_{ab} = \partial_a X^\mu\partial_b X^\nu\eta_{\mu\nu}\,. \tag{2.7}$$

γ^{ab} (with $\gamma = \det(\gamma_{ab})$ is a world-sheet (intrinsic) metric with signature (1,1). The equations of motion will be algebraic in γ, since no derivative appears in the action. Hence, it is just an auxiliary field, which can be eliminated. The equation of motion of this action is just the Laplace equation, which is an advantage comparing to the Nambu-Goto action. To study the model we need the equations of motion for γ_{ab}. Therefore we write up the variations

$$\gamma_{ab} \mapsto \gamma_{ab} + \delta\gamma_{ab}\,, \quad \gamma^{ab} \mapsto \gamma^{ab} - \delta\gamma^{ab}\,, \quad \delta\gamma^{ab} = \gamma^{ac}\gamma^{bc}\delta\gamma_{cd}\,. \tag{2.8}$$

For the determinant it holds that

$$\det(\gamma+\delta\gamma) = \det[\gamma(\mathbb{1}+\gamma^{-1}\delta\gamma)] = \det\gamma\det(\mathbb{1}+\gamma^{-1}\delta\gamma) = \det(\gamma)(1+\mathrm{Tr}(\gamma^{-1}\delta\gamma))\,, \tag{2.9}$$

using

$$\det(1+\epsilon) = \exp(\mathrm{Tr}\ln(1+\epsilon)) \approx \exp(\mathrm{Tr}(\epsilon)) \approx 1+\mathrm{Tr}(\epsilon)\,. \tag{2.10}$$

Hence, it holds that

$$\det(\gamma+\delta\gamma) - \det(\gamma) \equiv \delta\det(\gamma) = \det\gamma\gamma^{ab}\delta\gamma_{ab} = -\gamma_{ab}\delta\gamma^{ab}\,. \tag{2.11}$$

$$\delta\sqrt{-\det(\gamma)} = -\frac{1}{2}\sqrt{-\det(\gamma)}\gamma_{ab}\delta\gamma^{ab}\,. \tag{2.12}$$

Now, we calculate the variation of the action:

$$\delta S_P = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}\left\{\frac{1}{2}\gamma^{ab}\gamma^{cd}h_{cd} - h_{ab}\right\}\delta\gamma^{ab} \equiv \frac{1}{4\pi}\int \mathrm{d}^2\sigma\,\sqrt{-\gamma}T_{ab}\delta\gamma^{ab}\,. \tag{2.13}$$

From the vanishing variation we obtain the field equations. These tell us that the energy-momentum tensor has to vanish:

$$T_{ab} \equiv \frac{4\pi}{\sqrt{-\gamma}}\frac{\delta S_P}{\delta\gamma^{ab}} = \frac{1}{\alpha'}\left\{\frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} - h_{ab}\right\} \overset{!}{=} 0\,. \tag{2.14}$$

Then, it follows that

$$h_{ab} = \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd}\,, \tag{2.15}$$

$$\det(h_{ab}) = \frac{1}{4}\det(\gamma_{ab})(\gamma^{cd}h_{cd})^2 \Rightarrow \sqrt{-h} = \frac{1}{2}\sqrt{-\gamma}\gamma^{cd}h_{cd}\,. \tag{2.16}$$

h_{ab} and γ_{ab} are conformally related metrics. So, the Polykov-action is equivalent to the Nambu-Goto action.

2.1.1 The symmetries of the Polyakov action

Global symmetries

- D-dimensional Poincaré invariance:
  Under the transformation

$$X^\mu(\tau,\sigma) \mapsto \Lambda^\mu \;\; X^\nu(\tau,\sigma) + a^\mu \,, \tag{2.17}$$

the action is invariant ($\gamma_{ab}(\tau,\sigma) \mapsto \gamma_{ab}(\tau,\sigma)$).

Local symmetries

- reparametrization invariance:

$$X'^\mu(\tau,\sigma) = X^\mu(\tau,\sigma)\,, \quad \text{(world sheet scalars)} \tag{2.18}$$

$$\partial_a \sigma'^c \partial_b \sigma'^d \gamma'_{cd}(\tau',\sigma') = \gamma_{ab}(\tau,\sigma)\,, \quad \text{(symmetric second rank tensor)} \tag{2.19}$$

$$\mathrm{d}s^2 = \gamma_{ab}(\tau,\sigma)\,\mathrm{d}\sigma^a\,\mathrm{d}\sigma^b = \gamma'_{ab}(\tau',\sigma')\mathrm{d}\sigma'^a\,\mathrm{d}\sigma'^b\,. \tag{2.20}$$

Infinitesimal versions of the transformation: $\sigma'^a = \sigma^a - \xi^a$.

$$\delta x^\mu = \xi^a \partial_a X^\mu\,, \quad \delta\gamma_{ab} = \nabla_a \xi_b + \nabla_b \xi_a\,, \quad \nabla_a \xi_b = \partial_a \xi_b - \Gamma_{ab}{}^c \xi_c\,. \tag{2.21}$$

- two-dimensional Weyl invariance:

$$X'^\mu(\sigma,\tau) = X^\mu(\sigma,\tau)\,, \quad \gamma'_{ab} = \exp(2\omega(\tau,\sigma))\gamma_{ab}\,, \quad \delta\gamma^{ab} = -2\omega\gamma^{ab}\,. \tag{2.22}$$

2.1.2 Consequences of these symmetries

- A consequence of diffeos (reparametrization invariance) is $\nabla^a T_{ab} = 0$ (on-shell). Proof (general):

$$\delta_\xi S = \int \mathrm{d}\sigma \left\{ \frac{\delta S}{\delta\gamma^{ab}} \delta\gamma^{ab} + \frac{\delta S}{\delta\phi}\delta\phi \right\}\,. \tag{2.23}$$

The second term vanishes via the equations of motion for ϕ. The first term is (by definition of the energy-momentum tensor) $\sim \sqrt{-\gamma}T_{ab}$. One obtains:

$$\int \mathrm{d}\sigma\, \sqrt{-\gamma}T_{ab}(\nabla^a\xi^b + \nabla^b\xi^a) = 2\int \mathrm{d}\sigma\,\sqrt{-\gamma}T^{ab}\nabla^a\xi^b = -2\int \mathrm{d}\sigma\,\sqrt{-\gamma}(\nabla^b T_{ab})\xi^b\,, \tag{2.24}$$

which follows from integration by parts. □

- From Weyl invariance it follows that $T^a{}_a = \gamma^{ab}T_{ab} = 0$ (on-shell). Proof (general):

$$S[\exp(2\omega)\gamma_{ab}, \exp(d_1\omega)\phi_1] = S[\gamma_{ab},\phi]\,, \tag{2.25}$$

should hold because of Weyl invariance.

$$0 = \delta S = \int \mathrm{d}\sigma \left\{ -2\frac{\delta S}{\delta\gamma^{ab}}\gamma^{ab} + \sum_i d_i \frac{\delta S}{\delta\phi_i} \right\} \delta\omega\,. \tag{2.26}$$

The second term vanishes due to the field equations of ϕ. With $\delta S/\delta\gamma^{ab} \sim T_{ab}$ is follows that

$$\gamma^{ab}T_{ab} = 0\,, \tag{2.27}$$

hence the tracelessness of the energy-momentum tensor. □

We now use the reparametrization invariance to eliminate two of three degrees of freedom of γ_{ab} (gauge fixing).

$$\sqrt{-\gamma}\gamma^{ab} = \eta^{ab}\,. \tag{2.28}$$

Locally, one can always choose

$$\gamma_{ab} \sim \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{2.29}$$

This gauge is called the **conformal gauge**. Then, the action is given by

$$S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^2 - X'^2) , \tag{2.30}$$

and the energy-momentum tensor

$$T_{ab} = -\frac{1}{\alpha'} \left\{ \partial_a X \partial_b X - \frac{1}{2} \eta_{ab} (\partial_c X \partial^c X) \right\} = 0 . \tag{2.31}$$

(One has to derive the energy-momentum tensor in an ungauged theory and then fix the gauge to arrive at this result.)

$$\begin{aligned} \delta S_P &= -\frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \int_0^l \mathrm{d}\sigma \, \partial_a X^\mu \partial^a \delta X_\mu = \\ &= \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \int_0^l \mathrm{d}\sigma \, \Box X^\mu \delta X_\mu - \frac{1}{2\pi\alpha'} \int \mathrm{d}\tau \, \left[\partial_\sigma X^\mu \delta X_\mu \right]_0^l . \end{aligned} \tag{2.32}$$

The second term is a boundary term. Hence, the boundary term has to vanish. From the vanishing of the first term it follows that

$$\Box X^\mu = 0 = (-\partial_\tau^2 + \partial_\sigma^2) X^\mu , \tag{2.33}$$

which is the two-dimensional wave equation.

i.) A closed string is periodic: $X^\mu(\tau, \sigma + l) = X^\mu(\tau, \sigma)$. The solution of the wave equation is

$$X^\mu(\tau,\sigma) = q^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + \mathrm{i} \sqrt{\frac{\alpha'}{2}} \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \frac{1}{n} \left\{ \tilde{\alpha}_n^\mu \exp(-\frac{2\pi \mathrm{i} n (\tau + \sigma)}{l} \right) + \alpha_n^\mu \exp\left(-\frac{2\pi \mathrm{i} n (\tau - \sigma)}{l} \right) , \tag{2.34}$$

which is a Fourier decomposition in left and right moving waves. It holds that $\alpha_{-n}^\mu = (\alpha_n^\mu)^*$ and furthermore

$$q^\mu(\tau) = \frac{1}{l} \int_0^l \mathrm{d}\sigma \, X^\mu(\tau,\sigma) = q^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau , \tag{2.35}$$

describes the center-of-mass motion and

$$p^\mu(\tau) = \int_0^l \mathrm{d}\sigma \, \Pi^\mu(\tau,\sigma) = p^\mu , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\tau X^\mu)} = \frac{1}{2\pi\alpha'} \dot{X}^\mu , \tag{2.36}$$

is the center-of-mass momentum. We still need to impose $T_{ab} = 0$ and therefore $(\dot{X} \pm X')^2 = 0$ (Virasoro contraints). One has to impose these constraints for finding classical string solutions.

ii.) An open string needs to satisfy

$$\partial_\sigma X^\mu \delta X_\mu |_{\sigma=0}^{\sigma=l} = 0 . \tag{2.37}$$

There are two solutions, namely the Neumann boundary conditions (N) (with free string endpoints)

$$\partial_\sigma X^\mu(\tau)|_{\partial\Sigma} = 0 , \tag{2.38}$$

and the Dirichlet boundary conditions (D) (with fixed string endpoints)

$$\delta X^\mu(\tau)|_{\partial\Sigma} = 0 . \tag{2.39}$$

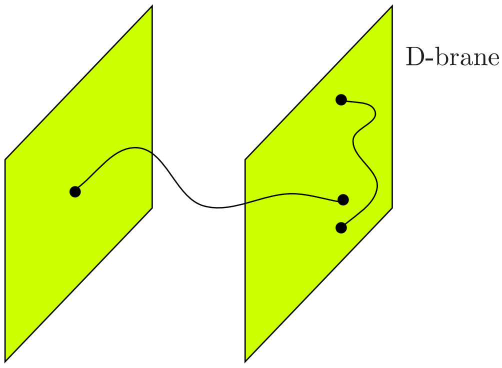
The meaning of these boundary conditions is as follows. (N) preserves space-time Poincaré invariance:

$$X^\mu \mapsto \Lambda^\mu{}_\nu X^\nu + a^\mu \,, \quad \partial_\sigma X^\mu \mapsto \Lambda^\mu{}_\nu \partial_\sigma X^\nu \,. \tag{2.40}$$

From Noether's theorem, this means that space-time momentum is conserved along N directions.

$$\partial_\tau P^\mu = \partial_\tau \left\{ \frac{1}{2\pi\alpha'} \int_0^l \mathrm{d}\sigma \, \dot{X}^\mu \right\} = \frac{1}{2\pi\alpha'} \int_0^l \mathrm{d}\sigma \, \ddot{X}^\mu = \frac{1}{2\pi\alpha'} \int_0^l \mathrm{d}\sigma X''^\mu = \frac{1}{2\pi\alpha'} [X'^\mu(\tau,\sigma)]_0^l = 0 \,. \tag{2.41}$$

(D) breaks space-time Poincaré invariance; the momentum is not conserved: $\partial_\tau P^\mu \neq 0$. Hence, momentum flows off the end of the open string. But where does it flow to?



Open strings sit on so-called D-branes. Hence, the momentum flows into these D-branes. One also talks of D$_P$-branes, where P is the number of space-dimensions of the brane. (Hence, we have a $(P+1)$-dimensional world volume.) Here, one has (NN) boundary conditions along $\mu = 0, \ldots, P$ and (DD) boundary conditions along $\mu = P+1, \ldots, D-1$. $P = 1$ is called a D-string, $P = 0$ a D-particle, etc. The solutions of the equations of motion for (NN) are given by

$$X(\tau,\sigma) = q + \frac{2\pi\alpha'}{l} p\tau + \mathrm{i}\sqrt{2\alpha'} \sum_{\substack{n\neq 0\\ n\in\mathbb{Z}}} \frac{1}{n}\alpha_n \cos\left(\frac{n\pi\sigma}{l}\right) \exp\left(-\frac{im\pi\tau}{l}\right) \,, \tag{2.42}$$

so the waves are reflected at the endpoint. For (DD) one has

$$X|_{\sigma=0} = q_i \,, \quad X|_{\sigma=l} = q_f \,, \tag{2.43}$$

and

$$X(\tau,\sigma) = q_i + \frac{1}{l}(q_f - q_i)\sigma + \sqrt{2\alpha'} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\alpha_n \sin\left(\frac{n\pi\sigma}{l}\right) \exp\left(-\frac{\mathrm{i}n\pi\tau}{l}\right) \,. \tag{2.44}$$

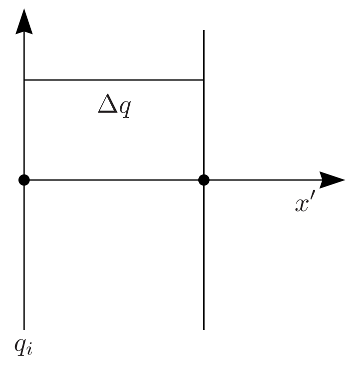
For (ND) one has

$$\partial_\sigma X|_{\sigma=0} = 0 \,, \quad X|_{\sigma=l} = q_f \,, \tag{2.45}$$

and

$$X(\tau,\sigma) = q_f + \mathrm{i}\sqrt{2\alpha'} \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\alpha_r \cos\left(\frac{n\pi\sigma}{l}\right) \exp\left(-\frac{\mathrm{i}n\pi\tau}{l}\right) \,. \tag{2.46}$$

Simple classical open-string solution:



$$X^0 = \frac{2\pi\alpha'}{l} p\tau\,, \quad X^1 = q_1 + \frac{\Delta q}{l}\sigma\,, \quad X^2 = X^3 = \ldots = X^{D-1} = 0\,. \tag{2.47}$$

From $\dot{X}^2 + X'^2 = 0$ it follows that $X^0 = \Delta q / l\tau$ and $\dot{X} \cdot X' = 0$.

$$\begin{aligned} S &= -\frac{1}{4\pi\alpha'} \int_0^l \mathrm{d}\sigma \int \mathrm{d}\tau \, (\dot{X}^2 - X'^2) = -\frac{1}{4\pi\alpha'} \int_0^l \mathrm{d}\sigma \int \mathrm{d}\tau \, 2\dot{X}^2 = \\ &= -\frac{1}{2\pi\alpha'} \int_0^l \mathrm{d}\sigma \int \mathrm{d}\tau \left(\frac{\Delta q}{l}\right)^2 . \end{aligned} \tag{2.48}$$

We will now obtain an interpretation of the dimension-full constant α' by looking at the energy:

$$E = P^0 = \frac{1}{2\pi\alpha'} \int_0^l \dot{X}^2 \, \mathrm{d}\sigma = \frac{1}{2\pi\alpha'} \Delta q = T \Delta q\,. \tag{2.49}$$

$T = 1/(2\pi\alpha')$ is called string tension. The energy of the string is proportional to its length and the constant of proportionality (string tension) can be written using α'.

Comments

One could also have defined the action in the way

$$S_P = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \sqrt{-\gamma} \left\{ \gamma^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} + {}^{(2)}R(\gamma) + \Lambda \right\} . \tag{2.50}$$

$R(2)$ is the curvature scalar of two-dimensional metric. Terms $R(2)$ lead to an organization of string perturbation theory in orders of a topological expansion (sphere, torus, double-torus, etc.). However, it can be written as

$$\sqrt{-\gamma}^{(\gamma)} R(\gamma) = \partial_\mu k^\mu\,, \tag{2.51}$$

and therefore are only boundary terms, which do not contribute to the equations of motion. That is why we did not include such terms. Furthermore, we did not include the term Λ (cosmological constant), because it is not Weyl-invariant.

2.2 Quantization of the bosonic string

We have to introduce $\hbar (= 1)$. There is

- the canonical quantization,
- the path-integral quantization and
- the BPST-quanization.

Here, we will do a canonical quantization in light-cone gauge. It allows to solve the Virasaro constraints explicitly and thereby eliminate the unphysical degrees of freedom. (So, only physical degrees of freedom will propagate.) It is a non-covariant gauge, but it has the advantage that it is the easiest way to obtain the spectrum of the string and to derive the number of dimensions. Canonical means that be replace the Poisson brackets by commutators and by using light-cone gauge we quantize only the physical degrees of freedom. Naive canonical transformations (equal time commutator):

$$\{X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')\}_{\mathrm{PB}} = \delta(\sigma - \sigma')\eta^{\mu\nu} \mapsto [X^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma')] = \mathrm{i}\delta(\sigma - \sigma')\eta^{\mu\nu}\,, \tag{2.52}$$

with

$$\Pi^\mu(\tau,\sigma) = \frac{1}{2\pi\alpha'} \dot{X}^\mu(\tau,\sigma)\,. \tag{2.53}$$

From this, one obtains

$$[\alpha^\mu_n, \alpha^\nu_m] = m\delta_{m+n,0}\eta^{\mu\nu}\,, \quad [q^\mu, p^\nu] = \mathrm{i}\eta^{\mu\nu}\,, \tag{2.54}$$

for the open string. This is true for all boundary conditions. (For the (ND) string the α_n^μ will be half-integer modes.) For the closed string one obtains additionally:

$$[\widetilde{\alpha}_n^\mu, \widetilde{\alpha}_m^\nu] = n\eta^{\mu\nu}\delta_{m+n,0} \,. \tag{2.55}$$

From hermiticity of X^μ it follows that $(\alpha_n^\mu)^\dagger = \alpha_{-n}^\mu$ and $(\widetilde{\alpha}_n^\mu)^\dagger = \widetilde{\alpha}_{-n}^\mu$. α_n^μ $(n > 0)$ are annihilation operators and α_n^μ $(n < 0)$ are creation operators. So, one has an infinite collection of harmonic oscillators labeled by the index n. We define the Fock space vacuum $|0\rangle$ by $\alpha_{n>0}^\mu|0\rangle = 0$. For states in Fock space it holds that

$$\|\alpha_{-1}^\mu|0\rangle\|^2 = \langle 0|\alpha_{+1}^\mu\alpha_{-1}^\mu|0\rangle = \eta^{\mu\nu} = \begin{cases} -1 & \text{for} \quad \mu = 0 \\ +1 & \text{for} \quad \mu = 1, \ldots, D \end{cases} \,. \tag{2.56}$$

The bad thing is, that for $\mu = 0$ the state has a negative norm. But, we still have to impose the constraints

$$\langle \text{phys}|T_{ab}|\text{phys}'\rangle = 0 \,, \tag{2.57}$$

and then we solve the constraints explicitly (light cone gauge). We have to introduce light cone coordinates:

$$\sigma^\pm = \tau \pm \sigma \,, \quad X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \,, \tag{2.58}$$

and X^i for $i = 1, \ldots, D-2$ are the transverse coordinates. Then it holds that

$$X \cdot X = \sum_i (X^i)^2 - 2X^+X^- \,. \tag{2.59}$$

Writing down the Virasora constraints in these coordinates leads to

$$\partial_+ X \cdot \partial_+ X = 0 = \partial_- X \cdot \partial_- X \,. \tag{2.60}$$

We use this representation and set $X^+ = (2\pi\alpha' p^+ \tau)/l$ to obtain $X^-(X^i)$.

$$S_P = \frac{1}{4\pi\alpha'} \int \mathrm{d}\tau\, \mathrm{d}\sigma \left\{ \sum_{i=1}^{D-2} \left[(\dot{X}^i)^2 - (X'^i)^2 \right] \right\} \,, \tag{2.61}$$

$$H_{\text{light}-\text{cone}} = \frac{1}{4\pi\alpha'} \int \mathrm{d}\sigma \left\{ \sum_{i=1}^{D-2} \left[(\partial_\tau X)^2 + (\partial_\sigma X)^2 \right] \right\} = \frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma \sum_i \left\{ (\partial_+ X^i)^2 + (\partial_- X^i)^2 \right\} \,. \tag{2.62}$$

From $T_{++} = 0$ one gets

$$\partial_\pm X \cdot \partial_\pm X = \sum_i (\partial_\pm X^i)^2 - 2\partial_\pm X^+ \partial_\pm X^- = 0 \,. \tag{2.63}$$

Hence, X^- can be written in terms of X^i. For the open string one finds

$$H_{\text{light}-\text{cone}} = \frac{\pi}{2l} \sum_{i=1}^{D-2} \sum_{n \neq 0} \alpha_n^i \alpha_n^i + \text{zero modes} \,, \tag{2.64}$$

with the zero modes given by

$$\frac{\pi\alpha'}{l} \sum_i p^i p^i \,, \tag{2.65}$$

for (NN),

$$\frac{1}{4\pi\alpha' l}(\Delta q)^2 \,, \quad (\Delta q)^2 = \sum_{i=1}^{D-2} (q_f^i - q_i^i)^2 \,, \tag{2.66}$$

for (DD) and 0 for (DN) and (ND), repectively. For the closed string we obtain

$$H_{\text{light-cone}} = \frac{\pi}{l} \sum_{i=1}^{D-2} \sum_{n \neq 0} (\alpha_n^i \alpha_{-n}^i + \widetilde{\alpha}_n^i \widetilde{\alpha}_n^i) + \frac{\pi\alpha'}{l} \sum_{i=1}^{D-2} p^i p^i \,. \tag{2.67}$$

We write the Hamiltonian of a harmonic oscillator in a normal ordered form. We will also want to that for our Hamiltonians. Then we get an infinite sum of the form

$$\sum_{n=1}^{\infty} n\,. \tag{2.68}$$

We must make sense about of this expression. The correct value of this number will be important for determining the critical dimension of the string. Using $[\alpha_n^i, \alpha_m^j] = n\delta_{n+m,0}\delta^{ij}$ we arrive at

$$\frac{1}{2}\sum_{n\neq 0}\sum_{i=1}^{D-2}\alpha_n^i\alpha_{-n}^i = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n>0}(\alpha_{-n}^i\alpha_n^i + \alpha_n^i\alpha_{-n}^i) = \sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + A\,, \tag{2.69}$$

with

$$A = \frac{1}{2}(D-2)\sum_{n=1}^{\infty} n\,. \tag{2.70}$$

This sum needs to be regularized, because it is infinite.

- Method 1:

$$H_{\text{light-cone}}^{(\text{open})} = \frac{\pi}{l}\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \frac{\pi}{2l}(D-2)\sum_{n>0} n\,. \tag{2.71}$$

We introduce $1/\varepsilon$ as an UV cut-off:

$$\frac{\pi}{l}\sum_{n>0} n \mapsto \frac{\pi}{l}\sum_{n=1}^{\infty} n \exp\left(-\varepsilon\left(\frac{\pi}{l}n\right)\right)\,. \tag{2.72}$$

This can be understood as follows:

$$\sum_n E_n \mapsto \sum_n E_n \exp\left(-\frac{E_n}{\Lambda}\right)\,. \tag{2.73}$$

What we need is the expression

$$\sum_{n=1}^{\infty} nq^n = \frac{q}{(1-q)^2}\,, \quad q = \exp\left(-\frac{\varepsilon\pi}{l}\right)\,, \tag{2.74}$$

and by using that, one obtains

$$\frac{\pi}{l}\sum_{n=1}^{\infty} n \exp\left(-\frac{\varepsilon\pi n}{l}\right) \approx \frac{l}{\pi\varepsilon^2} - \frac{1}{12}\frac{\pi}{l} + \mathcal{O}(\varepsilon)\,, \tag{2.75}$$

and for the light-cone Hamiltonian:

$$H_{\text{light-cone}}^{(\text{open})} = \frac{\pi}{l}\left\{\sum_{i=1}^{D-2}\sum_{n>0}\alpha_{-n}^i\alpha_n^i - \frac{D-2}{24}\right\} + \frac{l}{\pi\varepsilon^2}\frac{D-2}{2} + \text{zero mode part}\,. \tag{2.76}$$

The third term is proportional to the length of the string

$$l = \int_0^l \mathrm{d}\sigma\,, \tag{2.77}$$

and is divergent as $\varepsilon \mapsto 0$. To absorb the divergent piece, one has to add a cosmological constant (in two dimensions, has nothing to do with cosmological constant in general relativity) to S_p. Adjusting Λ one can absorb the divergent part. At the end, the theory will be Weyl invariant.

- Method 2: ζ-function regularization

  We define

$$\zeta(s,\nu) = \sum_{n=0}^{\infty} (n+\nu)^{-s}\,, \tag{2.78}$$

which ist defined for $\mathrm{Re}(s) > 1$. For other values of s it is defined by analytic continuation.

$$\sum_{n=0}^{\infty}(n+\nu) \equiv \zeta(-1,\nu) = -\frac{1}{12}(1-6\nu(1-\nu)) = \begin{cases} -1/12 & \text{for} \quad \nu = 0 \\ -1/12 + 1/8 & \text{for} \quad \nu = 1/2 \end{cases}\,. \tag{2.79}$$

Hence, the normal ordering constant is just

$$A = -\frac{D-2}{24} + \frac{d}{16}\,, \tag{2.80}$$

where d is the number of (DN) plus the number of (ND) directions.

Chapter 3

The excitation spectrum of the bosonic string – critical dimension

States in the 1-string Hilbert space $\mathcal{H}_1$ are denoted as $|0,k\rangle$ defined by

$$p^i|0,k\rangle = k^i|0,k\rangle\,, \quad \alpha_n^i|0,k\rangle = 0\,, \quad \text{for } n > 0\,. \tag{3.1}$$

A general state in $\mathcal{H}_1$ can be constructed as

$$|N,k\rangle = \prod_{i=1}^{D-2}\prod_{n=1}^{\infty}(\alpha_{-n}^i)^{N_n}|0,k\rangle\,, \quad N_n \in \mathbb{N}_0\,. \tag{3.2}$$

The mass of the state is given by

$$m^2 = -p^\mu p_\mu = 2p^+p^- - p^ip^i = \frac{l}{\pi\alpha'}H_{\text{light-cone}} - p^ip^i\,. \tag{3.3}$$

For the open string it holds that

$$\alpha' m^2 = N + A + \frac{\alpha'}{(2\pi\alpha')^2}(\Delta q)^2\,, \quad A = -\frac{D-2}{24} + \frac{d}{16}\,, \quad N = \sum_{i=1}^{D-2} N_i\,, \tag{3.4}$$

with

$$N_i = \begin{cases} \sum\limits_{n>0} \alpha_{-n}^i\alpha_n^i \quad \text{for} \quad n \in \mathbb{Z}\,, \quad \text{(NN), (DD)} \\ \sum\limits_{n>0} \alpha_{-n}^i\alpha_n^i \quad \text{for} \quad n \in \mathbb{Z}+1/2\,, \quad \text{(ND), (DN)} \end{cases}\,. \tag{3.5}$$

The same for the closed string:

$$\alpha' m^2 = 2(N+\widetilde{N}) + A + \widetilde{A}\,, \tag{3.6}$$

with

$$A = \widetilde{A} = \frac{D-2}{24}\,, \quad N = \sum_{n>0}\sum_i \alpha_{-n}^i\alpha_n^i\,, \quad \widetilde{N} = \sum_{n>0}\sum_i \widetilde{\alpha}_{-n}^i\widetilde{\alpha}_n^i\,. \tag{3.7}$$

3.1 The excitation spectrum of the open string

The level is the sum of the modes which we have created out of the vacuum by applying the creation operator.

- level 0: $|0,k\rangle$, $m^2 = \dfrac{2-D}{24\alpha'}$
  For $D > 2$ this is a tachyon. (It has a negative mass square, which results in the instability of the theory. Then the question arises, in what it decays.)

- level 1: $\alpha^i_{-1}|0,k\rangle$, $m^2 = \frac{1}{\alpha'}\left(1+\frac{2-D}{24}\right) = \frac{26-D}{24\alpha'}$

  For $m > 0$ we go to the rest frame of the particle. $(p^\mu) = (m, 0, \ldots, 0)$ is invariant under $\mathrm{O}(D-1)$ (stability group/litte group of massive particles in D dimensions). For $m = 0$ we cannot got the rest frame, because the particle moves with the speed of light. $(p^\mu) = (E, 0, \ldots, 0, E)$ is invariant under $\mathrm{E}(D-2) \supset \mathrm{O}(D-2)$ (little group of massless particles in D dimensions).

  $\alpha^i_{-1}|0,k\rangle$ transforms as a vector of $\mathrm{O}(D-2)$. Hence, it must be massless and therefore we obtain $D = 26$ (critical dimension of bosonic string).

- level 2: $\alpha^i_{-2}|0,k\rangle$, $\alpha^i_{-1}\alpha^j_{-1}|0,k\rangle$

  These states have the same mass ($\alpha' m^2 = 1$). The first vector is a vector of SO(24). We split the first state into a traceless piece and a piece with non-vanishing trace:

$$\left(\alpha^i_{-1}\alpha^j_{-1} - \frac{1}{24}\alpha^k_{-1}\alpha^k_{-1}\right) + \frac{1}{24}\delta^{ij}\alpha^k_{-1}\alpha^k_{-1} \,. \tag{3.8}$$

  They transform differently under Lorentz transformations.

  Hence, the representations of SO(24) must combine to representations of SO(25) $\supset$ SO(24).

3.1.1 Comment

Consider a D_P-brane. We have $D-2-p$ scalars with respect to the Lorentz group which acts on the D_P-brane. α^i_{-1} $(i = 1, \ldots, p-1)$ parallel to the D_P-brane and α^a_{-1} $(a = p, \ldots, D-1)$ perpendicular to the brane.

3.1.2 Comment on spin of string excitations

Spin operator:

$$J^{ij} = \int_0^l \mathrm{d}\sigma\,(X^i\Pi^j - X^j\Pi^i) \equiv S^{ij} + L^{ij} \,, \tag{3.9}$$

with

$$S^{ij} = -\mathrm{i}\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n) \,, \quad L^{ij} = x^i p^j - x^j p^i \,. \tag{3.10}$$

At level N the maximal eigenvalue of a particular spin component (e.g. S^{23}) is N. The state with maximum spin is $(\alpha^2_{-1} + \mathrm{i}\alpha^3_{-1})^N|0,k\rangle$. With $m^2 = 1/\alpha'(N-1)$ it follows that $S^{23} \le N = 1 + \alpha' m^2$.

3.2 The excitation spectrum of the closed string

$\sigma \mapsto \sigma + \delta$ should be a symmetry. Considering the operator

$$U_\delta = \exp\left(\frac{2\pi\mathrm{i}}{l}\delta(N - \widetilde{N})\right) \,, \tag{3.11}$$

one can show that

$$U_\delta \alpha^i_n U^\dagger_\delta = \alpha^i_n \exp\left(-\frac{2\pi\mathrm{i}}{l}\delta_n\right) \,, \quad U_\delta \widetilde{\alpha}^i_n U^\dagger_\delta = \widetilde{\alpha}^i_n \exp\left(\frac{2\pi\mathrm{i}}{l}\delta_n\right) \,, \tag{3.12}$$

and hence

$$U_\delta X^i(\tau,\sigma) U^\dagger_\delta = X^i(\tau, \sigma+\delta) \,. \tag{3.13}$$

Require $U_\delta|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$, from which follows that $N = \widetilde{N}$ on physical states (level matching condition).

- ground state: $N = \widetilde{N} = 0$, $|0,0,k\rangle$, $m^2 = -\frac{D-2}{6\alpha'}$

  That is again a tachyon.

- first excited state: $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0,0,k\rangle$, $\alpha' m^2 = \dfrac{26-D}{6}$

  There are $(D-2)^2$ polarization states. These cannot be grouped into a nontrivial representation of $\mathrm{SO}(D-1)$. Hence, they are massless and $D=26$. Be v^{ij} a massless state.

$$v^{ij} = \frac{1}{2}\left(v^{ij} + v^{ji} - \frac{2}{D-2}\delta^{ij}v^{kk}\right) + \frac{1}{2}(v^{ij} - v^{ji}) + \frac{1}{D-2}\delta^{ij}v^{kk}\,, \tag{3.14}$$

  whereas the first part is symmetric and traceless, the second one is antisymmetric and the third one a singlet. All massive states can be grouped into representations of SO(25).

The remaining diffeo after fixing the light-cone gauge:

$$\sigma \mapsto \sigma' = l - \sigma\,, \quad \tau \mapsto \tau' = \tau\,. \tag{3.15}$$

- no $1/\infty$ tension of this diffeo
- it repeats periodically $\sigma \mapsto \sigma + nl \Rightarrow \sigma' \mapsto \sigma' - (n-1)l$
- is maps endpoints of the open string to endpoints

$$\sigma = 0 \mapsto \sigma = l\,, \quad \sigma = l\,, \quad \sigma = 0\,. \tag{3.16}$$

- Their diffeos reversed the orientation of the world sheet.
- It is generated by the world parity operator Ω.

$$\Omega X^\mu(\tau,\sigma)\Omega^\dagger = X^\mu(\tau, l-\sigma)\,. \tag{3.17}$$

  From $\Omega^2 = 1$ there follow the eigenvalues ± 1. For an open string ist holds that

$$\Omega\alpha^\mu_n\Omega^\dagger = (-1)^n\alpha^\mu_n\,, \tag{3.18}$$

  for (NN) and for a closed string

$$\Omega\alpha^\mu_n\Omega^\dagger = \widetilde{\alpha}^\mu_n\,, \quad \Omega\widetilde{\alpha}^\mu_n\Omega^\dagger = \alpha^\mu_n\,. \tag{3.19}$$

  Fix the phase of Ω by requiring

$$\Omega|0;k\rangle = |0;k\rangle\,, \tag{3.20}$$

  for an open string and

$$\Omega|0,0;k\rangle = |0,0;k\rangle\,, \tag{3.21}$$

  for a closed string. From this one obtains

$$\Omega|N;k\rangle = (-1)^N|N;k\rangle\,, \quad \Omega|N,\widetilde{N};k\rangle = |\widetilde{N},N;k\rangle\,, \tag{3.22}$$

  for an open and closed string, respectively. Now we can consider two types of strings

  - unoriented strings: $\Omega|\mathrm{phys}\rangle = |\mathrm{phys}\rangle$ (gauging of Ω)
  - oriented string: no restriction

Chapter 4

Superstrings

The goals are to describe space-time spinors (i.e. fermions) and to remove tachyonic degrees of freedem. There are two ways to do that:

- Green-Schwarz formalism (which will not discussed in detail)

  One introduces $X^\mu(\sigma,\tau)$, which is a world-sheet scalar and space-time vector, and $\theta^A(\sigma,\tau)$, where θ is a space-time spinor and A the associated spinor index.

- Newen-Schwarz-Ramond (RNS) formalism

  One introduces $X^\mu(\sigma,\tau)$ and $\Psi^\mu_\alpha(\sigma,\tau)$, whereas Ψ is a world sheet spinor and a space-time vector with spinor index α.

  Recall that in the Polyakov action, two-dimensional scalars couple to two-dimensional gravity. Two-dimensional diffeo and Weyl invariance allow for elimination of the two-dimensional metric γ_{ab}. The generalization is to couple (X^μ,Ψ^μ) to two-dimensional supergravity.

  $$\gamma_{ab} \mapsto (\gamma_{ab}, X^a_\alpha)\,, \quad X^\mu \mapsto (X^\mu,\Psi^\mu)\,, \tag{4.1}$$

  where X^a_α is the world sheet gravitino. Consider $\sigma^a \mapsto \sigma^a + \xi^a$:

  - diffeo $\mapsto$ diffeo plus SUSY transformation ($\xi^a \mapsto (\xi^a, \varepsilon_a)$)
  - Weyl symmetry $\mapsto$ super Weyl symmetry

  One can use these symmetries to eliminate γ_{ab}, X_α (super conformal gauge).

  $$S = -\frac{1}{4\pi\alpha'}\int \mathrm{d}^2\sigma\,\left\{\partial_a X^\mu \partial^a X_\mu - \mathrm{i}\alpha' \overline{\Psi}^\mu \varrho^a \partial_a \Psi_\mu\right\}\,, \tag{4.2}$$

  where ϱ^a are the two-dimensional Dirac matrices obeying

  $$\{\varrho^a,\varrho^b\} = \eta^{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,. \tag{4.3}$$

  An explicit representation of these matrices — the so-called Majorana basis — is given by

  $$\varrho^0 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \quad \varrho^1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,. \tag{4.4}$$

  In this representation, the compponents of $\overline{\Psi} = (\psi, \widetilde{\psi})$ can be chosen as real. Then it holds that

  $$\overline{\Psi} = \Psi^\dagger \varrho^0 = \Psi^\intercal \varrho^0\,, \tag{4.5}$$

  and

  $$S = -\frac{1}{\pi\alpha'}\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\left\{\partial_+ X^\mu \partial^- X_\mu + \frac{\mathrm{i}\alpha'}{2}(\Psi^\mu \partial_+ \Psi_\mu + \widetilde{\Psi}^\mu \partial_- \widetilde{\Psi}_\mu)\right\}\,. \tag{4.6}$$

4.1 Residual symmetries of a gauge fixed action

Consider holomorphic diffeos

$$\delta_\xi X = \xi^+\partial_+ X + \xi^-\partial_- X\,, \tag{4.7}$$

$$\delta_\xi \widetilde{\psi} = \xi^+\partial_+\psi + \frac{1}{2}(\partial_+\xi^+)\widetilde{\psi} + \xi^-\partial_-\widetilde{\psi}\,, \tag{4.8}$$

$$\delta_\xi \psi = \xi^-\partial_-\psi + \frac{1}{2}(\partial_-\xi^-)\psi + \xi^+\partial_+\psi\,, \tag{4.9}$$

and holomorphic supersymmetries

$$\delta_\eta X = \sqrt{\frac{\alpha'}{2}}(\widetilde{\eta}\widetilde{\psi} + \eta\psi)\,, \quad \delta_\eta\widetilde{\psi} = \mathrm{i}\sqrt{\frac{2}{\alpha'}}\,\widetilde{\eta}\partial_+ X\,, \quad \delta_\eta\psi = \mathrm{i}\sqrt{\frac{2}{\alpha'}}\,\eta\partial_- X\,, \tag{4.10}$$

whereas $\xi^\pm = \xi^\pm(\sigma^\pm)$ and $\widetilde{\eta} = \widetilde{\eta}(\sigma^+)$, $\eta = \eta(\sigma^-)$. Verification for $\widetilde{\eta} = 0$ (i.e. $\delta\widetilde{\psi} = 0$):

$$\begin{aligned}
\delta_\eta S &= \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\partial_+(\delta X)\partial_- X + \partial_+ X\partial_-(\delta X) + \frac{\mathrm{i}\alpha'}{2}((\delta\psi)\partial_+\psi + \psi\partial_+(\delta\psi))\right\} = \\
&= \frac{1}{\pi\alpha'}\int \mathrm{d}^2\sigma \left\{\sqrt{\frac{\alpha'}{2}}\partial_+(\eta\psi)\partial_- X + \sqrt{\frac{\alpha'}{2}}\partial_+ X\partial_-(\eta\psi) - \sqrt{\frac{\alpha'}{2}}(\eta\partial_- X\partial_+\psi + \psi\partial_+(\eta\partial_- X))\right\} = \\
&\overset{\text{p.I.}}{=} \frac{1}{\pi\sqrt{2\alpha'}}\int \mathrm{d}^2\sigma\ \{-2\partial_+\partial_- X(\eta\psi) + \partial_+(\eta\partial_- X)\psi - \psi\partial_+(\eta\partial_- X)\} = \\
&= \frac{1}{\pi}\frac{1}{\sqrt{2\alpha'}}\int \mathrm{d}^2\sigma\ \{-2\partial_+\partial_- X(\eta\psi) + \partial_+\partial_- X(\eta\psi) - \psi\partial_- X\partial_+\eta + \partial_+\partial_- X\psi\eta - \psi\partial_- X\partial_+\eta\} = \\
&= \frac{1}{\pi}\frac{1}{\sqrt{2\alpha'}}\int \mathrm{d}^2\sigma\, \psi\partial_- X\partial_+\eta = 0\,,
\end{aligned} \tag{4.11}$$

if $\partial_+\eta = 0$. □

We also learn that $T_F \sim \psi\partial_- X$ is conserved. We need to impose the constraints

$$T_F \sim \psi\partial_- X = 0\,, \tag{4.12a}$$

$$\widetilde{T}_F \sim \widetilde{\psi}\partial_+ X = 0\,. \tag{4.12b}$$

Good exercise: Do the same for holomorphic diffeos $\sigma^+ \mapsto \sigma^+ + \xi^+$, $\sigma^- \mapsto \sigma^-$. Then it follows that

$$T_{\pm\pm} \sim (\partial_\pm X)(\partial_\pm X) + \frac{\mathrm{i}\alpha'}{2}\widetilde{\psi}^\mu\partial_\pm\widetilde{\psi}_\mu \overset{!}{=} 0\,. \tag{4.13}$$

Eqs. (4.12a) and (4.13) are called super Virasoro constraints. It is an exercise to show that these transformations satisfy the **SUSY algebra**:

$$[\delta_\lambda, \delta_\eta]X = \xi^+\partial_+ X + \xi^-\partial_- X\,, \quad \xi^- = 2\mathrm{i}\eta\lambda\,, \quad \xi^+ = 2\mathrm{i}\widetilde{\eta}\widetilde{\lambda}\,, \tag{4.14}$$

$$[\delta_\lambda, \delta_\eta]\psi = \xi^-\partial_-\psi + \frac{1}{2}(\partial_-\xi^-)\psi\,. \tag{4.15}$$

The equations of motion follow from the variation of the action with respect to the bosonic and fermionic fields, respectively.

$$\frac{\delta S}{\delta X^\mu} = 0 \Rightarrow \Box X^\mu = 0\,, \tag{4.16}$$

with corresponding boundary conditions (for open strings) and periodic boundary conditions (for closed strings).

$$\frac{\delta S}{\delta\psi^\mu} = 0\,. \tag{4.17}$$

From

$$\begin{aligned}\delta S \sim = & \int \mathrm{d}\sigma\,\mathrm{d}\tau\,\left\{\delta\psi\partial_+\psi_+\psi\partial_+\delta\psi+\delta\widetilde{\psi}\partial_-\widetilde{\psi}+\widetilde{\psi}\partial_-\delta\widetilde{\psi}\right\} = \\ = & \int \mathrm{d}\sigma\,\mathrm{d}\tau\,\left\{-2(\partial_+\psi)\delta\psi-2(\partial_-\widetilde{\psi})\delta\widetilde{\psi}+\partial_+(\psi\delta\psi)+\partial_-(\widetilde{\psi}\delta\widetilde{\psi})\right\} = \\ = & -2\int \mathrm{d}\sigma\,\mathrm{d}\tau\,\left\{\partial_+\psi\delta\psi+\partial_-\widetilde{\psi}\delta\widetilde{\psi}\right\}+\frac{1}{2}\int \mathrm{d}\tau\,\left[(\delta\psi)\psi-(\delta\widetilde{\psi})\widetilde{\psi}\right]\Big|_{\sigma=0}^{\sigma=l}\,,\end{aligned} \tag{4.18}$$

it follows that

$$\partial_+\psi^\mu = 0\,,\quad \partial_-\widetilde{\psi}^\mu = 0\,, \tag{4.19}$$

plus (periodic) boundary conditions for an (closed) open string. We need to require

$$\psi^\mu\delta\psi_\mu - \widetilde{\psi}^\mu\delta\widetilde{\psi}_\mu\Big|_{\sigma=0}^{\sigma=l} = 0\,. \tag{4.20}$$

The solutions are given by:

$$\psi(\delta\psi)|_0^l = 0\,,\quad \widetilde{\psi}(\delta\widetilde{\psi})|_0^l = 0\,, \tag{4.21}$$

for a **closed** string with ψ, $\widetilde{\psi}$ decoupled and

$$\psi(\delta\psi) - \widetilde{\psi}(\delta\widetilde{\psi})|_{\sigma=0} = 0\,,\quad \psi(\delta\psi) - \widetilde{\psi}(\delta\widetilde{\psi})|_{\sigma=l} = 0\,, \tag{4.22}$$

for an open string. An open string has to satisfy the boundary conditions separately at each end-point. We require at each end-point

$$\psi^\mu = \pm\widetilde{\psi}^\mu\,,\quad \left(\delta\psi^\mu = \pm\delta\widetilde{\psi}^\mu\right)\,. \tag{4.23}$$

Without loss of generality one can choose $\psi^\mu(\tau,0) = \widetilde{\psi}^\mu(\tau,0)$ and $\psi^\mu(\tau,l) = \eta\widetilde{\psi}^\mu(\tau,l)$ with a phase $\eta = \pm 1$. The choice $\eta = 1$ is called Ramond (R) sector and the choice $\eta = -1$ Newen-Schwarz (NS) sector.
The solutions of the equations of motion are

$$\widetilde{\psi}^\mu = \sum_r \sqrt{\frac{2\pi}{l}}\psi_r^\mu \exp\left(-\frac{\mathrm{i}\pi r(\tau+\sigma)}{l}\right)\,, \tag{4.24a}$$

$$\psi^\mu = \sum_i \sqrt{\frac{2\pi}{l}}\psi_r^\mu \exp\left(-\frac{\mathrm{i}\pi r(\tau+\sigma)}{l}\right)\,, \tag{4.24b}$$

where one has to distinguish between the two sectors. In Eq. (4.24a) it holds that $r \in \mathbb{Z}+1/2$ (NS) and in (4.24b) it holds that $r \in \mathbb{Z}$ (R). Consider a **closed** string:

$$\psi\delta\psi|_0^l = 0\,,\quad \widetilde{\psi}\delta\widetilde{\psi}|_0^l = 0\,. \tag{4.25}$$

- ψ and $\widetilde{\psi}$ are independent.
- One can choose ψ and $\widetilde{\psi}$ either as periodic or antiperiodic.

$$\psi^\mu(\tau,\sigma+l) = \eta\psi^\mu(\tau,\sigma)\,,\quad \eta = \pm 1\,, \tag{4.26}$$

$$\widetilde{\psi}^\mu(\tau+l) = \widetilde{\eta}\widetilde{\psi}^\mu(\tau,\sigma)\,,\quad \widetilde{\eta} = \pm 1\,. \tag{4.27}$$

We consider the same boundary conditions, because this preserves space-time Lorentz invariance.

$$\psi(\tau,\sigma) = \sqrt{\frac{2\pi}{l}}\sum_r \psi_r \exp\left(-\frac{2\pi\mathrm{i}r(\tau-\sigma)}{l}\right)\,,\quad \widetilde{\psi}(\tau,\sigma) = \sqrt{\frac{2\pi}{l}}\sum_r \widetilde{\psi}_r \exp\left(-\frac{2\pi\mathrm{i}r(\tau+\sigma)}{l}\right)\,, \tag{4.28}$$

with $r \in \mathbb{Z}$ (R) and $r \in \mathbb{Z}+1/2$ (NS). Now, one can choose the following signs of η, $\widetilde{\eta}$:

- $\eta = \widetilde{\eta} = -1$ (NS,NS), $\eta = \widetilde{\eta} = 1$ (R,R)
  These choices correspond to space-time bosons.
- $\eta = -\widetilde{\eta} = 1$ (R,NS), $\eta = -\widetilde{\eta} = -1$ (NS,R)
  These choices correspond to fermions.

4.2 Quantization of the fermionic string: critical dimension

Here, we will use light-cone gauge only. We use the symmetries to set $\gamma^{ab} = \eta^{ab}$, $\chi^a_\alpha = 0$, $X^+ \sim \tau$, $\psi^\dagger = \widetilde{\psi}^\dagger = 0$. (We will skip the proof that is it indeed possible to choose that gauge.) The constraints are $T_{++} = T_{--} = 0$, $T_{F,+} = T_{F,-} = 0$.

4.2.1 Quantization of fermions

- We have to impose anti-commutation relations, because of the opposite statistics compared to bosons.
- One has to use Dirac brackets (system of second class constraints). Note that e.g. the canonical momentum

$$\Pi = \frac{\partial^L \mathcal{L}}{\partial(\partial_0 \psi)} = -\frac{\mathrm{i}}{4\pi}\psi \sim \psi\,. \tag{4.29}$$

It is crucial to differ between left- and right-acting derivatives, because the fields anticommute. We define

$$\phi = \pi + \frac{\mathrm{i}}{4\pi}\psi \simeq 0\,, \tag{4.30}$$

which is called a second-class constraint. (They require a modification of the quantization procedure.) If one defines the Poisson bracket

$$\{\psi,\psi\}_{\mathrm{PB}} = 0\,, \quad \{\Pi,\Pi\}_{\mathrm{PB}} = 0\,, \{\psi,\Pi\}_{\mathrm{PB}} = -\delta(\sigma-\sigma')\,, \tag{4.31}$$

where both functions are evaluated at the same τ, one finds the following:

$$\{\phi,\phi\}_{\mathrm{PB}} = -\frac{\mathrm{i}}{2\pi}\delta(\sigma-\sigma') \equiv C = -\frac{\mathrm{i}}{2\pi}\mathbb{1}\,. \tag{4.32}$$

Then, one defines the Dirac bracket as

$$\{\psi,\psi\}_{\mathrm{DB}} \equiv \{\psi,\psi\}_{\mathrm{PB}} - \{\psi,\phi\}_{\mathrm{PB}} C^{-1} \{\phi,\psi\}_{\mathrm{PB}} = -2\pi\mathrm{i}\mathbb{1}\,. \tag{4.33}$$

Analogously:

$$\{\psi,\Pi\}_{\mathrm{DB}} = \{\psi,\Pi\}_{\mathrm{PB}} - \{\psi,\phi\}_{\mathrm{PB}} C^{-1} \{\phi,\Pi\}_{\mathrm{PB}} = -\frac{1}{2}\mathbb{1}\,. \tag{4.34}$$

From this follows that

$$\{\psi^i(\tau,\sigma),\psi^j(\tau,\sigma')\}_{\mathrm{DB}} = -2\pi\mathrm{i}\delta(\sigma-\sigma')\delta^{ij}\,, \tag{4.35}$$

$$\{\widetilde{\psi}^i(\tau,\sigma)\widetilde{\psi}^j(\tau,\sigma')\}_{\mathrm{DB}} = -2\pi\mathrm{i}\delta(\sigma-\sigma')\delta^{ij}\,. \tag{4.36}$$

The quantization now proceeds by replacing the Dirac bracket by the anti-commutator: $\{\bullet,\bullet\}_{\mathrm{DB}} \mapsto -\mathrm{i}[\bullet,\bullet]_+$:

$$\{\psi^i(\tau,\sigma),\psi^j(\tau,\sigma')\} = 2\pi\delta(\sigma-\sigma')\delta^{ij}\,, \quad \{\widetilde{\psi}^i(\tau,\sigma),\widetilde{\psi}^j(\tau,\sigma')\} = 2\pi\delta(\sigma-\sigma')\delta^{ij}\,, \tag{4.37}$$

$$\{\psi^i_r,\psi^j_s\} = \delta_{r+s,0}\delta^{ij}\,, \quad \{\widetilde{\psi}^i_r,\widetilde{\psi}^j_s\} = \delta_{r+s,0}\delta^{ij}\,, \quad \{\psi^i_r,\widetilde{\psi}^j_s\} = 0\,. \tag{4.38}$$

The next step is to express $H_{\mathrm{light-cone}}$ in terms of transverse oscillators. Since bosons and fermions decouple, one has

$$H_{\mathrm{light-cone}} = H^{(\mathrm{bose})}_{\mathrm{light-cone}} + \frac{\mathrm{i}}{4\pi}\int_0^l \mathrm{d}\sigma\,(\psi^i\dot{\psi}^i + \widetilde{\psi}^i\dot{\widetilde{\psi}}^i)\,. \tag{4.39}$$

Inserting the mode-expansions in the second term leads to

$$\frac{\mathrm{i}}{4\pi}\int_0^l \mathrm{d}\sigma\,(\psi^i\dot{\psi}^i + \widetilde{\psi}^i\dot{\widetilde{\psi}}^i) = \frac{\pi}{2l}\sum_r r\psi^i_{-r}\psi^i_r\,, \tag{4.40}$$

for the open string and then one finds:

$$H^{(\text{open})}_{\text{light-cone}} = \frac{\pi}{2l} \sum_{i=1}^{D-2} \left\{ \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n + \sum_r r\psi^i_{-r}\psi^i_r \right\} + \frac{\pi\alpha'}{l} \sum_i p^i p^i \,, \tag{4.41}$$

with the last term für (NN) boundary conditions. One then obtains for the mass operator

$$(m^2)^{(\text{open})} = \frac{1}{\alpha'} \left\{ \sum_{n>0} \alpha^i_{-n}\alpha^i_n + \sum_{r>0} r\psi^i_{-r}\psi^i_r \right\} + A \,, \tag{4.42}$$

with $r \in \mathbb{Z}$ for (R) and $r \in \mathbb{Z} + 1/2$ for (NS). For the closed string one gets

$$(m^2)^{(\text{closed})} = \frac{2}{\alpha'} \left\{ \sum_{n>0} (\alpha^i_{-n}\alpha^i_n + \widetilde{\alpha}^i_{-n}\widetilde{\alpha}^i_n) + \sum_{r>0} (r\psi^i_{-r}\psi^i_r + r\widetilde{\psi}^i_{-r}\widetilde{\psi}_r) \right\} + A + \widetilde{A} \,. \tag{4.43}$$

These are the expressions before doing the normal ordering. Now, we need to determine the normal ordering constants A, $\widetilde{A}$:

$$\begin{aligned} \frac{1}{2}\sum_i \sum_{n\neq 0} \alpha^i_{-n}\alpha^i_n + \frac{1}{2}\sum_i \sum_{r\neq 0} r\psi^i_r\psi^i_r &= \\ &= \frac{1}{2}\sum_i \sum_{n>0} (\alpha^i_{-n}\alpha^i_n + \alpha^i_n\alpha^i_{-n}) + \frac{1}{2}\sum_i \sum_{r>0} r(\psi^i - -r\psi^i_r - \psi^i_r\psi^i_{-r}) \end{aligned} \tag{4.44}$$

Using

$$\alpha^i_n\alpha^i_{-n} = n + \alpha^i_{-n}\alpha^i_n \,, \quad \psi^i_r\psi^i_{-r} = 1 - \psi^i_{-r}\psi^i_r \,, \tag{4.45}$$

one obtains

$$\sum_{n>0}\sum_i \alpha^i_{-n}\alpha^i_n + \sum_i \sum_{r>0} r\psi^i_{-r}\psi^i_r + \frac{1}{2}(D-2) \left\{ \sum_{n>0} - \sum_r r \right\} . \tag{4.46}$$

Here, we need ζ-function renormalization

$$\sum_{n>0} n = \zeta(-1,0) = -\frac{1}{12} \,, \quad \sum_{r>0} r = \begin{cases} \zeta(-1,0) = -1/12 & \text{for} \quad (\text{R}) \\ \zeta(-1,1/2) = 1/24 & \text{for} \quad (\text{NS}) \end{cases} \,, \tag{4.47}$$

and one ends up with

$$A^{(\text{open})} = \begin{cases} 0 & \text{for} \quad (\text{R}) \\ -(D-2)/16 & \text{for} \quad (\text{NS}) \end{cases} \,. \tag{4.48}$$

It must hold that

$$-\frac{D-2}{16} + \frac{d}{8} \,, \tag{4.49}$$

für d (ND+DN) boundary conditions.

4.3 Excitation spectrum of the open fermionic string

We only choose (NN), (DD) boundary conditions. Let us look at the (NS) sector first:

$$\alpha^i_n|0;k\rangle = \psi^i_r|0;k\rangle = 0 \,, \quad n>0,\, r>0 \,. \tag{4.50}$$

- We have a ground state $|0,k\rangle$ with $\alpha' m^2 = -(D-2)/16$.
- The first excited state is $\psi^i_{-1/2}|0;k\rangle$ with $\alpha' m^2 = \frac{1}{2} - \frac{D-2}{16} = -\frac{D-10}{16}$.

  Since it has only $D-2$ degrees of freedom, it must be a massless vector and from that one obtains $D_{\text{crit}} = 10$.

- The higher excited states with $m^2 > 0$ fall into the represenations of SO(9).

Now we are coming to the (R) sector: Note that fermions have zero modes ψ_0^i.

$$\{\psi_0^i, \psi_0^j\} = \delta^{ij}, \quad i = 1, \ldots, 8. \tag{4.51}$$

Note:

$$\frac{1}{\sqrt{2}}\Gamma^i = \psi_0^i, \quad \{\Gamma^i, \Gamma^j\} = 2\delta^{ij}. \tag{4.52}$$

One can show that in even dimensions there is a unique representation of the Clifford algebra in terms of Dirac matrices. Hence, $\Gamma^i = \sqrt{2}\psi^i$ are 16×16 Dirac matrices of SO(8). (In general: In $D = 2n$ dimensions the dimensionality of the matrices is given by 2^n.) The mass operator is

$$\alpha' m^2 = \sum_i^{\infty} \alpha_{-n}^i \alpha_n^i + \sum_{n>0} n\psi_{-n}^i \psi_n^i, \tag{4.53}$$

with $[m^2, \psi_0^i] = 0$. From this we find that the ground state in the (R) sector is degenerate and has $m^2 = 0$. We will denote these states by $|A\rangle$, where $A = 1, \ldots, 16$ labels the degeneracy.

$$\psi_0^i|A\rangle = \frac{1}{\sqrt{2}}(\Gamma^i)_A{}^B|B\rangle, \tag{4.54}$$

where A, B are spinor indices of Γ^i. Hence, the ground state of the (R) sector transforms like a space-time spinor of SO(8). (That is why it must be massless.) All the states in the (R) sector are space-time spinors and therefore fermions.
The 16-dimensional spinor representation of SO(8) is reducible. Define

$$\Gamma^9 = \Gamma^1 \ldots \Gamma^8, \quad (\Gamma^9)^2 = 1, \quad \{\Gamma^1, \Gamma^9\} = 0, \tag{4.55}$$

from what follows that the eigenvalues of Γ^9 are ± 1.

$$\mathrm{Tr}(\Gamma^9) = \mathrm{Tr}(\Gamma^1\Gamma^2 \ldots \Gamma^8) = -\mathrm{Tr}(\Gamma^2\Gamma^8\Gamma^2) = -\mathrm{Tr}(\Gamma^1\Gamma^2 \ldots \Gamma^8) = -\mathrm{Tr}(\Gamma^9) = 0. \tag{4.56}$$

Hence, Γ^9 has got 8 eigenvalues $+1$ and 8 eigenvalues -1. Γ^9 itself commutes with $M^{ij} \sim [\Gamma^i, \Gamma^j]$ (the generators of SO(8)), what — by Schur's lemma — means that Γ^9 is constant on each irreducible representation. So, the representation splits as $\mathbf{16} = \mathbf{8}_s + \mathbf{8}_c$ into the two irreducible represenations of SO(8). One splits the 16-dimensional space into the two eigenspaces of Γ^9. Then, we get

$$|A\rangle = |a\rangle \oplus |\dot{a}\rangle, \quad \Gamma^9|a\rangle = |a\rangle, \quad \Gamma^9|\dot{a}\rangle = -|\dot{a}\rangle, \tag{4.57}$$

with $A = 1, \ldots, 16$ and $a = 1, \ldots, 8$ and $\dot{a} = 1, \ldots, 8$.

4.4 Massless states of open fermionic string

- (NS) sector:
  - $|0\rangle$ with $\alpha' m^2 = -1/2$
  - $\psi_{-1/2}^i|0\rangle$ with $\alpha' m^2 = 0$ (massless vector of SO(8))
- (R) sector:
  - $|a\rangle$, $|\dot{\alpha}\rangle$ (massless spinors, one left- and one right-handed)

4.5 Spectrum of closed fermionic string

We need to impose the level matching condition.

$$\exp(\mathrm{i}\pi\delta(N - \widetilde{N})), \quad N = \sum_{n>0} \alpha_{-n}^i \alpha_n^i + \sum_{r>0} r\psi_{-r}^i \psi_r^i. \tag{4.58}$$

$\alpha' m^2$	states and their SO(8) representation constant	little group with respect to little group	representation constant	$(-1)^F$	$(-1)^{\widetilde{F}}$
		(NS,NS)			
-2	$\|0\rangle \otimes \|0\rangle$ $\mathbf{1}$ $\mathbf{1}$	SO(9)	$\mathbf{1}$	-1	-1
0	$\widetilde{\psi}^i_{-1/2}\|0\rangle \otimes \widetilde{\psi}^j_{-1/2}\|0\rangle$ $\mathbf{8}_v$ $\mathbf{8}_v$	SO(8)	$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	1	1
		(R,R)			
0	$\|a\rangle \otimes \|b\rangle$ $\mathbf{8}_s$ $\mathbf{8}_s$	SO(8)	$\mathbf{1}+\mathbf{28}+\mathbf{35}_+$	1	1
	$\|\dot{a}\rangle \otimes \|\dot{b}\rangle$ $\mathbf{8}_c$ $\mathbf{8}_c$	SO(8)	$\mathbf{1}+\mathbf{28}+\mathbf{35}_-$	-1	-1
	$\|a\rangle \otimes \|\dot{b}\rangle$ $\mathbf{8}_s$ $\mathbf{8}_c$		$\mathbf{8}_v+\mathbf{56}_v$	1	-1
	$\|\dot{a}\rangle \otimes \|b\rangle$ $\mathbf{8}_c$ $\mathbf{8}_s$		$\mathbf{8}_v+\mathbf{56}_v$	-1	1
		(R,NS)			
	$\|a\rangle \otimes \psi^i_{-1/2}\|0\rangle$ $\mathbf{8}_s$ $\mathbf{8}_v$	SO(8)	$\mathbf{8}_c+\mathbf{56}_c$	1	1
	$\|\dot{a}\rangle \otimes \psi^i_{-1/2}\|0\rangle$ $\mathbf{8}_c$ $\mathbf{8}_s$		$\mathbf{8}_s+\mathbf{56}_s$	-1	1
		(NS,R)			
	$\widetilde{\psi}^i_{-1/2}\|0\rangle \otimes \|a\rangle$ $\mathbf{8}_v$ $\mathbf{8}_s$	SO(8)	$\mathbf{8}_c+\mathbf{56}_c$	1	1
	$\widetilde{\psi}^i_{-1/2}\|0\rangle \otimes \|0\rangle$ $\mathbf{8}_v$ $\mathbf{8}_c$		$\mathbf{8}_s+\mathbf{56}_s$	1	-1

There are several problems with this theory:

1) tachyons,
2) lack of modular invariance, invariance under large diffeos,
3) non-local operator product expansion of vertex operators.

The cure of all these problems is given by the Gliozzi-Scherk-Olive-projection (GSO). Define operators F and $\widetilde{F}$ such that

$$\{(-1)^F, \psi^i\} = 0 = \{(-1)^{\widetilde{F}}, \widetilde{\psi}^i\}\,. \tag{4.59}$$

For the (NS) sector, this operator is given by

$$F = \sum_{r>0} \psi^i_{-r}\psi^r_r - 1\,, \quad \widetilde{F} = \ldots \tag{4.60}$$

The first term measures the number of world-sheet fermions. A state

$$\alpha^{i_1}_{-n_1}\ldots\alpha^{i_N}_{-n_N}\psi^{j_1}_{-r_1}\ldots\psi^{j_M}_{-r_M}|0\rangle_{\rm NS}\,, \tag{4.61}$$

has $(-1)^F = (-1)^{M+1}$ and hence $(-1)^F|0\rangle_{\rm NS} = -|0\rangle_{\rm NS}$. For the (R) sector we have

$$(-1)^F = \Gamma^9 \exp\left(\mathrm{i}\pi\sum_{m>0}\psi^i_{-m}\psi^i_m\right)\,, \quad \Gamma^9|a\rangle = |a\rangle\,, \quad \Gamma^9|\dot{a}\rangle = -|\dot{a}\rangle\,. \tag{4.62}$$

A state

$$\alpha^{i_1}_{-n_1}\ldots\alpha^{i_N}_{-n_N}\psi^{j_1}_{-m_1}\ldots\psi^{j_M}_{-m_M}|a\rangle\,, \tag{4.63}$$

has

$$(-1)^F = (-1)^M(-1)^{\sum_j \delta_{m_j,0}}\,. \tag{4.64}$$

Another state

$$\alpha^{i_1}_{-n_1} \dots \alpha^{i_N}_{-n_N} \psi^{j_1}_{-m_1} \dots \psi^{j_M}_{-m_M} |\dot{a}\rangle \,, \tag{4.65}$$

has

$$(-1)^F = -(-1)^M (-1)^{\sum_j \delta_{m_j,0}} \,. \tag{4.66}$$

Two consistent GSO-projections:

- For NS-states require $(-1)^F = (-1)^{\widetilde{F}} = 1$.
- For R-states:
  - $(-1)^F = (-1)^{\widetilde{F}} = 1$ (IIB-theory)

(NS,NS)	(R,R)	(NS,R)	(R,NS)
$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	$\mathbf{1}+\mathbf{28}+\mathbf{35}_+$	$\mathbf{8}_c+\mathbf{56}_c$	$\mathbf{8}_c+\mathbf{56}_c$
128 space-time bosons		128 space-time fermions	
chiral spectrum, $N=2$ space-time SUSY in $D=10$, IIB SUGRA			

  - (NS,NS): $\mathbf{1}$: Dilaton, $\mathbf{28}$: anti-symmetric tensor, $\mathbf{35}_v$: graviton
  - (NS,R)+(R,NS): $\mathbf{8}_c$: dilatino (superpartner of dilaton)
  - (NS,R)+(R,NS): $\mathbf{56}_c$: gravitino

  - $(-1)^F = -(-1)^{\widetilde{F}} = -1$ (IIA-theory)

(NS,NS)	(R,R)	(NS,R)	(R,NS)
$\mathbf{1}+\mathbf{28}+\mathbf{35}_v$	$\mathbf{8}_v+\mathbf{56}_v$	$\mathbf{8}_s+\mathbf{56}_s$	$\mathbf{8}_c+\mathbf{56}_c$
128 space-time bosons		128 space-time fermions	
non-chiral spectrum, $N=2$ space-time SUSY in $D=10$, IIA SUGRA			

Neither of these theories has a tachyonic state. IIA and IIB SUGRA are low energy effective field theories of the massless string states. Be $\langle\phi\rangle$ the vacuum expectation value of the dilaton. Then, the string coupling constant, which measures the number of loops on the world sheet, is given by $g_s = \exp(\langle\phi\rangle)$.

$$G_N^{(10)} = g_s^2 l_s^8 \,, \quad \frac{1}{G_N^{(10)}} \sim \frac{\exp(-2\langle\phi\rangle)}{l_s^8} \,. \tag{4.67}$$

The states of the standard model should be searched for among the massless string excitations. Other string excitations are of order Planck scale and much to heavy to be produced in collider experiments.

The type-II theory has a world-sheet parity operator which exchanges left- and right-movers, as symmetry. Therefore, we can gauge this symmetry, which leads to the theory of unoriented closed strings. Which states survive the projection? The field content of $N=1$ SUGRA in $D=10$ is given by:

- (NS,NS): $\mathbf{1}+\mathbf{35}$,
- (NS,R)+(R,NS): $\mathbf{8}_c+\mathbf{56}_c$,
- (R,R): $\mathbf{28}$.

Typ IIB/(Ω projection) does not lead to a consistent theory. The cure is to add open strings and 32 D9-branes and 1 orientifold plane. The type I string contains oriented plus non-oriented open plus closed strings. The gauge symmetry is SO(32). The effective low-energy field theory of massless excitations is Type I SUGRA coupled to $N=1$ SUSY with gauge group SO(32).

	SUSY	
Type IIA	$N=2$	closed, oriented
Type IIB	$N=2$	closed, oriented
Type I	$N=1$	closed plus open, oriented plus non-oriented
heterotic $E_8 \times E_8$	$N=1$	closed, oriented
heterotic SO(32)	$N=1$	closed, oriented

L	R	
bosonic	fermonic	
$D_L = 26$	$D_R = 10$	$D_L - D_R = 16$, $T^{16} = \mathbb{R}^{16}/\Gamma_{16}$ (lattice, which defines a torus)

Chapter 5

String Compactification

The consistency of the theory has to require that the string lives in Minkowski space-time. We have to find a way to formulate string theory in non-trivial background. We will only consider the bosonic sector of the superstring. We can write the Polyakov action in the following form:

$$S_P = -\frac{1}{4\pi\alpha'}\int\limits_{\Sigma} \mathrm{d}^2\sigma\,\sqrt{-\gamma}\left\{\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \mathrm{i}\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(x) + \alpha'\Phi(x)^{(2)}R(\gamma)\right\}\,. \tag{5.1}$$

We have coupled the world sheet to a symmetric and antisymmetric tensor field and the dilaton. The backgrounds are not arbitrary. They are constraint by the requirement of conformal invariance. Note that in the case of $G_{\mu\nu} = \eta_{\mu\nu}$, $B_{\mu\nu} = 0$ and $\Phi = \text{const.}$ this is a free field theory. For $G_{\mu\nu}(x)$, $B_{\mu\nu}(x)$ and $\Phi(x)$ is is an interacting conformal field theory. The conformal invariance of the interacting theory requires all β-functions to vanish. The β-functions follow from lowest order perturbation theory (one-loop Feynman diagrams) with respect to α'. To obtain the β-functions one does a so-called background expansion of the metric:

$$G_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\varrho\nu\sigma}(x^0)(x-x^0)^\varrho(x-x^0)^\sigma + \dots\,, \tag{5.2}$$

where at the point x^0 the metric is flat and the Christoffel symbols vanish. Then, one has to calculate the corresponding Feynman diagrams. The results are:

$$\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi - \frac{\alpha'}{4}H_{\mu\varrho\sigma}H_\nu{}^{\varrho\sigma} + \mathcal{O}(\alpha'^2)\,, \tag{5.3a}$$

$$\beta^B_{\mu\nu} = -\frac{\alpha'}{2}\nabla^\varrho H_{\varrho\mu\nu} + \alpha'\nabla^\varrho\Phi H_{\varrho\mu\nu} + \mathcal{O}(\alpha'^2)\,, \tag{5.3b}$$

$$\beta^\Phi = \frac{1}{6}(D-26) - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'(\nabla\Phi)^2 - \frac{\alpha'}{24}H_{\mu\nu\varrho}H^{\mu\nu\varrho} + \mathcal{O}(\alpha'^2)\,, \tag{5.3c}$$

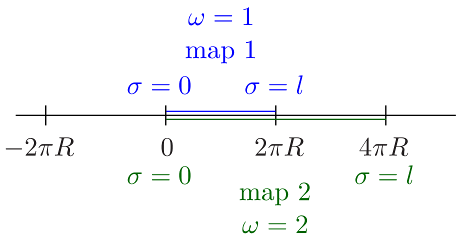
for a bosonic string and with

$$H_{\mu\nu\varrho} = \partial_\mu B_{\nu\varrho} + \partial_\nu B_{\varrho\mu} + \partial_\varrho B_{\mu\nu}\,. \tag{5.4}$$

$\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ corresponds to the equations of motion for the background fields. Some exact solutions (to all orders in α') are known:

- For constant Φ, $B_{\mu\nu}$ and flat space-time ($G_{\mu\nu} = \eta_{\mu\nu}$) the equations are solved to all orders of perturbation theory.
- Plane gravitational waves: $G_{\mu\nu}(x)$, $G_{\mu\nu}(x)$, $\Phi = \text{const.}$
- Calabi-Yau manifolds: $C_{\mu\nu}(x)$, $B_{\mu\nu} = 0$, $\Phi = \text{const.}$

Now we want to look for solutions to $\beta^G_{\mu\nu} = \beta^B_{\mu\nu} = \beta^\Phi = 0$ for $\mathcal{M}^{10} = \mathcal{M}^{3,1}\times K_6$. where $\mathcal{M}^{3,1}$ is the four-dimensional Minkowski space-time and K_6 a compact six-dimensional manifold. The simplest example for K_6 is a flat 6-torus: $T^6 = \mathbb{R}^6/\Lambda^6$, where Λ^6 is a six-dimensional lattice (periodic identifications). Flat means that $R_{\mu\nu} = 0$ or $G_{ij} = \delta_{ij}$. The simplest torus compactification (for a closed bosonic string) is on $T^1 = S^1 = \mathbb{R}/2\pi R\mathbb{Z}$. So, we compactify $X \equiv X^{25}$. What does this compactification mean? We identify $X = X + 2\pi R\omega$ with $\omega \in \mathbb{Z}$. This periodicity has two effects:

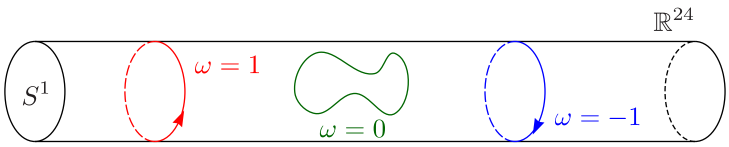


- It leads to a quantization of momentum.

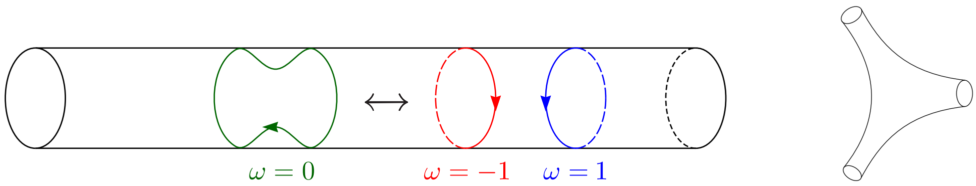
  $\exp(2\pi \mathrm{i} Rp)$: $q \mapsto q + 2\pi R$. Hence, the p eigenvalues are quantized: $p = n/R$ with $n \in \mathbb{Z}$.

- winding:

  $X(\tau, \sigma)$ for $0 \leq \sigma \leq l$ is a map from an object with the topology of a circle to another circle: $S^1 \mapsto S^1$ $(\pi^1(S^1) = \mathbb{Z})$. Such maps are characterized by a winding number ω.

  e.g. 

  The periodicity condition is $X(\sigma + l) = X(\sigma) + 2\pi R\omega$. These are new states that enter the theory (twisted states).



  Any S^1-compactified closed string theory must include winding states with $\omega \neq 0$.

$$X(\tau, \sigma) = q + \frac{2\pi\alpha'}{l}\frac{n}{R}\tau + \omega\frac{2\pi R}{l}\sigma + \mathrm{i}\sqrt{\frac{\alpha']}{2}}\sum_{n\neq 0}\frac{1}{n}\left\{\alpha_n \exp\left(-\frac{2\pi \mathrm{i} n(\tau-\sigma)}{l}\right) + \widetilde{\alpha}_n \exp\left(-\frac{2\pi \mathrm{i} n(\tau+\sigma)}{l}\right)\right\}. \tag{5.5}$$

  Write $X = X_L + X_R$ $(l = 2\pi)$ with

$$X_L = x_l + \frac{\alpha'}{2}P_L(\tau + \sigma) + \widetilde{\text{osc}}\,, \tag{5.6a}$$

$$X_R = x_r + \frac{\alpha'}{2}P_R(\tau - \sigma) + \widetilde{\text{osc}}\,. \tag{5.6b}$$

$$x_{L,R} = \frac{1}{2}(q \pm c)\,, \quad P_{L,R} = \frac{n}{R} \pm \frac{\omega R}{\alpha'}\,. \tag{5.7}$$

  – mass operator:

$$m^2 = (p^0)^2 - \sum_{i=1}^{24}(p^i)^2 = \frac{2}{\alpha'}(N + \widetilde{N} - 2) + \left(\frac{n}{R}\right)^2 + \left(\frac{\omega R}{\alpha'}\right)^2 = \frac{2}{\alpha'}(N + \widetilde{N} - 2) + \frac{1}{2}(P_L^2 + P_R^2)\,. \tag{5.8}$$

  – level matching condition:

$$\frac{4}{\alpha'}(N - \widetilde{N}) + P_L^2 - P_R^2 = 0 \Rightarrow n\omega + N - \widetilde{N} = 0\,. \tag{5.9}$$

Now, let us look at the spectrum:

i) Tachyon: $N = \widetilde{N} = n = \omega = 0$, $\alpha' m^2 = -4$

   For $n \neq 0$ tachyons depend on the value of R/l_s.

ii) massless states: $N = \widetilde{N} = 1$, $n = \omega = 0$

- $\alpha^i_{-1}\widetilde{\alpha}^j_{-1}|0,0;k\rangle$ (i, $j = 1, \ldots, 23$) gives the 25-dimensional graviton, anti-symmetric tensor and the dilaton.
- $(\alpha^i_{-1}\widetilde{\alpha}^{25}_{-1} + \alpha^{25}_{-1}\widetilde{\alpha}^i_{-1})|0,0;k\rangle$, $(\alpha^i_{-1}\widetilde{\alpha}^{25}_{-1} - \alpha^{25}_{-1}\widetilde{\alpha}^i_{-1})|0,0;k\rangle$ are called Kaluza-Klein vectors $h_{i,25}$, $B_{i,25}$ (vectors under compactified space-time).
- $\alpha^{25}_{-1}\widetilde{\alpha}^{25}_{-1}|0,0;k\rangle$: Kaluza-Klein scalar (25-component of the space-time metric $h_{25,25}$).

Comments

- These massless states also occur in compactified field theories.
- Question: What couples to these gauge bosons?

Let us look at the following:

- $n = \pm\omega = \pm 1$
  * $n = \omega = \pm 1$, $m^2 = \dfrac{1}{R^2} + \left(\dfrac{R}{\alpha'}\right)^2 + \dfrac{2}{\alpha'}(N + \widetilde{N} - 2)$, $0 = N - \widetilde{N} + 1$
  For $N = 0$, $\widetilde{N} = 1$ one finds two vectors $\widetilde{\alpha}^i_{-1}|\pm 1, \pm 1\rangle$ and two scalars $\widetilde{\alpha}^{25}_{-1}|\pm 1, \pm 1\rangle$.
- $n = -\omega = \pm 1$
  For $N = 1$ and $\widetilde{N} = 0$ one finds two vectors $\alpha^i_{-1}|\pm 1, \mp 1\rangle$ and two scalars $\alpha^{25}_{-1}|\pm 1, \mp 1\rangle$. The mass squared

$$m^2 = \frac{1}{R^2} + \left(\frac{R}{\alpha'}\right)^2 - \frac{2}{\alpha'}\,, \tag{5.10}$$

  has its minimum for $R = \sqrt{\alpha'}$. This means that there are additional massless vectors and scalars at $R = \sqrt{\alpha'} = l_s$. (These are the only ones.)

Look at

$$\int \mathrm{d}\sigma \int \mathrm{d}\tau\, G_{25,\mu}(\partial_\sigma X^\mu \partial_\tau X^{25} + \partial_\sigma X^\mu \partial_\tau X^{25})\,, \tag{5.11}$$

and obtain with $X^{25} = \omega R\sigma = \alpha'(n/R)\tau$, $X^\mu = x^\mu(\tau) = q^\mu + \alpha' p^\mu \tau$:

$$\int \mathrm{d}\tau \int_0^l \mathrm{d}\sigma\,(G_{\mu,25} - \partial_\tau x^\mu) = \frac{n}{R}\int \tau\, \partial_\tau x^\mu A_\mu\,. \tag{5.12}$$

That looks like a charged particle couples to a gauge field, whereas the charge is its momentum quantum number.

$$\int \mathrm{d}\tau \int \mathrm{d}\sigma\, B_{25,\mu}\partial_\sigma X^{25}\partial_\tau X^\mu ßsim\omega \int \widetilde{A}_\mu \partial_\tau x^\mu\,. \tag{5.13}$$

The two Kaluza gauge bosons, which arise from compactification, couple to different charges, namely to the momentum quantum number and to the winding number of a compactified string. Here we have two vectors and two scalars that are charged under these gauge bosons. We can take two linear combinations of the Kaluza-Klein vector particles: $\alpha^{25}_{-1}\widetilde{\alpha}^i_{-1}|0\rangle$ and $\alpha^i_{-1}\widetilde{\alpha}^{25}_{-1}|0\rangle$, where the first one transforms under $\mathrm{U}(1)_L$ and the second under $\mathrm{U}(1)_R$. The first couples to $n - \omega$ and the second to $n + \omega$. Hence, the charges of the massless states are $(\pm 2, 0)$ and $(0, \pm 2)$. There are additional massless scalars with charges $(\pm 2, 0)$ and $(0, \pm 2)$. Here again a list of all massless states:

		$\mathrm{U}(1)_L$-$\mathrm{U}(1)_R$-charges
vectors	$\alpha^{25}_{-1}\widetilde{\alpha}^1_{-1}|0,0\rangle$	$(0,0)$
	$\alpha^i_{-1}\widetilde{\alpha}^{25}_{-1}|0,0\rangle$	$(0,0)$
	$\widetilde{\alpha}^i_{-1}|\pm 1, \pm 1\rangle$	$(0,\pm 2)$
	$\alpha^i_{-1}|\pm 1, \mp 1\rangle$	$(\pm 2, 0)$
scalars	$\alpha^{25}_{-1}\alpha^{25}_{-1}|0,0\rangle$	$(0,0)$
	$\widetilde{\alpha}^{25}_{-1}|\pm 1, \pm 1\rangle$	$(0,\pm 2)$
	$\alpha^{25}_{-1}|\pm 1, \mp 1\rangle$	$(\pm 2, 0)$
	$|\pm 2, 0\rangle$	$(\pm 2, \pm 2)$
	$|0, \pm 2\rangle$	$(\pm 2, \mp 2)$

Note that for generic R the four charged gauge bosons are massive as are all eight charged scalars. Four scalars have the same mass as the gauge bosons:

$$m^2 = \left[\frac{R^2 - \alpha'}{R\alpha'}\right]^2 . \tag{5.14}$$

This leads us to the string-Higgs effect. From the scalars $|0, \pm 2\rangle$, $|\pm 2, 0\rangle$ two become massive and two become tachyonic.

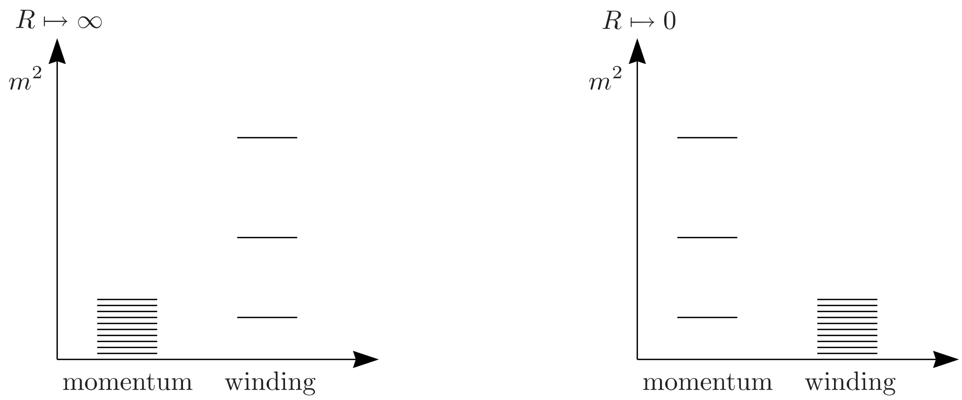
5.1 T-duality

5.1.1 Closed strings

Consider the spectrum of a closed string on S^1_R:

$$m^2 = \frac{n^2}{R^2} + \frac{\omega^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \widetilde{N} - 2) . \tag{5.15}$$

For $R \mapsto \infty$ the winding modes become infinitely heavy, because it costs energy to stretch the string, since it is under tension. For the momentum modes we get a continuous spectrum, which is a sign for a non-compact dimension. (For example, an physicist, who would live in a (2+1)-dimensional world, would observe copies of (our four-dimensional) electron with continuous spectrum of masses. This would lead him him/her to the conclusion that there must be an extra non-compact dimension. The physicist would observe the fourth component of the four-momentum, which is continuous. For $R \mapsto \infty$ the situation is reversed.



One can show that the spectrum is invariant under $R \mapsto \sqrt{\alpha'}/R$, $n \leftrightarrow \omega$ (T-duality transformation).

$$P_L = \frac{n}{R} + \frac{\omega R}{\alpha'} \mapsto P_L , \quad P_R = \frac{n}{R} - \frac{\omega R}{\alpha'} \mapsto -P_R . \tag{5.16}$$

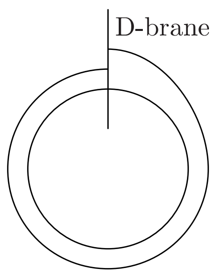
Also the action and hence the whole theory is invariant (duality of the theory). You never find this behaviour in a particle theory. If one also exchanges $\alpha_n \mapsto -\alpha_n$ and $\widetilde{\alpha}_n \mapsto \widetilde{\alpha}_n$ then it holds that

$$X_L \mapsto X_L , \quad X_R \mapsto -X_R , \quad X = X_L + X_R \mapsto X = X_L - X_R . \tag{5.17}$$

Under these operations the conformal field theory (the full algebra) does not change, which is quite an amazing property of string theory. The duality is non-perturbative in α' (loop expansion), but perturbative in g_s (topolocial expansion).

5.1.2 Open strings

We start with an open string with (NN) boundary conditions and compactify that on a circle. For an open string there is no winding number that is a topological invariant. Illustration:



Then, we get

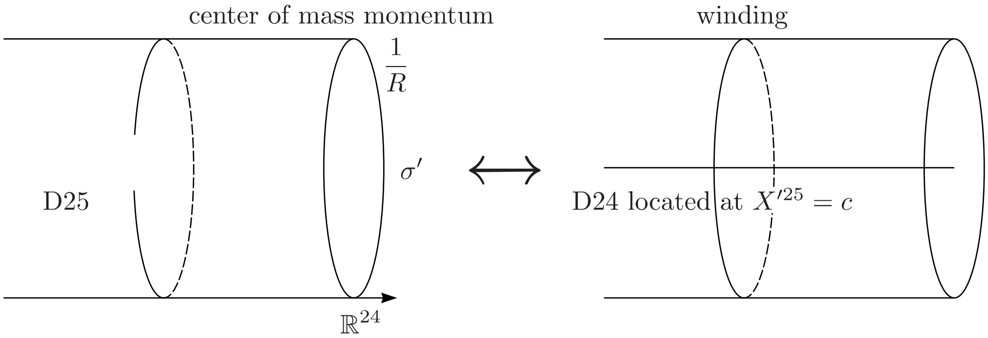
$$X(\tau,\sigma) = q + 2\pi\alpha' p\tau + \mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n\cos(n\sigma)\exp(-\mathrm{i}n\tau) =$$
$$= q + \alpha' p(\tau+\sigma) + \alpha' p(\tau-\sigma) + \mathrm{i}\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n}{2n}\left[\exp(-\mathrm{i}\pi(\tau-\sigma)) + \exp(-\mathrm{i}\pi(\tau+\sigma))\right] =$$
$$= X_L + X_R\,. \tag{5.18}$$

Now exchange $X_L \mapsto X_L$, $X_R \mapsto -X_R$:

$$X \mapsto X' = X_L - X_R = c + 2\alpha' p\sigma + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{1}{n}\alpha_n\sin(n\sigma)\exp(-\mathrm{i}n\tau)\,. \tag{5.19}$$

That is the mode expansion of a string with (DD) conditions.

$$q_f - q_i = 2\pi\alpha'\frac{n}{R} = 2\pi\alpha' R_{\mathrm{dual}}\,, \quad R_{\mathrm{dual}} = \frac{\alpha'}{R}\,. \tag{5.20}$$



$$\partial_\sigma X = \partial_\sigma(X_L + X_R) = \partial_\tau X_L - \partial_\tau X_R = \partial_\tau X'\,. \tag{5.21}$$

Recall:

$$\frac{1}{G_N^{(10)}} = \frac{1}{l_s^8 g_s^2}\,. \tag{5.22}$$

We compactify on a circle S_R:

$$\frac{1}{G_N^{(9)}} = \frac{2\pi R}{l_s^8 g_s^2} \xrightarrow{\text{T-duality}} \frac{2\pi}{l_s^6 R g_s'^2}\,. \tag{5.23}$$

Newton's constant can be measured and therefore we obtain

$$\frac{2\pi}{l_s^6 R g_s^2} = \frac{2\pi R}{l_s^8 g_s^2} \Rightarrow \boxed{g_s' = \frac{l_s}{R}g_s\,.} \tag{5.24}$$

Wrap a DP-brane around T^p. This is a particle in uncompactified space-time with mass

$$m = T_p\prod_{i=1}^{p}(2\pi R_i)\,. \tag{5.25}$$

Now use T-duality on the p-th circle ($T = \tau_p h(g_s)$):

$$m = \tau_p h(g_s)\prod_{i=1}^{p-1}(2\pi R_i)\cdot(2\pi R_p) = \tau_{p-1}h\left(\frac{g_s\sqrt{\alpha'}}{R_p}\right)\prod_{i=1}^{p-1} = \tau_{p-1}h(g_s')\prod_{i=1}^{p-1}(2\pi R_i)\,. \tag{5.26}$$

Comparison gives

$$h(g_s') \sim \frac{R_p}{\sqrt{\alpha'}}h(g_s) \Rightarrow h(g_s) \sim \frac{1}{g_s} \Rightarrow h\left(\frac{g_s\sqrt{\alpha'}}{R_p}\right) \sim \frac{R_p}{\sqrt{\alpha'}g_s}\,. \tag{5.27}$$

Hence, we can write the tension of a D_p brane as

$$\boxed{T_p \sim \frac{1}{g_s l_s^{p+1}}\,.} \tag{5.28}$$

Be M the mass of N D-branes:

$$V \sim G_N M \sim g_s^2 \frac{N}{g_s} = N g_s\,. \tag{5.29}$$

For $Ng_s \ll 1$ the back reaction on geometry can be neglected. However, for $Ng_s \gg 1$ this is not the case. We consider a D3 brane in M^{10}. For $g_sN \ll 1$ there is no back reaction. For $g_sN \gg 1$ there is a strong back reaction such that the space-time is deformed to $\mathrm{AdS}_5 \times S^5$ (by putting many of these branes one on another) with

$$S^5 : \sum_{i=1}^{6} X_i^2 = R^2\,, \quad \mathrm{AsS}_5 : -(X^0)^2 - (X^5)^2 + \sum_{i=1}^{4}(X_i)^2 = -R^2\,. \tag{5.30}$$

Since Dirichlet boundary conditions break space-time Lorentz invariance, the momentum is not conserved along the string and is transferred to its brane. However, the brane will not move very much due to this momentum transfer, because it is very heavy. Recall that in IIA-theory we had the fields A_μ, $A_{\mu\nu\varrho}$ and in IIB-theory the fields A_0, $A_{\mu\nu}$, and $A^\dagger_{\mu\nu\varrho\sigma}$. What is the significance of these higher-rank tensors? A string can have a coupling to these antisymmetric tensor fields, whereas the coupling constant is the tension of the string. The object, which are charged under these fields, is the D-brane. A D0-brane has a one-dimensional world volume and couples to $A^{(1)}$, whereas a D2-brace with a two-dimensional world volume, couples to $A^{(2)}$ (IIA). A D(-1)-brane only lives in one point in space-time and couples to $A^{(0)}$, a D1-brane couples to $A^{(2)}$ and a D3-brane to $A^{(4)}$ (IIB). In a IIA-theory there exist D0-branes with tension $T \sim 1/(g_s l_s)$.

There are theories in particle physics that have monopole solutions. These monopoles are charged. However, it turns out that there is no force between these. That is because there is a Higgs field that leads to attraction. (Boson fields with odd spin can lead to both repelling and attraction (photon, gluon, weak bosons) and boson fields with even spin only lead to attraction (Higgs, graviton).) The attraction and the repelling effects cancel out independently of the distance. A similar effect leads to the fact that the net force between D-branes is zero.

There are marginal bound states of N D0-branes (marginal, because the mass of a bound state is $m_N = Nm = N/g_s$). If g_s becomes large, the mass spectrum will become continuous. The interpretation of this effect it that as $g_s \mapsto \infty$, the theory becomes 11-dimensional (11-dim SUGRA, low energy limit of M-theory).