STRING-STEILKURS 2008: CONFORMAL FIELD THEORY

S. Fredenhagen

29.09 – 2.10.2008
Albert-Einstein-Institut in Golm

Letzte Aktualisierung und Verbesserung: October 4, 2008

Mitschrift der Vorlesungsreihe CONFORMAL FIELD THEORY
von Herrn S. FREDENHAGEN auf dem String-Steilkurs 2008
von MARCO SCHRECK.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

Contents

Chapter 1

Introduction to conformal field theory in two dimensions

Conformal invariance means that there is some invariance under a group which is bigger than the Poincaré group. This implies that correlators are heavily restricted and that 1-, 2- and 3-point functions are fixed. (This is true in any dimension.) Furthermore, there is no S-matrix. The important objects are the correlators and there is a correspondence between field and state. In two dimensions there are additional features:

- There is a factorization into left- and right-moving modes.
- An infinite-dimensional Lie algebra organizes the spectrum (Virasoro algebra).

Today we are starting with the discussion of conformal transformations. Then we will see, how the invariance under conformal transformations restricts the form of the correlators.
Conformal transformations are those which preserve angles. For example, this is the case for rotations, translations and scaling. A general definition would be

$$\phi : \mathcal{M} \to \widetilde{\mathcal{M}}\,, \quad \phi^* \widetilde{g} = \lambda \cdot g\,, \tag{1.1}$$

where λ is a positive function on $\mathcal{M}$. For $\mathcal{M} = \widetilde{\mathcal{M}} = \mathbb{R}^n$ this property is given by

$$\eta_{\mu\nu} \frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} \frac{\partial \widetilde{x}^\nu}{\partial x^\beta} = \lambda(x) \eta_{\alpha\beta}\,. \tag{1.2}$$

We consider an infinitesimal transformation $x^\mu \mapsto \widetilde{x}^\mu = x^\mu + \omega^\mu(x)$:

$$\eta_{\mu\nu}(\delta^\mu{}_\alpha + \partial_\alpha \omega^\mu)(\delta^\nu{}_\beta + \partial_\beta \omega^\nu) = \eta_{\alpha\beta} + (\partial_\beta \omega_\alpha + \partial_\alpha \omega_\beta) + \mathcal{O}(\omega^2)\,. \tag{1.3}$$

Hence, it has to hold that

$$\partial_\beta \omega_\alpha + \partial_\alpha \omega_\beta \sim \eta_{\alpha\beta}\,. \tag{1.4}$$

For $d \geq 3$ it follows from (1.4) that ω is at most quadratic in x. If ω is constant, the above transformation is just a translation. For linear ω it is a rotation, scaling (dilatation): $\widetilde{x}^\mu = x^\mu + \lambda x^\mu$. If ω is quadratic, we obtain a transformation, which we might not have guessed: a special conformal transformation. Let us look at the inversion:

$$x^\mu \mapsto \frac{x^\mu}{|x|^2}\,. \tag{1.5}$$

One can check that this is indeed a conformal transformation. The special conformal transformations are combinations of an inversion, a translation and again an inversion:

$$x^\mu \mapsto \frac{x^\mu}{|x|^2} \mapsto \frac{x^\mu}{|x|^2} + a^\mu \mapsto \frac{\frac{x^\mu}{|x|^2} + a^\mu}{\left|\frac{x^\mu}{|x|^2} + a^\mu\right|^2} = \frac{x^\mu + a^\mu |x|^2}{1 + 2a \cdot x + |x|^2 |a|^2}\,. \tag{1.6}$$

Infinitesimally, this is given by

$$\omega^\mu = a^\mu |x|^2 - 2(a \cdot x) x^\mu\,. \tag{1.7}$$

The infinitesimal conformal transformations form a Lie algebra so(1,$d+1$). (For $\mathbb{R}^{p,q}$ we get so$(p+1,q+1)$.) To make inversion well-defined, consider $S^d \simeq \mathbb{R}^d \cup \{\infty\}$. Note: Similar for Minkowski theory on $\mathbb{R}^{1,d-1}$, but we have to go to a covering space of the conformal compactification of $\mathbb{R}^{1,d-1} \to \mathbb{R} \times S^{d-1}$. All this was rather general for any dimension, but now we will restrict the whole to two dimensions. Consider complex numbers $z = x + \mathrm{i}y$ on the complex plane (compactified version: Riemann sphere). In terms of z and $\overline{z}$:

$$(\eta_{\alpha\beta}) = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, \quad (\eta^{\alpha\beta}) = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{1.8}$$

For

$$(\omega^\mu) = \begin{pmatrix} \omega^z \\ \omega^{\overline{z}} \end{pmatrix}, \tag{1.9}$$

the z-z-component of Eq. (1.4) is given by $2\partial_z\omega_z = 0$ and as a result of that, $\omega^{\overline{z}} = 2\omega_z$ is anti-holomorphic: $\partial_z\omega^{\overline{z}} = 0$. Similarly, ω^z is holomorphic: $\partial_{\overline{z}}\omega^z = 0$. Hence, any holomorphic ω satisfies Eq. (1.4). Consider global conformal transformations mapping $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$:

$$z \mapsto w(z) = \frac{az+b}{cz+d}. \tag{1.10}$$

These are the so-called Möbius transformations and they exactly have one zero (pole). The conformal group of $\overline{\mathbb{C}}$ is

$$\mathcal{C}(\overline{\mathbb{C}}) = \mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2, \tag{1.11}$$

with

$$\mathrm{SL}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc = 1 \right\}, \quad \mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}. \tag{1.12}$$

It holds that $\mathrm{PSL}(2,\mathbb{C}) \sim \mathrm{SO}^{\uparrow}_{+}(1,3)$. Special conformal transformations:

$$z \mapsto w(z) = \frac{z}{1+az}. \tag{1.13}$$

1.1 Correlators

A field $\phi(x)$ has to transform in a representation of the conformal group. This is characterized by

- a scaling dimension Δ,
- and a representation of the Lorentz/rotation group,

which specify how $\phi(0)$ transforms. In two dimensions there is a scaling dimension Δ and spin s.

$$z \mapsto \lambda z, \quad z \mapsto \exp(\mathrm{i}\theta)z, \tag{1.14}$$

$$\phi \mapsto \phi', \quad \phi'(z,\overline{z}) = \lambda^\Delta \exp(\mathrm{i}s\theta)\phi(\lambda\exp(\mathrm{i}\theta)z, \lambda\exp(-\mathrm{i}\theta)\overline{z}). \tag{1.15}$$

For a general conformal transformation $z \mapsto w(z)$ it holds that

$$\phi'(z,\overline{z}) = \left(\frac{\partial w}{\partial z}\right)^h \left(\frac{\partial\overline{w}}{\partial\overline{z}}\right)^{\overline{h}} \phi(w(z),\overline{w}(\overline{z})), \quad \Delta = h + \overline{h}, \quad s = h - \overline{h}. \tag{1.16}$$

Let us see, how this transformation property restricts the correlators. We will begin with the one-point function:

$$\langle\phi(z,\overline{z})\rangle = \langle\phi(0,0)\rangle = \mathrm{const.} \tag{1.17}$$

This holds in every translation-invariant field theory. Under scaling/rotation:

$$\langle\phi(z,\overline{z})\rangle = \alpha^h\overline{\alpha}^{-h}\langle\phi(\alpha z,\overline{\alpha z})\rangle. \tag{1.18}$$

From $\langle\phi\rangle \neq 0$ it follows that $h = \overline{h} = 0$. Let us consider a two-point function:

$$\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\rangle = \langle\phi_1(z_1 - z_2, \overline{z}_1 - \overline{z}_2)\phi_2(0,0)\rangle =: f(z_1 - z_2)\,. \tag{1.19}$$

From scale/rotation invariance we obtain

$$f(z) = Cz^{-h_1-h_2}\overline{z}^{-\overline{h}_1-\overline{h}_2}\,. \tag{1.20}$$

Under special conformal transformations:

$$z \mapsto w(z) = \frac{z}{1+az}\,, \quad \frac{\mathrm{d}w}{\mathrm{d}z} = \frac{1}{1+az} - \frac{za}{(1+az)^2} = \frac{1}{(1+az)^2}\,. \tag{1.21}$$

From that it is clear that

$$f(z) = (1+az)^{-2h_1}(1+\overline{a}\,\overline{z})^{-2\overline{h}_1}f(w) = \left(\frac{w}{z}\right)^{2h_1}\left(\frac{\overline{w}}{\overline{z}}\right)^{2\overline{h}_1} f(w)\,. \tag{1.22}$$

Hence, for a non-vanishing two-point function it has to hold that $h_1 = h_2$ and $\overline{h}_1 = \overline{h}_2$.
For a three-point function there is a unique Möbius transformation that sends three given distinct points z_1, z_2, z_3 to 0, 1, ∞, respectively:

$$z \mapsto \frac{z - z_1}{z - z_3}\cdot\frac{z_2 - z_3}{z_2 - z_1}\,. \tag{1.23}$$

As a result of that the coordinate dependence is fixed.

$$\begin{aligned}\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\rangle = C\cdot\frac{1}{(z_1-z_2)^{n_1+n_2-n_3}}\cdot\frac{1}{(z_2-z_3)^{n_2+n_3-n_1}}\times\\ \times\frac{1}{(z_3-z_1)^{n_3+n_1-n_2}}\cdot\frac{1}{(\overline{z}_1-\overline{z}_2)^{\cdots}}\cdot\ldots\,,\end{aligned} \tag{1.24}$$

whereas the constant C is called structure constant of the three-point function.

1.2 Left-right factorization

The solutions of the two-dimensional field equation

$$(\partial_\tau^2 - \partial_\sigma^2)X(\tau,\sigma) = 0\,, \tag{1.25}$$

of some field X split into a left-moving and a right-moving part:

$$X(\tau,\sigma) = X_R(\tau+\sigma) + X_L(\tau-\sigma)\,. \tag{1.26}$$

There is a Noether current associated to transformations $X \mapsto X + \text{const}$. That is given by $J_\alpha = \partial_\alpha X$. The current is conserved: $\partial^\alpha J_\alpha = 0$. From

$$J_+ = \partial_+ X = \partial_+ X_R(\sigma^+)\,, \quad J_- = \partial_- X_L(\sigma^-)\,, \quad \sigma^\pm = \tau \pm \sigma\,, \tag{1.27}$$

it follows that

$$\partial_+ J_- + \partial_- J_+ = 0\,. \tag{1.28}$$

So, there are two conserved, chiral currents. This is a generic feature of scale-invariant two-dimensional field theory. The energy-momentum tensor $T_{\alpha\beta}$ as a Noether current corresponding to translational symmetries imposes conserved charges and momenta:

$$\int \mathrm{d}\tau\, T_{0\beta} = P_\beta\,. \tag{1.29}$$

P_β has scaling dimension 1 (from the conformal algebra). Hence, $T_{\alpha\beta}$ has scaling dimension $\Delta = 2$. Coordinates z, $\overline{z}$:

$$T_{zz}\quad (s=2)\,,\quad T_{\overline{z}\overline{z}}\quad (s=-2)\,,\quad T_{z\overline{z}} = T_{\overline{z}z}\quad (s=0)\,. \tag{1.30}$$

$$\langle T_{zz}(w,\overline{w})T_{zz}(0,0)\rangle = \frac{C}{w^4} \Rightarrow \langle \partial_w T_{zz} \partial_{\overline{w}} T_{zz} \rangle = 0 \Rightarrow \partial_{\overline{w}} T_{zz} \equiv 0\,. \tag{1.31}$$

In a scale-invariant two-dimensional field theory the following holds: For a conserved current J_μ whose charge

$$Q^{(\tau)} = \int \mathrm{d}\sigma\, J_0(\tau,\sigma)\,, \tag{1.32}$$

is conserved and scale-invariant, there exists a dual current $\widetilde{J}_\mu = \varepsilon_{\mu\nu}J^\nu$ which is also conserved. They split into chiral currents. Example: In a free-field theory with the conserved current $J_\mu = \partial_\mu X$ there also exist the independently conserved currents $J_+ = \partial_+ X$, which only depends on $\tau+\sigma$ and $J_- = \partial_- X$, which only depends on $\tau-\sigma$. We have also found out, that there is a holomorphic and anti-holomorphic part of the energy-momentum tensor in a theory parameterized by $z = x + \mathrm{i}y$: $\partial_{\overline{z}} T_{zz} = 0$ and $\partial_z T_{\overline{z}\,\overline{z}} = 0$. From conservation $\partial^\mu T_{\mu\nu} = 0$ one obtains

$$\partial_{\overline{z}} T_{zz} + \partial_z T_{\overline{z}z} = 0 \Rightarrow \partial_z T_{\overline{z}z} = 0 = \partial_{\overline{z}} T_{z\overline{z}}\,. \tag{1.33}$$

Since $T_{\mu\nu}$ is symmetric, $T_{z\overline{z}}$ is constant.

$$\langle T_{z\overline{z}}(w,\overline{w})T_{z\overline{z}}(0,0)\rangle = \frac{C}{w^2 w^{-2}} = 0 \Rightarrow T_{z\overline{z}} \equiv 0\,. \tag{1.34}$$

That means that $T_{\mu\nu}$ is traceless: ${T_\mu}^\mu = 0$.

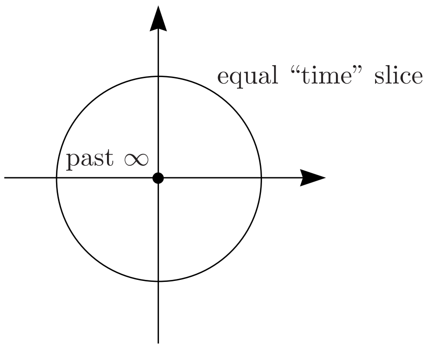
Minkowski space-time		Euclidian "space-time"
correlators (distribution) (satisfying Wightman axioms)	analytic continuation →	Euclidian Green's functions (satisfying Osterwalder-Schrader axioms, i.p. reflection positivity)
expectation values ↕ reconstruction		expectation values of "time-ordered" fields ↑
fields as operator-valued distributions acting on a Hilbert space		fields as operator-valued functions acting on a Hilbert space

The Osterwalder-Schrader axioms are a remnant of unitarity. They have to be fulfilled such that the Euclidian theory can be continued back to Minkowski space-time. The Wightman axioms in Minkowski space-time do not only hold for time-ordered, but for general correlators.

Chapter 2

Radial quantization of Euclidian theory

Starting with a Minkowski theory on $\mathbb{R} \times S^1$, (τ, σ) with $\sigma \in [0, 2\pi[$ (covering space of conformal compactified Minkowski space) one does the analytic continuation $\tau \mapsto -\mathrm{i}t$. The coordinates $z = \exp(\mathrm{i}(\tau + \sigma))$, $\overline{z} = \exp(\mathrm{i}(\tau - \sigma))$ are then replaced by $z = \exp(t)\exp(\mathrm{i}\delta)$ and $\overline{z} = \exp(t)\exp(-\mathrm{i}\delta)$. They span the whole complex plane.



Time-ordering now corresponds to radial ordering. Euclidian time translation $t \mapsto t + a$ results in $z \mapsto \exp(a)z$, which is a scaling (dilation). The dilation operator has replaced the role of the Hamiltonian. Time reflection $t \mapsto -t$ just corresponds to the inversion: $z \mapsto z/|z|^2$.

2.1 Conformal invariance once more

Classically, $T_{\mu\nu}x^\nu$ is the current associated to scaling. If it is conserved, it holds that

$$\partial^\mu(T_{\mu\nu}x^\nu) = T_{\mu\nu}\eta^{\mu\nu} = \mathrm{Tr}(T) = 0\,. \tag{2.1}$$

The current for a general conformal transformation is given by $T_{\mu\nu}\omega^\nu(x)$.

$$\partial^\mu(T_{\mu\nu}\omega^\nu) = T_{\mu\nu}\partial^\mu\omega^\nu = \frac{1}{2}T_{\mu\nu}(\partial^\mu\omega^\nu + \partial^\nu\omega^\mu) \sim \mathrm{Tr}(T) = 0\,. \tag{2.2}$$

Conserved charges are given by the integration of the underlying current over an equal-time slice:

$$Q_{\omega,\overline{\omega}} = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,\omega(z)T(z) + \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}\overline{z}\,\overline{\omega}(\overline{z})\overline{T}(\overline{z})\,. \tag{2.3}$$

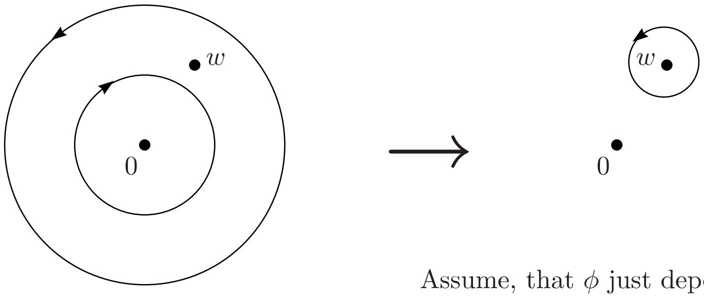
We look at the transformation of a scaling field ϕ under the transformation ω:

$$\begin{aligned}\delta_\omega\phi(w) &= \left[\frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,T(z), \phi(w)\right] = [Q_{\omega,\overline{w}}, \phi(w)] = \\ &= \frac{1}{2\pi\mathrm{i}}\left\{\oint\limits_{|z|>|w|}\mathrm{d}z - \oint\limits_{|z|<|w|}\mathrm{d}z\right\}\mathcal{R}(T(z)\phi(w))w(z) = \frac{1}{2\pi\mathrm{i}}\oint\limits_{w}\mathrm{d}z\,\mathcal{R}(T(z)\phi(w))\omega(z) = \\ &\overset{!}{=} h(\partial_z\omega)(w)\phi(w) + \omega(w)\partial\phi(w)\,,\end{aligned} \tag{2.4}$$

where $\mathcal{R}$ means radial ordering:

$$\mathcal{R}(\psi(z)\phi(w)) = \begin{cases} \psi(z)\phi(w) & \text{for} \quad |z| > |w| \\ \phi(w)\psi(z) & \text{for} \quad |w| > |z| \end{cases} . \tag{2.5}$$

So, the transformation of a field ϕ under a scaling transformation can be written in terms of a contour integral over a radial-ordered term. This equation is sometimes called **conformal Ward identity**.



Assume, that ϕ just depends on z ($h = 0$).

$$\phi'(z) = (1 + \partial_z \omega)^h \phi(z + \omega(z))$$
$$\delta\phi(z) = \phi'(z) - \phi(z) = h(\partial_z \omega)\phi(z) + \omega\partial_z\phi + \mathcal{O}(\omega^2)$$

That means:

$$\mathcal{R}(T(z)\phi(w)) = \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial_z\phi(w) + \text{regular terms} . \tag{2.6}$$

2.2 Primary fields

Fields ϕ with operator product expansion (OPE) as in Eq. (2.6) are called primary fields. Scaling fields are **quasi-primary fields**:

$$\mathcal{R}(T(z)\phi(w)) = \ldots + \frac{0}{(z-w)^3} + \frac{h}{(z-w)^2}\phi(w) + \frac{1}{z-w}\partial\phi(w) + \ldots . \tag{2.7}$$

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} , \quad L_n = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \, z^{n+1} T(z) , \tag{2.8}$$

From these arguments we can compute the commutator of L_n and such a field $\phi(w)$ (with $\omega(z) = z^{n+1}$):

$$[L_n, \phi(w)] = h(n+1)w^n\phi(w) + w^{n+1}\partial\phi(w) = w^n[n(n+1) + w\partial_w]\phi(w) . \tag{2.9}$$

Quasi-primary fields are the ones for $n = 0, \pm 1$ and primary fields are the ones for all n. Let us now look at the operator product expansion of TT:

$$\mathcal{R}(T(z)T(w)) = \frac{\frac{C}{2}}{(z-w)^4} + \frac{0}{(z-w)^3} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) . \tag{2.10}$$

There are no higher-order terms allowed, if we assume that there are no fields of negative scaling dimension.

$$\langle T(z)T(0)\rangle = \frac{\frac{C}{2}}{z^4} . \tag{2.11}$$

Now, we can do the following computation:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} , \tag{2.12}$$

where c is called **central charge**. Compare to the algebra of $l_m = -z^{m+1}\partial_z$:

$$[l_m, l_n] = (m-n)l_{m+n} . \tag{2.13}$$

The second term in Eq. (2.12) is the central extension. Eq. (2.12) is called **Virasoro algebra** and Eq. (2.13) is called **Witt algebra**. The Virasora algebra is useful to organize the spectrum.

Chapter 3

Conformal transformations of the energy-momentum tensor

$$\delta_\omega T(\omega) = \frac{1}{2\pi \mathrm{i}} \oint\limits_w \mathrm{d}z \, \mathcal{R}(T(z)T(w))\omega(z) = 2(\partial\omega)T + \omega\partial T + \frac{1}{2\pi \mathrm{i}} \int\limits_w \mathrm{d}z \left(\frac{\frac{C}{2}}{(z-w)^4}\omega(z) \right) =$$
$$= 2\omega' T + \omega\partial T + \frac{C}{12}\omega'''(w) \tag{3.1}$$

Under a large conformal transformation $z \mapsto w(z)$ the transformation of the energy-momentum tensor is given by:

$$T \mapsto T'(z) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 T(w(z)) + \frac{c}{12}\{w, z\}\,, \tag{3.2}$$

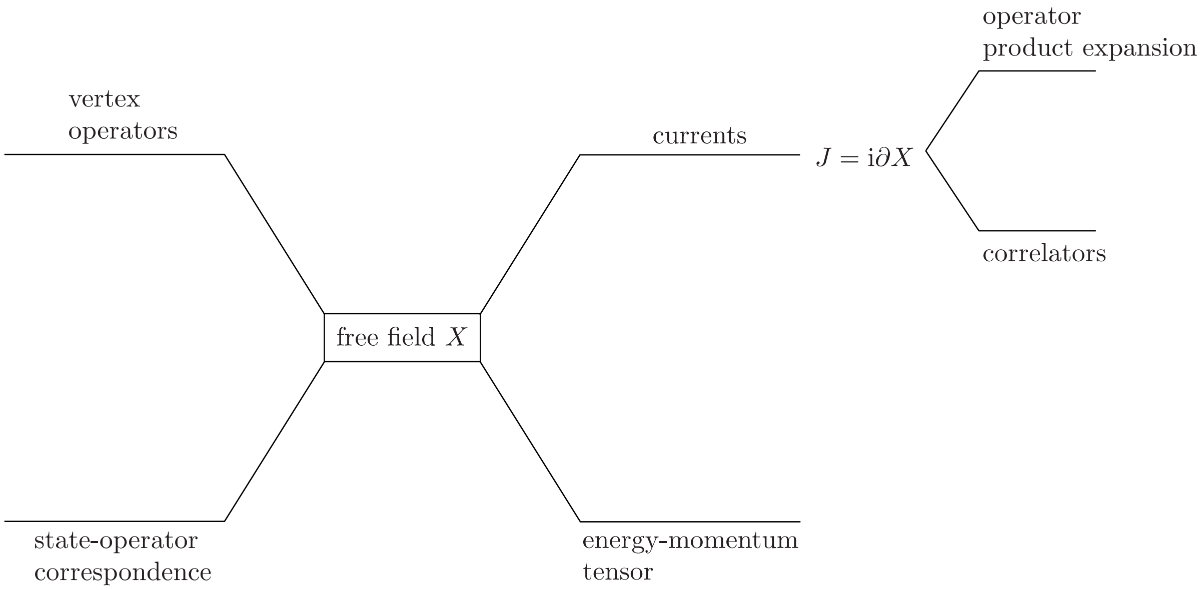
with $\{\bullet, \bullet\}$ denoting the Schwarz derivative defined by

$$\{w, z\} = \frac{w'''w' - \frac{3}{2}(w'')^2}{(w')^2}\,. \tag{3.3}$$

The Schwarz derivative tells how much a function deviates from the Möbius transformation. Hence, it vanishes for

$$w(z) = \frac{az + b}{cz + d}\,. \tag{3.4}$$

Consider a free scalar (massless) field $X(\tau, \sigma)$ on $\mathbb{R} \times S^1$. The generating currents for translations in the target space of coordinates are given by $J = \mathrm{i}\partial X$.



$$\mathcal{L} = -\frac{1}{4\pi\alpha'}\left(\partial_\alpha X \partial_\beta X \eta^{\alpha\beta}\right) . \tag{3.5}$$

The energy-momentum tensor

$$T_{\alpha\beta} = \frac{1}{2\pi\alpha'}\left(\partial_\alpha X \partial_\beta X - \frac{1}{2}\eta_{\alpha\beta}\partial_\gamma X \partial^\gamma X\right) , \quad \sigma^\pm = \tau \pm \sigma , \tag{3.6}$$

has the following properties (light-cone coordinates):

1) T-symmetric and traceless: $T_{+-} = T_{-+} = 0$,
2) T-conserved: $\partial_+ T_{--} = 0 = \partial_- T_{++}$.

The solution of the equations of motion is given by

$$X(\sigma^-, \sigma^+) = x + \alpha' p\tau + \mathrm{i}\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left(\frac{1}{n}a_n \exp(-\mathrm{i}n\sigma^-) + \frac{1}{n}\overline{a}_n \exp(-\mathrm{i}n\sigma^+)\right) . \tag{3.7}$$

From the condition that X is real one can state

$$x^* = x , \quad p^* = p , \quad a_n^* = a_{-n} , \quad \overline{a}_n^* = \overline{a}_{-n} . \tag{3.8}$$

We quantize the theory by introducing the commutation relations

$$[a_m, a_n] = m\delta_{m+n,0} = [\overline{a}_m, \overline{a}_n] , \quad [x, p] = \mathrm{i} . \tag{3.9}$$

Let $|k\rangle$ be the ground state with properties

$$p|k\rangle = k|k\rangle , \quad a_m|k\rangle = \overline{a}_m|k\rangle = 0 \, \forall\, m > 0 . \tag{3.10}$$

Apply a_n, $\overline{a}n$ $(n < 0)$ on the ground state:

$$a_{n_1} \ldots a_{n_r}\overline{a}_{m_1} \ldots \overline{a}_{m_s}|k\rangle , \quad n_i < 0,\, m_i < 0 . \tag{3.11}$$

These span the Fock space $\mathcal{F}_k$. The reality conditions

$$x^\dagger = x , \quad p^\dagger = p , \quad a_n^\dagger = a_{-n} , \quad \overline{a}_n^\dagger = \overline{a}_{-n} , \tag{3.12}$$

specify the scalar product up to a constant of normalization. By using

$$z = \exp(\mathrm{i}\sigma^-) = \exp(\mathrm{i}(\tau - \sigma)) \xrightarrow{\tau\mapsto -\mathrm{i}t} \exp(t)\exp(-\mathrm{i}\delta) , \tag{3.13}$$

we calculate the chiral current:

$$J(z) = \mathrm{i}\frac{\partial}{\partial z}X(z, \overline{z}) = \frac{1}{2}\frac{\partial}{\partial\sigma^-}X(z, \overline{z}) = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} a_n z^{-n-1} , \quad a_0 = \sqrt{\frac{\alpha'}{2}}p . \tag{3.14}$$

For determining the correlators we first need the operator product expansion. $J(z)$ has the conformal weight $n = 1$.

$$J(z)J(w) = \frac{C}{(z-w)^2} + \frac{0}{z-w} + \text{regular terms} . \tag{3.15}$$

The second term vanishes because of the symmetry between z and w. There still remains to determine the constant C.

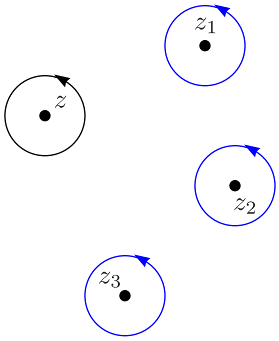
$$\langle 0|J(z)J(w)|0\rangle = \frac{C}{(z-w)^2} . \tag{3.16}$$

We split J into the sum $J^{(+)} + J^{(-)}$, whereas $J^{(+)}$ contains the a_n with $n \geq 0$ and the $J^{(-)}$ all a_n with $n < 0$. By using $[a_m, a_n] = m\delta_{m+n,0}$ one obtains:

$$\begin{aligned}
[J^{(+)}(z), J(w)] &= \frac{\alpha'}{2}\sum_{m\geq 0}\sum_{n\in\mathbb{Z}}[a_m, a_n]z^{-m-1}w^{-n-1} = \frac{\alpha'}{2}\sum_{m\geq 0} m z^{-m-1}w^{m-1} = \\
&= \frac{\alpha'}{2}\frac{\partial}{\partial w}\sum_{m\geq 0}\left(\frac{w}{z}\right)^m \frac{1}{z} \overset{|w|<|z|}{=} \frac{\alpha'}{2}\frac{\partial}{\partial w}\left(\frac{1}{1-\frac{w}{z}}\right)\frac{1}{z} = \frac{\alpha'}{2}\frac{1}{(z-w)^2} .
\end{aligned} \tag{3.17}$$

Then, we obtain:

$$\langle 0|J(z)J(w)|0\rangle = \langle 0|J^{(+)}(z)J(w)|0\rangle = \langle 0|[J^{(+)}(z), J(w)]|0\rangle = \frac{\alpha'}{2}\frac{1}{(z-w)^2} , \quad (\langle 0|0\rangle = 1) . \tag{3.18}$$



We calculate

$$\langle 0|J(z)J(z_1)\dots J(z_n)|0\rangle\,, \tag{3.19}$$

by using

$$J(z) = \frac{1}{2\pi \mathrm{i}} \oint_z \mathrm{d}w\, \frac{J(w)}{w-z} = -\frac{1}{2\pi\mathrm{i}} \sum_{i=1}^{n} \oint_{z_i} \mathrm{d}w\, \frac{1}{w-z}\,, \tag{3.20}$$

from deforming the integration contour and obtain with the above operator product expansion:

$$\begin{aligned} -\frac{1}{2\pi\mathrm{i}} \sum_{i=1}^{n} \oint_{z_i} \mathrm{d}w\, \frac{1}{w-z} \langle 0|J(z_1)\dots J(w)J(z_i)\dots J(z_n)|0\rangle = \\ = \sum_{i=1}^{n} \frac{\alpha'}{2} \frac{1}{(z_i-z)^2} \langle 0|J(z_1)\dots \widehat{J(z_1)}\dots J(z_n)|0\rangle\,, \end{aligned} \tag{3.21}$$

where $\widehat{\bullet}$ means that the corresponding operator is left out. Now to the energy-momentum tensor:

$$T_{zz} = \frac{1}{2\pi\alpha'} \partial_z X \partial_z X = -\frac{1}{2\pi\alpha'} JJ\,, \quad T(z) = \frac{1}{\alpha'} : J(z)J(z) : \,. \tag{3.22}$$

We calculate the normal ordering product:

$$\begin{aligned} : J(z)J(z) : &= \lim_{w\mapsto z} \left(J(z)J(w) - \text{singular part of OPE}\right) = \lim_{w\mapsto z} \Big(J(z)J(w) - [J^{(+)}(z), J(w)]\Big) = \\ &= J^{(-)}(z)J(z) + J(z)J^{(+)}(z)\,. \end{aligned} \tag{3.23}$$

"The annihilators are sorted to the right."

$$T(z)T(w) = \frac{\frac{c}{2}}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{z-w} \partial T(w) + \text{regular terms}\,. \tag{3.24}$$

One obtains by considering the two non-vanishing contractions

$$\langle T(z)T(w)\rangle = \frac{1}{(\alpha')^2} \langle : J(z)J(z) :: J(w)J(w) : \rangle = \frac{2}{\alpha'^2} \cdot \left(\frac{\alpha'}{2} \frac{1}{(z-w)^2}\right)^2 = \frac{1}{2} \frac{1}{(z-w)^4}\,, \tag{3.25}$$

and hence, the central charge c is equal to 1.

3.1 Vertex operators

The operator $\exp(\mathrm{i}kx)$ shifts the momentum by k.

$$\exp(\mathrm{i}kx) \mapsto V_k(z,\overline{z}) =: \exp(\mathrm{i}kX(z,\overline{z})) := \exp(\mathrm{i}kX^{(-)}(z,\overline{z})) \exp(\mathrm{i}kX^{(+)}(z,\overline{z}))\,. \tag{3.26}$$

We split $X = X^{(-)} + X^{(+)}$ with $X^{(-)}$ containing the a_n für $n < 0$ and $X^{(+)}$ containing the a_n für $n > 0$.

$$T(z)V_k(w,\overline{w}) = \frac{\frac{\alpha'}{4}k^2}{(z-w)^2} V_k(w,\overline{w}) + \frac{1}{z-w} \partial_w V_k(w,\overline{w}) + \text{regular terms}\,, \tag{3.27}$$

where V_k is the primary field of conformal weight $h = (\alpha'/4)k^2$.

$$J(z)V_k(w,\overline{w}) = \frac{\frac{\alpha'}{2}k}{z-w} V_k(w,\overline{w}) + \text{regular terms}\,. \tag{3.28}$$

3.2 State-operator correspondence

Consider the limit z, $\overline{z} \mapsto 0$ (Euclidian infinite parts).

$$\lim_{z,\overline{z}\mapsto 0} V_k(z,\overline{z})|0\rangle = \lim_{z\mapsto 0} \exp\left\{ \mathrm{i}k \left(x + \sum_{n<0} \ldots z^{-n} + \ldots \overline{z}^{-n} \right) \right\} |0\rangle = \exp(\mathrm{i}kx)|0\rangle = |k\rangle\,. \tag{3.29}$$

$$\lim_{z\mapsto 0} J(z)|0\rangle = \lim_{z\mapsto 0} \sum_{n<0} a_n z^{-n-1}|0\rangle \sqrt{\frac{\alpha'}{2}} = \sqrt{\frac{\alpha'}{2}} a_{-1}|0\rangle\,, \tag{3.30}$$

which gives us the first excited state of the vacuum. Doing a similar computation one finds

$$\lim_{z\mapsto 0} J^{(n)}(z)|0\rangle = \lim_{z\mapsto 0} \left(\frac{\partial}{\partial z}\right)^n J(z)|0\rangle = \sqrt{\frac{\alpha'}{2}}\, n!a_{-n-1}|0\rangle\,. \tag{3.31}$$

Given is a Fock state

$$|\phi\rangle = a_{n_1} \ldots a_{n_r} \overline{a}_{m_1} \ldots \overline{a}_{m_s} |k\rangle\,, \tag{3.32}$$

with $n_i < 0$, $m_i < 0$. There is a field

$$V(|\phi\rangle, z, \overline{z}) =: \frac{\sqrt{\frac{2}{\alpha'}}}{(-n_1-1)!} J^{-n_1-1}(z) \ldots \frac{\sqrt{\frac{2}{\alpha'}}}{(-n_r-1)!} J^{-n_r-1}(z) \ldots \overline{J}(\overline{z}) \ldots V_k(z,\overline{z}) :\,, \tag{3.33}$$

with

$$V(|\phi\rangle; 0, 0)|0\rangle = |\phi\rangle\,. \tag{3.34}$$

3.3 Uniqueness of the state-operator correspondence

Assume that the fields are, in particular, translation-invariant, $\phi \mapsto \phi'(z) = \phi(z+a)$. Assume ϕ_1 and ϕ_2 create the same state. That means $\phi_1(0)|0\rangle = \phi_2(0)|0\rangle$. That means $\phi_1(z)|0\rangle = \phi_2(z)|0\rangle$ and

$$\langle 0|\phi_1(w_1)\psi_2(w_2)(\phi_1(z) - \phi_2(z))|0\rangle = 0\,, \quad |w_i| > |z|\,. \tag{3.35}$$

Using analytic continuation we bring ψ_1 to future infinity and ψ_2 to past infinity:

$$\langle \psi_1|\phi_1(z) - \phi_2(z)|\psi_2\rangle = 0 \Rightarrow \phi_1(z) = \phi_2(z)\,. \tag{3.36}$$

3.4 Duality theorem

We want to proof that

$$\mathcal{R}(\psi(z)\phi(w)) = V(\phi(z-w)|\phi\rangle; w)\,. \tag{3.37}$$

For that we need to check on $|0\rangle$ at $w = 0$.

$$\psi(z)\psi(0)|0\rangle = \psi(z)|\phi\rangle = V(\psi(z)|\phi\rangle, 0)|0\rangle\,. \tag{3.38}$$

Application (with $a_0 = \sqrt{\alpha'/2}p$):

$$J(z)V_k(w,\overline{w}) = V\left(\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} a_n (z-w)^{-n-1}|k\rangle; w, \overline{w} \right) = \frac{\alpha'}{2} k \frac{1}{z-w} V(|k\rangle, w, \overline{w}) + \text{regular terms}\,. \tag{3.39}$$

Chapter 4

Modular invariance

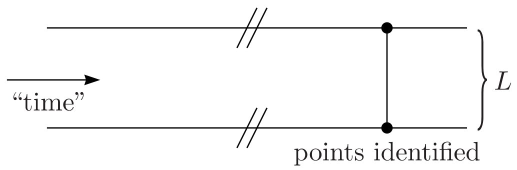
Consider the map

$$z \mapsto w = \frac{L}{2\pi} \log(z)\,. \tag{4.1}$$

What does this map do? Take $z = r\exp(\mathrm{i}\vartheta)$. Then, it follows that

$$w = \frac{L}{2\pi}\log(r) + \mathrm{i}\frac{L}{2\pi}\vartheta\,, \tag{4.2}$$

so the real part runs from zero to infinity and the imaginary part runs from zero to L. The image of the complex plane under this map is



$$T_{\text{plane}}(z) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 T_{\text{cyl}}(w) + \frac{c}{12}\{w, z\}\,. \tag{4.3}$$

By using

$$w' = \frac{L}{2\pi z}\,, \quad w'' = \frac{L}{2\pi z^2}\,, \quad w''' = \frac{2L}{2\pi z^3}\,, \tag{4.4}$$

we can compute the Schwarz derivative:

$$\{w, z\} = \frac{w''' \cdot w' - \frac{3}{2}(w'')^2}{(w')^2} = \frac{\frac{2}{z^3}\cdot\frac{1}{z} - \frac{3}{2}\cdot\frac{1}{z^4}}{\left(\frac{1}{z}\right)^2} = \frac{1}{2z^2}\,. \tag{4.5}$$

Hence, the energy-momentum tensor in the plane is given by

$$T_{\text{plane}}(z) = \left(\frac{L}{2\pi z}\right)^2 T_{\text{cyl}}(w) + \frac{c}{24z^2}\,. \tag{4.6}$$

We want to use the relation between the energy-momentum tensors to see how the Hamiltonians are related. The Hamiltonian is the generator of “time” translations:

$$\begin{aligned} H_{\text{cyl}} &= \frac{1}{2\pi}\int_0^L \mathrm{d}\sigma\,[T_{\text{cyl}}(\tau + \mathrm{i}\sigma) + \overline{T}_{\text{cyl}}(\tau - \mathrm{i}\sigma)] = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}w\, T_{\text{cyl}}(w) + \ldots = \\ &= \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,\frac{\mathrm{d}w}{\mathrm{d}z}\left[\left(\frac{2\pi z}{L}\right)^2 T_{\text{plane}}(z) - \frac{c}{24z^2}\left(\frac{2\pi z}{L}\right)^2\right] + \ldots = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}z\,\left[\frac{2\pi z}{L}T_{\text{plane}}(z) - \frac{\pi c}{12Lz}\right]\ldots = \frac{1}{L}\left(2\pi L_0 - \frac{\pi c}{12\ldots}\right. \end{aligned} \tag{4.7}$$

so that the final result is:

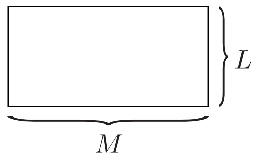
$$H_{\text{cyl}} = \frac{2\pi}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right)\,. \tag{4.8}$$

The Hamiltonian is shifted by an off-set $-c/12$. Translations in imaginary direction:

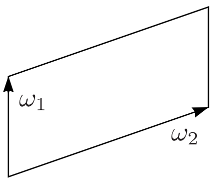
$$P = \frac{2\pi}{L}(L_0 - \overline{L}_0)\,. \tag{4.9}$$

We cut the cylinder at length M and put periodic boundary conditions on it, such that we get a torus. The partition function then is:

$$Z = \mathrm{Tr}_{\mathcal{H}} \exp(-H_{\mathrm{cyl}} \cdot M) = \mathrm{Tr}_{\mathcal{H}} \exp\left\{-\frac{2\pi M}{L}\left(L_0 + \overline{L}_0 - \frac{c}{12}\right)\right\}. \tag{4.10}$$



$= \mathrm{i}L$



$$\begin{aligned} Z(\omega_1, \omega_2) &= \mathrm{Tr}_{\mathcal{H}} \exp\left(-H_{\mathrm{cyl}} \cdot \mathrm{Re}(\omega_2)\right) \exp\left(-P\mathrm{iIm}(\omega_2)\right) = \\ &= \mathrm{Tr}_{\mathcal{H}} \exp\left\{-L_0\left(\frac{2\pi\mathrm{Re}(\omega_2)}{L} + \frac{2\pi\mathrm{iIm}(\omega_2)}{L}\right) - \overline{L}_0\left(\frac{2\pi\mathrm{Re}(\omega_2)}{L} - \frac{2\pi\mathrm{iIm}(\omega_2)}{L}\right) + \frac{\pi c}{6L}\mathrm{Re}(\omega_2)\right\} = \\ &= \mathrm{Tr}_{\mathcal{H}} \exp\left\{2\pi\mathrm{i}\tau\left(L_0 - \frac{c}{24}\right)\right\} \exp\left\{2\pi\mathrm{i}\overline{\tau}\left(\overline{L}_0 - \frac{c}{24}\right)\right\}, \end{aligned} \tag{4.11}$$

with

$$\mathrm{i}\tau = \frac{-\mathrm{Re}(\omega_2) - \mathrm{iIm}(\omega_2)}{L} = -\frac{\omega_2}{L} \Rightarrow \tau = -\frac{\omega_2}{\mathrm{i}L} = -\frac{\omega_2}{\omega_1}\,. \tag{4.12}$$

The ratio ω_2/ω_1 determines the shape of the torus. This description is important near phase transitions, where the correlation length goes to infinity and the theory becomes scale invariant. Finite size effects: Send $M \mapsto \infty$ and keep L fixed.

$$Z = \mathrm{Tr}_{\mathcal{H}}\left\{q^{L_0 - c/24} - \overline{q}^{\overline{L}_0 - c/24}\right\} \overset{M \text{ large}}{\sim} \exp\left(\frac{\pi c}{6}\frac{M}{L}\right), \quad q = \exp(2\pi\mathrm{i}\tau)\,. \tag{4.13}$$

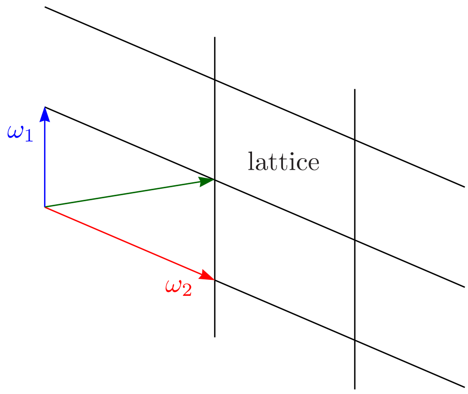
Hence, the lowest L_0-eigenvalue (which is taken to be zero) dominates. The free energy per length is given by

$$\frac{\log(Z)}{M} \mapsto \frac{\pi c}{6L}\,. \tag{4.14}$$

Changing L of the cylinder the free energy will change. How it changes, is determined by the central charge. An application of this description in solid state physics is the two-dimensional Ising model. From the computation of the free energy one can extract the value 1/2 for the central charge of this model. Primary fields:

$$\langle \phi(w_1, \overline{w}_1)\phi(w_2, \overline{w}_2)\rangle = \left(\frac{\pi}{L}\frac{1}{\sinh\left(\frac{\pi}{L}(w_1 - w_2)\right)}\right)^{2h}\left(\frac{\pi}{L}\frac{1}{\sinh\left(\frac{\pi}{L}(\overline{w}_1 - \overline{w}_2)\right)}\right)^{2\overline{h}}. \tag{4.15}$$

Hence, for large distances correlation function is supressed exponentially with a characteristic length scale L. At short distances one obtains the same result as for correlation functions in the complex plane.



We have the freedom to choose ω_1 and ω_2:

$$\begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}\,, \quad ad - bc = 1\,, \tag{4.16}$$

with a, b, c, $d \in \mathbb{Z}$ $(\mathrm{SL}(2,\mathbb{Z})/\mathbb{Z}_2)$. The theory shall be invariant under the action of the modular group. The modular group is generated by

$$S : \tau \mapsto -\frac{1}{\tau}\,, \quad S = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\,, \quad T : \tau \mapsto \tau + 1\,, \quad T = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\,. \tag{4.17}$$

To check modular invariance, we just need to show

$$Z(\tau) = Z\left(-\frac{1}{\tau}\right) = Z(\tau+1)\,. \tag{4.18}$$

T-invariance:

$$q^{L_0 - c/24}\overline{q}^{L_0 - c/24} \mapsto \exp\left\{2\pi\mathrm{i}(L_0 - \overline{L}_0)\right\} q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - c/24}\,. \tag{4.19}$$

The spectrum of $L_0 - \overline{L}_0$ in $\subset \mathbb{Z} \ni h - \overline{h}$.

4.1 The Cardy formula

We assume that the spectrum is discrete.

$$Z = \mathrm{Tr}_{\mathcal{H}} q^{L_0 - c/24}\overline{q}^{\overline{L}_0 - c/24} = \sum_{h,\overline{h}} d_{h,\overline{h}} q^{h - c/24}\overline{q}^{\overline{h} - c/24}\,. \tag{4.20}$$

By treating q and $\overline{q}$ as independent we obtain:

$$d_{h,\overline{h}} = \frac{1}{(2\pi\mathrm{i})^2}\oint \mathrm{d}q \oint \mathrm{d}\overline{q}\, Z(q,\overline{q}) q^{-1-h+c/24}\overline{q}^{-1-\overline{h}+c/24}\,. \tag{4.21}$$

With $q = \exp(2\pi\mathrm{i}\tau)$, $\mathrm{d}q = 2\pi\mathrm{i}q\,\mathrm{d}\tau$ and assuming S-invariance we get:

$$d_h = \frac{1}{2\pi\mathrm{i}}\oint \mathrm{d}q\, Z(q) q^{-1-h+c/24} = \int \mathrm{d}\tau\, \exp\left\{-2\pi\mathrm{i}\tau\left(h - \frac{c}{24}\right)\right\} \underbrace{Z(\tau)}_{=Z(-1/\tau)}\,. \tag{4.22}$$

By using

$$Z(\widetilde{q}) = \mathrm{Tr}_{\mathcal{H}}\widetilde{q}^{L_0 - c/24}\,, \quad \widetilde{q} = \exp\left(-\frac{2\pi\mathrm{i}}{\tau}\right)\,, \tag{4.23}$$

we obtain in assuming $\mathrm{Im}(\tau)$ as small:

$$d_h = \int \mathrm{d}\tau\, \exp\left\{-2\pi\mathrm{i}\tau\left(h - \frac{c}{24}\right)\right\}\exp\left(\frac{2\pi\mathrm{i}}{\tau}\frac{c}{24}\right)\,. \tag{4.24}$$

For large h the integrand will we an strongly oscillating function. The main contribution then comes from stationary points

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\tau\left(h - \frac{c}{24}\right) + \frac{1}{\tau}\frac{c}{24}\right] = h - \frac{c}{24} + \frac{1}{\tau^2}\frac{c}{[}24 \Rightarrow \tau^2 = -\frac{\frac{c}{24}}{h - \frac{c}{24}} \approx -\frac{c}{24h} \Rightarrow \tau \approx \mathrm{i}\sqrt{\frac{c}{24h}}\,, \tag{4.25}$$

and is now given by:

$$d_h \sim \exp\left\{-2\pi\mathrm{i}\left(\mathrm{i}\sqrt{\frac{c}{24h}}\right)h + \frac{2\pi c}{24}\sqrt{\frac{24h}{c}}\right\} \sim \boxed{\exp\left(4\pi\sqrt{\frac{ch}{24}}\right)\,,} \tag{4.26}$$

which is called the **Cardy formula**.

$$\log(d_{h,\overline{h}}) = 2\pi\left(\sqrt{\frac{ch}{6}} + \sqrt{\frac{c\overline{h}}{6}}\right) + \dots\,. \tag{4.27}$$

The leading behaviour is just given by the central charge. Hence, the central charge has something to do with the number of states of a certain level and as a result of this with the density of states. Consider a BTZ (Bañados, Bunster [formerly known as Teitelboim], Zanelli) black hole in (2+1) dimensions. (Teitelboim discovered that he had a different father and changed his name. When he discovered the solution his same was Teitelboim.) The line element square (with angular momentum $J = 0$) is given by

$$\mathrm{d}s^2 = \left(\frac{r^2}{l^2} - 8GM\right) \mathrm{d}t^2 - \frac{1}{\frac{r^2}{l^2} - 8GM} \mathrm{d}r^2 - r^2 \, \mathrm{d}\phi^2 \,, \tag{4.28}$$

with M being the black hole mass and l has something to do with the cosmological constant. It is the radius of curvature of the asymptotic AdS_3. There is a dual CFT at the boundary of AdS. The central charge is related with the cosmological constant. We will just give the result, since we cannot derive it here:

$$c = \frac{3l}{2G} \,, \quad L_0 + \overline{L}_0 = lM \,. \tag{4.29}$$

For calculating the entropy take $h = \overline{h}$. For BTZ black holes with spin the difference between h and $\overline{h}$ is characterized by the spin.

$$S = 4\pi\sqrt{\frac{ch}{6}} = 4\pi\sqrt{\frac{3l \cdot l \cdot M}{2G \cdot 6 \cdot 2}} = 2\pi\sqrt{\frac{l^2 M}{2G}} \,. \tag{4.30}$$

The entropy is determined by the area of the horizon and is here just one-dimensional. The area is given by

$$A = 2\pi \cdot r_{\mathrm{Horizon}} = 2\pi\sqrt{8GMl^2} \,, \tag{4.31}$$

and hence the entropy can be written in the form

$$\boxed{S = \frac{A}{4G} \,.} \tag{4.32}$$

Of course, that is a (2+1)-dimensional problem, but the physical world is a (3+1)-dimensional one. However, higher-dimensional black holes can be related to BTZ black holes by some duality. With the help of conformal field theory one can compute entropies for a large number of black holes. Furthermore, conformal field theory finds further application in statistical physics and string theory.