String-Steilkurs 2008:
Supersymmetry and Supergravity

M. Zagermann

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Albert-Einstein-Institut in Golm

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von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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Chapter 1

Introduction to Supersymmetry

1.1 Symmetries in Particle Physics

1.1.1 The Poincaré group $\mathscr{P}$

The Poincaré is defined as the isometry group of four-dimensional Minkowski space. Its Lie algebra is spanned by two types of generators, namely P_m, M_{mn}, whereas m, $n = 0, 1, 2, 3$. P_m are the translation generators and M_{mn} generate the Lorentz transformations. These generators can be written in a more familiar form. The angular momentum operators $L_i \equiv \varepsilon_{ijk} M_{jk}$ ($i = 1, 2, 3$) generate the rotations and $K_i \equiv M_{0i}$ generate the boosts.

$$[P_m, P_n] = 0\,, \quad [P_m, M_{np}] = \mathrm{i}\eta_{mp} P_n - \mathrm{i}\eta_{mn} P_p\,, \quad \eta_{mn} = \mathrm{diag}(-,+,+,+)\,, \tag{1.1}$$

$$[M_{mn}, M_{pq}] = \mathrm{i}\eta_{mp} M_{nq} - \mathrm{i}\eta_{np} M_{mq} - (p \leftrightarrow q)\,. \tag{1.2}$$

The generators of the Poincaré group transform themselves under the Lorentz group. Defining

$$J_i^{\pm} \equiv \frac{1}{2}(L_i \pm \mathrm{i}K_i)\,, \tag{1.3}$$

one finds that

$$[J_i^{\pm}, J_j^{\pm}] = \mathrm{i}\varepsilon_{ijk} J_k\,, \quad [J_i^{+}, J_j^{-}] = 0\,. \tag{1.4}$$

The irreducible representations can be labeled by two half-integers (j, j'). One of these corresponds to the spin quantum number J_i^+ and the other one to J_i^-. The representations split into two types. P_m and M_{mn} themselves correspond to

$$P_m \simeq \left(\frac{1}{2}, \frac{1}{2}\right)\,. \tag{1.5}$$

$$M_{mn} \simeq (1,0) \oplus (0,1)\,, \quad \text{with } (1,0) \text{ for } J_i^+ \text{ and } (0,1) \text{ for } J_i^-\,. \tag{1.6}$$

Remark: Irreducible representations (j, j') with $j + j' =$ half-integer correspond to "double-valued" representations of the Lorentz group SO(3,1). (However, they are single-valued representations of the universal covering group $\widetilde{\mathrm{SO}}(3,1) \simeq \mathrm{SL}(2,\mathbb{C})$ of the Lorentz group.) Analogy: quantum mechanics of angular momentum: SO(3,1) $\leftrightarrow$ SO(3). (Spin 1/2: double-valued representation of SO(3), but they are single-valued representations of $\widetilde{\mathrm{SO}}(3) \simeq \mathrm{SU}(2)$.) Hence, (j, j') with $j + j' =$ half-integer correspond the **fermionic quantities**. You make the following observation: The known particle physics is invariant under

$$\mathrm{Lie}(\mathscr{P}) \oplus \mathrm{Lie}(G_{\mathrm{int}})\,, \tag{1.7}$$

with G_{int} being an "internal" symmetry group, which is given by

$$G_{\mathrm{SM}} = \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)\,. \tag{1.8}$$

The Lie algebra of the internal symmetry group is spanned by generators T_a, for which it holds that $[T_a, T_b] = \mathrm{i}f_{ab}{}^c T_c$ ($a = 1, \ldots, \dim(G_{\mathrm{int}})$). The dimension of the internal group is at least 12. "$\oplus$" means direct sum, so we have

$$[T_a, P_m] = [T_a, M_{mn}] = 0\,, \tag{1.9}$$

so the two algebras do not talk to each other. They do not transform under the Lorentz group; hence, the generators live in the trivial representation $(0,0)$. G_{int} **cannot** change the mass ($m^2 = -P_a P^a$) or the spin/helicity of a particle:

$$L_i T_a |s, m\rangle = T_a L_i |s, m\rangle = s T_a |s, m\rangle \,. \tag{1.10}$$

Similarly this can be proofed for the mass.

1.2 "Known" particle spectrum

spin/helicity	2	3/2	1	1/2	0
particles	"graviton"	?	photon $W^{\pm}$, Z gluons	leptons quarks	Higgs (?)

G_{int} can at most operate "**vertically**".

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \simeq \mathbf{2} \text{ of SU}(2)\,, \tag{1.11}$$

$$\begin{pmatrix} d_r \\ d_g \\ d_b \end{pmatrix}_R \simeq \mathbf{3} \text{ of SU}(3)\,, \quad \text{gluons} \simeq \mathbf{8} \text{ of SU}(3)\,. \tag{1.12}$$

Grand unified theories, as for example $G_{\text{int}} = \text{SO}(10) \supset G_{\text{SM}}$:

$$\begin{pmatrix} \text{quarks} \\ \text{leptons} \end{pmatrix} \simeq \mathbf{16} \text{ of SO}(10)\,, \tag{1.13}$$

$$\begin{pmatrix} \text{all gauge} \\ \text{bosons} \end{pmatrix} \subset \mathbf{45} \text{ of SO}(10)\,. \tag{1.14}$$

There arises one question: Can one also "unify" the particles of **different** spin? Can you, in particular, unify the graviton ($s = 2$) and the gauge bosons ($s = 1$)? The disappointing news is the Coleman-Mandula theorem (1967).

1.3 The Coleman-Mandula theorem

Under the (reasonable) assumptions that

- the S-matrix is nontrivial and analytic in the scattering angles
- and that there exist only a finite number of particles below a fixed mass,

symmetries, which are described by **Lie algebras**, must be of the following form:

$$\text{Lie}(\mathscr{P}) \oplus \text{Lie}(G_{\text{int}})\,. \tag{1.15}$$

This implies that G_{int} can never change the spin! Are there perhaps symmetries that do not form Lie algebras? Then, we could circumvent the Coleman-Mandula theorem and connect particles with different spin. Suppose that you have some general symmetry generator that acts additively on multi-particle states. Then, it must be of the following form:

$$G = \sum_{i,j} \int \mathrm{d}^3 p \int \mathrm{d}^3 q \, a_i^\dagger(p) K_{ij}(q,p) a_j(q) = a^\dagger * K * a\,, \tag{1.16}$$

where $a_i^\dagger(p)$ are creation and $a_j(q)$ annihilation operators. **Bosons** have creation and annihilation operators that satisfy a commutation relation of the form

$$[b_i(p), b_j^\dagger(q)] = \delta_{ij}\delta^{(3)}(p-q)\,. \tag{1.17}$$

Fermions satisfy an anticommutation relation

$$\{f_i(p), f_j^\dagger(q)\} = \delta_{ij}\delta^{(3)}(p-q)\,. \tag{1.18}$$

Hence, G can be split in $B + F$ with

$$B = b^\dagger * K_{bb} * b + f^\dagger * K_{ff} * f\,, \tag{1.19}$$

which transforms bosons and fermions under themselves or

$$F = f^\dagger * K_{fb} b + b^\dagger * K_{bf} * f\,, \tag{1.20}$$

which transforms bosons to fermions and vice versa.

$$[B_1, B_2] = B_3\,, \quad [F_1, B_2] = F_3\,, \quad [F_1, F_2] = \text{not bilinear in } a^\dagger,\, a\,, \tag{1.21}$$

so this is not a proper symmetry generator. **But**, it holds that

$$\{F_1, F_2\} = B_3\,. \tag{1.22}$$

So, allowing for Bose-Fermi symmetries F, results in a "**graded Lie algebra**" (or "**super Lie algebra**"). Schematically, it holds that

$$[G_1, G_2\} = G_3\,, \quad [G_1, G_2\} \equiv G_1 G_2 - (-1)^{\eta_1 \eta_2} G_2 G_1\,, \tag{1.23}$$

where $\eta(B) = 0$ and $\eta(F) = -1$. Graded Jacobi identity:

$$(-1)^{\eta_a \eta_c} [[G^a, G^b\} G^c\} + \text{cyclic} = 0\,. \tag{1.24}$$

Symmetries between bosons and fermions are called supersymmetries and are described by anti-commuting symmetry generators F. These are called SUSY generators or "odd" or "fermionic". The B generators are either called "even" or "bosonic". From now on we will write $F \to Q$.

1.4 The Super-Poincaré algebra

Three remarks:

- The supersymmetry generators Q change the spin $= (j, j')$. They cannot be the trivial representation (0,0), because then, the spin could not be changed.
- $Q^\dagger \simeq (j', j)$,
- $\{Q, Q^\dagger\} \simeq (j + j', j + j') \oplus \ldots$ (non-zero, bosonic).

The Coleman-Mandula theorem tells us that a state of the form $(j + j', j + j')$ must be either an internal symmetry generator with (0,0), or P_m with (1/2,1/2) or M_{mn} with (1,0)⊕(0,1). The first and the third possibility go out, what stays is the second one.

$$Q \simeq \left(\frac{1}{2}, 0\right) \text{ or } \left(0, \frac{1}{2}\right)\,, \quad \boxed{\{Q, Q^\dagger\} \sim P_m}\,. \tag{1.25}$$

Without loss of generality one can define $Q \mapsto (1/2, 0)$ and $Q^\dagger \mapsto (0, 1/2)$. We use the notation Q_A ($A = \{1, 2\}$, labels (1/2,0)) and $Q^\dagger_{\dot{A}}$ ($\dot{A} = \{1, 2\}$, labels (0,1/2)) and the 2×2-matrices $r(M_{ab})_{AB}$ and $\widetilde{r}(M_{ab})_{\dot{A}\dot{B}}$.

$$[M_{ab}, Q_A] = r(M_{ab})_{AB} Q_B\,, \quad [M_{ab}, Q^\dagger \dot{A}] = \widetilde{r}(M_{ab})_{\dot{A}\dot{B}} Q^\dagger_{\dot{B}}\,. \tag{1.26}$$

A further generalization is $Q_A \to Q^r_A$ (with $r = 1, \ldots, N$). $N = 1$ is denoted as "minimal" or "simple" SUSYV and $N > 1$ as "N-extended" SUSY. Using similar arguments than above plus the graded Jacobi identity one finds (Haag-Łopuszański-Sohnius, 1975):

$$\boxed{\{Q^r_A, (Q^\dagger_{\dot{B}})^s\} = 2\delta^{rs} (\sigma^m)_{A\dot{B}} P_m\,, \quad \sigma^0_{A\dot{B}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad \sigma^i_{A\dot{B}} = \text{Pauli matrices } (i = 1,\, 2,\, 3)\,.} \tag{1.27}$$

With $\eta_{mn} = \text{diag}(-, +, +, +)$ it holds that $(\sigma_0)_{A\dot{B}} = -(\sigma^0)_{A\dot{B}}$. Now, we are interested in $\{Q^r_A, Q^s_B\}$. From

$$\left(\frac{1}{2}, 0\right) \times \left(\frac{1}{2}, 0\right) = (0, 0) \oplus (1, 0)\,, \tag{1.28}$$

one can state that $\{Q^r_A, Q^s_B\}$ must either contain T_a or M_{mn}. However, M_{mn} is excluded, because then the equations

$$[P_m, Q^r_A] = [P_m, (Q^\dagger_{\dot{A}})^s] = 0\,, \tag{1.29}$$

could not be fulfilled. It has to hold that

$$\{Q^r_A, Q^s_B\} = e_{AB} Z^{rs}\,, \quad e_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{1.30}$$

where the Z^{rs} is some linear combination of the T_a: $Z^{rs} = -Z^{sr} = (ca)^{rs} T_a$. These linear combinations cannot be arbitrary and must commute with everything else of the algebra (using the graded Jacobi identity). These are then called "central charges". They, of course, commute with the generators of the Poincaré group, since they are built up of internal symmetry generators. Anyway, they must form an Abelian subgroup of the internal symmetry group. Note that for $N = 1$ SUSY the central charges have to vanish. It must be

$$[M_{mn}, Q^r_A] = r(M_{mn})_A{}^B Q^r_B\,, \tag{1.31}$$

where $r(M_{mn})_A{}^B$ is the $(1/2, 0)$-representation matrix of Lorentz transformations.

$$[T_a, Q^r_A] = (t_a)^r{}_s Q^s_A\,, \tag{1.32}$$

whereas the choice $(t_a)^r{}_s$ is possible. The T_a with $(t_a)^r{}_s \neq 0$ generate the "**R-symmetry group**". R-symmetries are internal symmetries that rotate the Q^r_A and therefore act nontrivially on supercharges. In general, the R-symmetry group is $\mathrm{U}(N)_R$, for $N = 1$ SUSY this is the $\mathrm{U}(1)_R$. The standard model gauge group has to commute with the supercharges.

Remarks

0) Adding also $[P, P]$, $[P, M]$, ... this is called a **super Poincaré algebra**.

1) Acting with Q and $Q^\dagger$ on a state raises or lowers the spin/helicity by 1/2.

2) Acting repeatedly with Q^r_A or $(Q^\dagger_{\dot{A}})^s$ creates new states of different spins/helicities: $\{|s\rangle, Q|s\rangle, Q^2|s\rangle, \ldots\}$. At a certain point any of this products of Q will be zero (because they anti-commute), so the sequence terminates after **finitely** many steps. The states so obtained form a so-called "**supermultiplet**" (representation of the SUSY algebra). They are "**superpartners**" of one another.

3) $[P, Q] = 0 = [P, Q^\dagger]$ implies that $[m^2, Q] = 0 = [m^2, Q^\dagger]$ ($m^2 = P_m P^m$, $m = 0, 1, 2, 3$). Q and $Q^\dagger$ cannot change the mass; hence, the masses within a supermultiplet are **all the same**. However, this is **not** observed in nature! So, SUSY can at best be a **broken** symmetry! The usual assumption is that the masses of the standard model superpartners are at least of $\mathcal{O}(100\,\mathrm{GeV})$.

Chapter 2

Representations of the super Poincaré algebra

Our focus will be on **massless** supermultiplets only because of

i) lack of time and, what is more important,

ii) SUSY is only a good approximation for $E \gg \Delta m \approx m_{\text{superpartner}} \ll m_{\text{SM}-\text{particle}}$. Hence, $m \approx 0$ is a good approximation for large E.

iii) In the standard model the masses all vanish before the electroweak symmetry breaking.

Choose a Lorentz frame such that $P^1 = P2^2 = 0$, with $P^m = (E, 0, 0, E)$. This implies that

$$\{Q^r_A, Q^{\dagger\,s}_{\dot{B}}\} = 2\delta^{rs}(\sigma_0 P^0 + \sigma_3 P^3)_{A\dot{B}} = 2E\delta^{rs}\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}_{A\dot{B}} . \tag{2.1}$$

From this, we can read off

$$\boxed{\{Q^r_1, Q^{\dagger\,s}_{\dot{1}}\} = 4E\delta^{rs}\,, \quad \{Q^r_2, Q^{\dagger\,s}_{\dot{2}}\} = 0\,,} \tag{2.2}$$

and furthermore (no sum!):

$$0 = \langle\phi|\{Q^r_2, (Q^\dagger_{\dot{2}})^r\}|\phi\rangle = \|(Q^\dagger_{\dot{2}})^r|\phi\rangle\|^2 + \|Q^r_2|\phi\rangle\|^2\,\forall\,|\phi\rangle \in \mathcal{H} \Rightarrow \boxed{Q^r_2 = Q^{\dagger\,r}_{\dot{2}} = 0\,.} \tag{2.3}$$

So, we can forget about the Q_2 and just consider the Q_1.

$$\{Q^r_1, Q^s_1\} = e_{11}Z^{rs} = 0\,. \tag{2.4}$$

For massless supermultiplets the central charge has no affect on the anti-commutator. However, one can also show (as an exercise) that $Z^{rs} = 0$. One can construct massless supermultiplets from acting with Q^r_1 or $Q^{\dagger\,r}_{\dot{1}}$.

$$\{Q^r_1, Q^{s\,\dagger}_1\} = 4E\delta^{rs}\,, \quad \{Q^r_1, Q^s_1\} = 0\,. \tag{2.5}$$

Redefining $q^r := (4E)^{-\frac{1}{2}}Q^r_1$ we arrive at

$$\{q^r, (q^s)^\dagger\} = \delta^{rs}\,, \quad \{q^r, q^s\} = 0\,. \tag{2.6}$$

Remark: q^r lowers helicity by 1/2 and $(q^r)^\dagger$ raises helicity by 1/2 (by calculating $[q^r_1, L_i]$).

2.1 Case 1: $N = 1$ SUSY

Our algebra is $\{q, q^\dagger\} = 1$ and $\{q, q\} = 0$ (hence $q^2 = 0 = (q^\dagger)^2$). Start with state of maximal helicity $|\lambda_{\text{max}}\rangle$. From that if follows

$$q^\dagger|\lambda_{\text{max}}\rangle = 0\,, \quad q|\lambda_{\text{max}}\rangle =: \left|\lambda_{\text{max}} - \frac{1}{2}\right\rangle\,. \tag{2.7}$$

By using the anti-commutator one ends up with

$$\left\langle \lambda_{\max} - \frac{1}{2} \middle| \lambda_{\max} - \frac{1}{2} \right\rangle = \langle \lambda_{\max} | q^\dagger q | \lambda_{\max} \rangle = \langle \lambda_{\max} | \lambda_{\max} \rangle \neq 0 \Rightarrow \left| \lambda_{\max} - \frac{1}{2} \right\rangle \neq 0 \,. \tag{2.8}$$

Anyway, one obtains:

$$q \left| \lambda_{\max} - \frac{1}{2} \right\rangle = q^2 | \lambda_{\max} \rangle = 0 \,, \tag{2.9}$$

and

$$q^\dagger \left| \lambda_{\max} - \frac{1}{2} \right\rangle = q^\dagger q | \lambda_{\max} \rangle = | \lambda_{\max} \rangle \,. \tag{2.10}$$

So, we have a complete supermultiplet

$$\left\{ | \lambda_{\max} \rangle, \left| \lambda_{\max} - \frac{1}{2} \right\rangle \right\} \,. \tag{2.11}$$

It is irreducible, so every massless particle of helicity λ has exactly one superpartner of helicity $\lambda \pm 1/2$.

2.1.1 Consequences for the standard model

Assume $[\mathrm{Lie}(G_{\mathrm{SM}}), Q] = 0$. (For SU(3), SU(2) this is automatically the case.) This implies that the superpartners are in the same [SU(3) × SU(2)× U(1)]-representation.

i) Quarks with $\lambda = \pm 1/2$ can be either have a superpartner with $\lambda = 0$ or $\lambda = 1$. The case $\lambda = 1$ would correspond to vector bosons in **3** of SU(3). That would be a non-unitary theory, so this possibility is ruled out. As a result of that it must hold that $\lambda_{\text{partner}} = 0$ and the superparnters must be scalar colour-triplet (in the **3** of SU(3)) particles, so-called "squarks".

ii) The leptons with $\lambda = \pm 1/2$ have superpartners with $\lambda_{\text{partner}} = 0$ because of the same reason as above for the quarks. These are called "sleptons".

iii.) Gauge bosons with $\lambda = \pm 1$ could either have $\lambda_{\text{partner}} = \pm 1/2$ or $\lambda_{\text{partner}} = \pm 3/2$. Particles with $\lambda = \pm 3/2$ cannot be coupled in a renormalizable way. So, this possibility is ruled out, if one claims renormalizability. Therefore, it must be $\lambda_{\text{partner}} = \pm 1/2$ and these superpartners are called "gauginos". These are fermions that live in the adjoint representation of the standard model gauge group G_{SM}.

iv.) From $\lambda = 0$ ist follows that $\lambda = \pm 1/2$. The partners are called "Higgsinos".

v.) For the graviton with $\lambda = \pm 2$ it follows that $\lambda_{\text{partner}} = \pm 3/2$ or $\lambda_{\text{partner}} = \pm 5/2$. Partners with spin $\pm 5/2$ are not known how to couple to gravity, so that is out. $\lambda = \pm 3/2$ causes problems with renormalizability as mentioned above. But that is not important here, because gravity anyway is not renormalizable! The superpartner of the graviton is called "gravitino".

2.2 Case 2: N-extended SUSY

$$\{q^r, {q^\dagger}^s\} = \delta^{rs} \,, \quad \{q^r, q^s\} = 0 \,. \tag{2.12}$$

From $|\lambda_{\max}\rangle$ one obtains N states of helicity $\lambda_{\max} - 1/2$

$$\{q|\lambda_{\max}, q^2|\lambda_{\max}, \dots, q^N|\lambda_{\max}\} \,, \tag{2.13}$$

$\binom{N}{2}$ states with helicity $\lambda_{\max} - 1$

$$\{q^1 q^2 |\lambda_{\max}\rangle, q^1 q^3 |\lambda_{\max}\rangle, \dots, q^{N-1} q^N |\lambda_{\max}\rangle\} \,, \tag{2.14}$$

and one state with $\lambda_{\max} - N/2 = \lambda_{\min}$:

$$\{q^1 q^2 \dots q^N |\lambda_{\max}\rangle \,. \tag{2.15}$$

- $N = 2$:

λ	2	3/2	1	1/2	0	-1/2	-1	-3/2	-2
	1	2	1						
		1	2	1					
				1	2	1			

- $N = 4$ (super Yang-Mills theory, $N = 4$ is the **largest** N which maintains renormalizability)

λ	2	3/2	1	1/2	0	-1/2	-1	-3/2	-2
	1	4	6	4	1				
			1	4	6	4	1		

- $N = 8$ (SUGRA $N = 8 = N_{\text{max}}$):

λ	2	3/2	1	1/2	0	-1/2	-1	-3/2	-2
	1	8	28	56	70	56	28	8	1

Perhaps, $N = 8$ SUGRA could be the theory of everything. It could be renormalizable or even finite, because different loop contributions cancel. However, it turns out, that $N = 8$ SUGRA has some problems.

  - The largest gauge group is $\mathrm{SO}(8) \not\supset \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$.
  - $N \geq 2$ only has non-chiral gauge interactions. Hence, they cannot describe the electroweak sector.

To summarize, $N = 1$ SUSY seems the only realistic option.

2	3/2	1	1/2	0
1	1			
		1	1	

It does not unify gravity with gauge interactions.

Chapter 3

Globally supersymmetric field theories

SUSY imposes strong constraints on

a) possible particle spectrum,

b) possible interactions.

These have a large impact on QFT and therefore we need **supersymmetric field theories**.

3.1 Fields and symmetry operators

Schematically it holds that $|x\rangle = \phi(x)|0\rangle$. Consider translations:

$$\phi(x+a)|0\rangle = |X+a\rangle = \exp(\mathrm{i}pa)|x\rangle = \exp(\mathrm{i}pa)\phi(x)|0\rangle = \exp(\mathrm{i}pa)\phi(x)\exp(-\mathrm{i}pa)\exp(\mathrm{i}pa)|0\rangle = \exp(\mathrm{i}pa)\phi(x)\exp(-\mathrm{i}pa)|0\rangle\,, \tag{3.1}$$

from which it follows that

$$\exp(\mathrm{i}pa)\phi(x)\exp(-\mathrm{i}pa) = \phi(x+a)\,. \tag{3.2}$$

Infinitesimally, this means:

i) $\delta_a\phi \equiv \phi(x+a) - \phi(x)|_{a\mapsto 0} = \mathrm{i}a^m[P_m, \phi(x)] = a^m\partial_m\phi(x)$

ii) $P_m \times \phi(x) := [P_m, \phi(x)] = -\mathrm{i}\partial_m\phi(x)$ (action of P_m on $\phi(x)$)

That must be consistent with the expression

$$0 = [P_m, P_n] \times \phi = (-\mathrm{i})^2[\partial_m, \partial_n]\phi = 0\,. \tag{3.3}$$

This is consistent with the representation of the symmetry algebra on the field $\phi(x)$.

3.2 SUSY analogon of the above

$$Q_A \times (\text{bosonic field}) = [Q_A, \text{bosonic field}] = \text{fermionic field}\,, \tag{3.4a}$$

$$Q_A \times (\text{fermionic field}) = \{Q_A, \text{fermionic field}] = \text{bosonic field}\,. \tag{3.4b}$$

This must be consistent with the SUSY algebra

$$\boxed{\{Q_A, Q^\dagger_{\dot{A}}\} \times \text{field} = 2\sigma^m_{A\dot{A}}P_m \times \text{field} = -2\mathrm{i}\sigma^m_{A\dot{A}}\partial_m(\text{field})\,.} \tag{3.5}$$

This constraint restricts the way the fields can occur on the right-hand of Eq. (3.4). Often, Eq. (3.4) is written in the form (i) by introducing "infinitesimal parameters" ε^A, $\varepsilon^{\dot{A}}$ ($A = 1$, 2). Then, we find

$$\delta_\varepsilon\phi = (\varepsilon^A Q_A + \varepsilon^{\dot{A}} Q^\dagger_{\dot{A}}) \times \phi = \begin{cases} \varepsilon^A[Q_A, \phi_B] + \ldots \\ \varepsilon^A\{Q_A, \phi_F\} + \varepsilon^{\dot{A}}\{Q^\dagger_{\dot{A}}, \phi\} \end{cases}\,. \tag{3.6}$$

Useful trick: Assume ε^A, $\varepsilon^{\dot{A}}$ to be **anti-commuting**: $\varepsilon^1\varepsilon^2 = -\varepsilon^2\varepsilon^1$ (Grassmann variables). One also requires that they anti-commute with fermionic operators, but commute with bosonic ones:

$$\varepsilon^A \phi_F = -\phi_F \varepsilon^A \,, \quad \varepsilon^A \phi_B = \phi_B \varepsilon^A \,. \tag{3.7}$$

As a result of that all brackets become commutators, if we write ε^A, $\varepsilon^{\dot{A}}$ **inside** the brackets:

$$\varepsilon^A \{Q_A, \phi_F\} = [\varepsilon^A Q_A, \phi_F] \,, \tag{3.8}$$

etc. Also:

$$[\varepsilon^A Q_A, \varepsilon^{\dot{A}} Q_{\dot{A}}] = -2\varepsilon^A \sigma^m_{A\dot{A}} \varepsilon^{\dot{A}} P_m \,. \tag{3.9}$$

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi = -2\varepsilon_1^A \sigma^m_{A\dot{A}} \varepsilon_2^{\dot{A}} \underbrace{P_m \times \phi}_{=-\mathrm{i}\partial_m \phi} \,. \tag{3.10}$$

This gives us a representation of SUSY algebra on fields. Then, we have to require that dynamics is also invariant under SUSY:

$$\delta_\varepsilon S[\phi_1, \dots, \phi_n] = 0 \,. \tag{3.11}$$

The is equivalent to the statement that the action is supersymmetric. There are two cases:

i.) $\varepsilon^A = \text{const.}$: global SUSY

ii.) $\varepsilon^A = \varepsilon^A(x)$: local SUSY (supergravity)

To write down an example, we switch to **four-component spinor notation**. From now on we will use the $*$ for Hermitian conjugate of a Hilbert space operator instead of the symbol †, because we would like to use † for Hermitian conjugation plus transposition:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}^\dagger = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}^{*,\mathsf{T}} = (Q_1^*, Q_2^*) \,. \tag{3.12}$$

Combine Q_A and $Q^*_{\dot{A}}$ into a four-component spinor as follows:

$$Q_\alpha = \begin{pmatrix} e \cdot Q^*_{(2)} \\ Q_{(2)} \end{pmatrix}_\alpha \,, \quad e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,, \tag{3.13}$$

with $\alpha = 1, 2, 3, 4$. Define matrices $(\gamma_m)_{\alpha\beta}$ ($m = 0, 1, 2, 3$ and $\alpha = 1, 2, 3, 4$):

$$\gamma_0 := \begin{pmatrix} 0 & \mathrm{i}\mathbb{1}_2 \\ \mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix} \,, \quad \gamma_j = \begin{pmatrix} 0 & -\mathrm{i}\sigma_j \\ \mathrm{i}\sigma_j & 0 \end{pmatrix} \,, \tag{3.14}$$

with $j = 1, 2, 3$. That is the so-called Weyl representation, which is very convenient, if one talks about two-component spinors. The γ matrices obey the Clifford algebra:

$$\{\gamma_m, \gamma_n\} = 2\eta_{mn} \mathbb{1}_4 \,. \tag{3.15}$$

We define

$$\gamma_5 := \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix} \,, \quad (\gamma_5)^2 = \mathbb{1}_4 \,, \quad \{\gamma_5, \gamma_m\} = 0 \,. \tag{3.16}$$

The Dirac conjugate is given by $\overline{\psi} \equiv -\mathrm{i}\psi^\dagger\gamma_0$. A four-component ψ with

$$\overline{\psi} = \psi^\mathsf{T} C \,, \quad C = \begin{pmatrix} e & 0 \\ 0 & -e \end{pmatrix} \,, \tag{3.17}$$

is called a "**Majorana spinor**". C is the charge conjugation matrix. Eq. (3.17) is equivalent to $\psi = \psi^c$. Hence, these are spinors that describe particles which are their own anti-particles. Because of this constraint, Majorana spinors only have two independent complex components:

$$\psi = \begin{pmatrix} e\chi^* \\ \chi \end{pmatrix} \,, \tag{3.18}$$

whereas χ is the two-component spinor containing the two independent degrees of freedom. A general Dirac spinor, which is needed to describe particles that are not their own antiparticles (as for example the electron), can be written as

$$\psi = \begin{pmatrix} e\chi^* \\ \lambda \end{pmatrix}, \quad \lambda \neq \chi \,. \tag{3.19}$$

Hence, Q_α is a Majorana spinor.

$$\boxed{\{Q_\alpha, \overline{Q}_\beta\} = -2\mathrm{i}\gamma^m_{\alpha\beta} P_m \,.} \tag{3.20}$$

In the following, all spinors are meant to be **anti-commuting Majorana** spinors. Let us consider the following example:

$$\overline{\psi}\chi = \chi^\intercal C\chi = \psi_\alpha C_{\alpha\beta}\chi_\beta = -\chi_\beta C_{\alpha\beta}\psi_\alpha = -\chi^\intercal C^\intercal \psi = \chi^\intercal C\psi = \overline{\chi}\psi \,. \tag{3.21}$$

We introduce **chiral projectors**

$$P_L = (\mathbb{1}_4 + \gamma_5) = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_R = \frac{1}{2}(\mathbb{1}_4 - \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix}. \tag{3.22}$$

With these one can write

$$\chi_L \equiv P_L\chi \,, \quad \chi_R \equiv P_R\chi \,, \quad \overline{\chi}_R \equiv \overline{\chi}P_R = \overline{\chi_L} \,, \quad \overline{\chi_L} \equiv \overline{\chi}P_L = \overline{\chi_R} \,. \tag{3.23}$$

- Exercise 1: Verify $\gamma_m^\intercal = -C\gamma_m C^{-1}$, $\gamma_5^\intercal = C\gamma_5 C^{-1}$ and $C^\intercal = -C$.
- Exercise 2: Use this to show $\overline{\psi}\chi = \overline{\chi}\psi$, $\overline{\psi}\gamma_m\chi = -\overline{\chi}\gamma_m\psi$ and $\overline{\psi}\gamma_m\gamma_5\chi = \overline{\chi}\gamma_m\gamma_5\psi$.
- Exercise 3: Show that $P_L + P_R = \mathbb{1}_4$, $\gamma_m P_L = P_R\gamma_m$, $P_L^2 = P_L$, $P_R P_L = 0$, $\overline{\psi_L}\chi_L = \overline{\chi_L}\psi_L$, $\overline{\psi_L}\gamma_5\chi_R = -\overline{\chi_R}\gamma_m\psi_L$ and finally $\partial\!\!\!/\partial\!\!\!/ = \gamma^m\partial_m\gamma^n\partial_n\phi(x) = \Box\phi(x)\mathbb{1}_4$.

With these preparations we are now ready to write down the simplest four-dimensional supersymmetric field theory: the free massless **Wess-Zumino model**. This model contains one complex scalar $\phi(x)$, one Majorana fermion $\chi(x)$ and another complex scalar $F(x)$.

$$\delta_\varepsilon\phi = \overline{\varepsilon}_L\chi_L \,, \quad \delta_\varepsilon\phi^* = \overline{\varepsilon}_R\chi_R \,, \tag{3.24a}$$

$$\delta_\varepsilon\chi_L = \frac{1}{2}(\partial\!\!\!/\phi)\varepsilon_R + \frac{1}{2}F\varepsilon_L \,, \quad \delta_\varepsilon\chi_R = \frac{1}{2}(\partial\!\!\!/\phi^*)\varepsilon_L + \frac{1}{2}F^*\varepsilon_R \,, \tag{3.24b}$$

$$\delta_\varepsilon F = \overline{\varepsilon}_R\partial\!\!\!/\chi_L \,, \quad \delta_\varepsilon F^* = \overline{\varepsilon}_L\partial\!\!\!/\chi_R \,. \tag{3.24c}$$

These satisfy:

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}] \begin{pmatrix} \phi \\ \chi \\ F \end{pmatrix} = \frac{1}{2}\overline{\varepsilon}_2\gamma^m\varepsilon_1\partial_m \begin{pmatrix} \phi \\ \chi \\ F \end{pmatrix} \sim \overline{\varepsilon}_2\gamma^m\varepsilon_1 P_m \times \begin{pmatrix} \phi \\ \chi \\ F \end{pmatrix}. \tag{3.25}$$

Hence, (ϕ, χ, F) represents the SUSY algebra! The Lagrangian of the Wess-Zumino model is given by

$$\mathcal{L} = -(\partial_m\phi)(\partial^m\phi^*) - \overline{\chi}\partial\!\!\!/\chi + FF^* \,. \tag{3.26}$$

Claim: $\mathcal{L}$ is SUSY-invariant modulo a total derivative. We want to proof that and first write up a lemma:

$$\overline{\chi}\partial\!\!\!/\chi = 2\overline{\chi_L}\partial\!\!\!/\chi_R + \text{total derivatives} \,. \tag{3.27}$$

The proof of this lemma is not so difficult. We use equations of the third exercise:

$$\begin{aligned} \overline{\chi}\partial\!\!\!/\chi &\overset{(1)}{=} \overline{\chi}\partial\!\!\!/(P_L + P_R)\chi \overset{(3)}{=} \overline{\chi}\partial\!\!\!/P_L\chi_L + \overline{\chi}\partial\!\!\!/P_R\chi_R = \overline{\chi_R}\partial\!\!\!/\chi_L + \overline{\chi_L}\partial\!\!\!/\chi_R = \\ &\overset{(6)}{=} -\overline{\chi_L}\overleftarrow{\partial\!\!\!/}\chi_R + \overline{\chi_L}\overrightarrow{\partial\!\!\!/}\chi_R \overset{\text{p.I.}}{=} 2\overline{\chi}_L\partial\!\!\!/\chi_R \,.\Box \end{aligned} \tag{3.28}$$

Now we can proceed with the proof of our claim (with the supercurrent $J_R^m = -\overline{\chi_R}\gamma^m(\partial\!\!\!/\phi)$)

$$\begin{aligned}
\delta_\varepsilon[-\overline{\chi}\partial\!\!\!/\chi] &\overset{\text{Lemma}}{=} \delta_\varepsilon[-2\overline{\chi_L}\partial\!\!\!/\chi_R] = -2\overline{\delta\chi_L}\partial\!\!\!/\chi_R - 2\overline{\chi_L}\partial\!\!\!/(\delta\chi_R) \overset{(6)+\text{p.I.}}{=} +2\overline{\chi_R}\overleftarrow{\partial\!\!\!/}(\delta\chi_L) + 2\overline{\chi_L}\overleftarrow{\partial\!\!\!/}(\delta\chi_R) = \\
&= 2\overline{\chi_R}\overleftarrow{\partial\!\!\!/}\left\{\frac{1}{2}(\partial\!\!\!/\phi)\varepsilon_R + \frac{1}{2}F\varepsilon_L\right\} + \text{h.c.} = \overline{\chi_R}\overleftarrow{\partial\!\!\!/}(\partial\!\!\!/\phi)\varepsilon_R + \overline{\chi_R}\overleftarrow{\partial\!\!\!/}\varepsilon_L F + \text{h.c.} = \\
&\overset{\text{p.I.}}{=} -\overline{\chi_R}(\partial\!\!\!/\partial\!\!\!/\phi)\varepsilon_R - \overline{\chi}_R\gamma^m(\partial\!\!\!/\phi)(\partial_m\varepsilon_R) + \overline{\chi_R}\overleftarrow{\partial\!\!\!/}\varepsilon_L F + \text{h.c.} = \\
&\overset{(7),(5),(6)}{=} -\overline{\varepsilon}_R\chi_R\Box\phi + J_R^m(\partial_m\varepsilon_R) - \overline{\varepsilon}_L\overrightarrow{\partial\!\!\!/}\varepsilon_R F + \text{h.c.} =: A + B + C + \text{h.c.}
\end{aligned} \tag{3.29}$$

As a further exercise it can be shown that

$$(A + \text{h.c.}) = -\delta_\varepsilon[-(\partial_m\phi)(\partial^m\phi^*)]\,, \quad (C + \text{h.c.}) = -\delta_\varepsilon(FF^*)\,. \tag{3.30}$$

The result is:

$$\delta_\varepsilon\mathcal{L} = J_R^m(\partial_m\varepsilon_R) + J_L^m(\partial_m\varepsilon_L) = 0\,, \tag{3.31}$$

for global SUSY (constant ε). Let us now have a look at the equations of motion for F. They are $F = 0$, hence, we could set $F = 0$ everywhere and $\mathcal{L}|_{F=0}$ would be invariant under $\delta\phi|_{F=0}$, $\delta\chi|_{F=0}$. So, why have we introduced the field F? $[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\phi$ would still be okey, but

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\chi = \frac{1}{2}(\overline{\varepsilon}_2\gamma^m\varepsilon_1)\partial_m\chi + [\ldots]\partial\!\!\!/\chi\,, \tag{3.32}$$

where the second term is bad. This second term vanishes, if the equations of motion are used. Eliminating the auxiliary field F the SUSY algebra only holds "**on-shell**" (i.e. after use of equations of motion.) With F, it holds **off-shell**. For the physical content one does not need the off-shell formulation, but one cannot use superspace formulation. For $N \geq 2$, off-shell formulations are often unknown.

Chapter 4

Introduction to Supergravity

There are three equivalent definitions:

i) These are locally supersymmetric field theories ($\varepsilon = \varepsilon(x)$).

ii) Supersymmetrizations of general relativity

iii) Only known field theories of interacting spin 3/2 fields.

In order to illustrate the equivalence of the three definitions we consider the locally supersymmetric Wess-Zumino model with the Lagrangian

$$\mathcal{L}_{\mathrm{WZ}} = -(|\partial\phi|^2 + \overline{\chi}\partial\!\!\!/\chi)\,. \tag{4.1}$$

The variation under local SUSY is given by

$$\delta_\varepsilon \mathcal{L}_{\mathrm{WZ}} = J_R^m(\partial_m \varepsilon_R) + J_L^m(\partial_m \varepsilon_L) \neq 0\,, \tag{4.2}$$

with the supercurrents

$$J_R^m \equiv -\overline{\chi_R}\gamma^m \partial\!\!\!/\phi\,, \quad J_L^m = (J_R^m) + \text{h.c.}\,. \tag{4.3}$$

Consider the Lagrangian of free quantum electrodynamics: $\mathcal{L}_{\text{free}}^{\text{QED}} = -\overline{\lambda}\partial\!\!\!/\lambda$. Under U(1) transformations $\lambda \mapsto \exp(\mathrm{i}\Lambda(x))\lambda$ the variation does not vanish:

$$\delta_{\mathrm{U}(1)}\mathcal{L}_{\text{free}} = J_{\text{em}}^m \partial_m \Lambda(x)\,. \tag{4.4}$$

To compensate this contribution one has to couple a vector field to the current and impose a certain transformation property of this vector field:

$$\mathcal{L}_{\text{int}} = -eJ_{\text{em}}^m A_m\,, \quad \delta_{\mathrm{U}(1)}A_m = \frac{1}{e}\partial_m \Lambda\,. \tag{4.5}$$

We do the same in the Wess-Zumino model. Add

$$\mathcal{L}'_{\mathrm{WZ}} = -\kappa J_R^m \psi_{m,R} - \kappa J_L^m \psi_{m,L}\,, \quad \delta_\varepsilon \psi_m = \frac{1}{\kappa}\partial_m \varepsilon\,, \quad [\kappa] = \frac{1}{\text{mass}}\,, \quad [\kappa]^{(D)} = \left(\frac{1}{\text{mass}}\right)^{\frac{D-2}{2}}\,. \tag{4.6}$$

The field ψ has a vector index m and a spinor index α. $\psi_{m,\alpha}$ is the "**gauge field** of local SUSY". However, it turns out that this is not enough.

$$\delta_\varepsilon[\mathcal{L}_{\mathrm{WZ}} + \mathcal{L}'_{\mathrm{WZ}}] = -\kappa(\delta J_R^m)\psi_{m,R} - \kappa(\delta J_L^m)\psi_{m,L}\,, \tag{4.7}$$

with

$$\delta J_R^m = \delta(-\overline{\chi_R}\gamma^m \partial\!\!\!/\phi) = \overline{\varepsilon}\gamma_m \eta^{mn}\left[|\partial\phi|^2 + \overline{\chi_R}\partial\!\!\!/\chi_L + \ldots\right]\,. \tag{4.8}$$

One realizes that

$$\eta^{mn}\left[|\partial\phi|^2 + \overline{\chi_R}\partial\!\!\!/\chi_L + \ldots\right] = T^{mn}\,. \tag{4.9}$$

Hence, the result is

$$\delta_\varepsilon(\mathcal{L}_{\mathrm{WZ}} + \mathcal{L}'_{\mathrm{WZ}}) \simeq \kappa\bar{\varepsilon}\gamma_m\psi_n T^{mn} + \dots\,, \tag{4.10}$$

and we have to add a second term

$$\mathcal{L}''_{\mathrm{WZ}} \sim g_{mn}T^{mn}\,, \quad \delta_\varepsilon g_{mn} \sim \kappa\bar{\varepsilon}\gamma_{(m}\psi_{n)}\,. \tag{4.11}$$

g_{mn} is some tensor field, which turns out to be the space-time metric. This implies that local SUSY implies the coupling to gravity. (In the literature one sometimes finds another hand-waving argument. We look at the SUSY algebra:

$$\{Q, \overline{Q}\} \sim P_m\,. \tag{4.12}$$

If one tries to make the left-hand side local, one also has to consider local translations, which is equivalent to general coordinate transformations ($\to$ gravity).)

- ψ_m is the superpartner of the metric ("gravitino").
- $\kappa = 1/M_{\mathrm{pl}} = \sqrt{8\pi G} = (10^8\ \mathrm{GeV})^{-1}$.

These considerations tell us that there should be a **pure** $N = 1$ **supergravity** theory. Just consider (g_{mn}, ψ_m), no (ϕ, χ), ... We expect:

$$\mathcal{L}_{\mathrm{pure}} = \mathcal{L}_{\mathrm{kin}}[g_{mn}] + \mathcal{L}_{\mathrm{kin}}[\psi_m] + \mathcal{L}_{\mathrm{int}}[g_{mn}, \psi_m]\,, \tag{4.13}$$

with

$$\boxed{\mathcal{L}_{\mathrm{kin}}[g_{mn}] = -\frac{1}{2\kappa^2}\sqrt{g}R\,.} \tag{4.14}$$

$\mathcal{L}_{\mathrm{kin}}[\psi_m]$ should be the spacetime covariant version of

$$\boxed{\mathcal{L}_{\mathrm{Ravita-Schwinger}} = -\frac{1}{2}\overline{\psi}_m\gamma^{mn\varrho}\partial_n\psi_p\,, \quad \gamma^{mnp} = \gamma^{[m}\gamma^n\gamma^{p]}\,.} \tag{4.15}$$

For spin-3/2 terms this is the only kinetic term which is suitable without having ghosts. We know that the variations are given by

$$\delta_\varepsilon g_{mn} \sim \kappa\bar{\varepsilon}\gamma_{(m}\psi_{n)}\,, \quad \delta_\varepsilon\psi_m \sim \frac{1}{\kappa}\partial_m\varepsilon\Big|_{\mathrm{cov.}} + \dots\,. \tag{4.16}$$

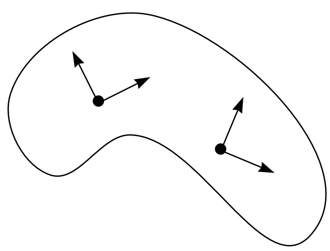
However, we have to know, how fermions are to be described in a curved space-time, since Eq. (4.15) comes from a Minkowski-like description.

4.1 Fermions in curved space-time

In Minkowski space-time vectors are described by vector representations of SO(3,1) and spinors by double-values spinor representations of SO(3,1). In curved space, vectors are described by

$$V^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\nu}V^\nu\,, \quad \frac{\partial x^{\mu'}}{\partial x^\nu} \in \mathrm{GL}(4,\mathbb{R}) \supset \mathrm{SO}(3,1)\,. \tag{4.17}$$

However, general linear groups do not have spinor representations. Hence, we choose a local inertial frame at each point. Instead of using tangent vector field ∂_μ we will use orthogonal fields (vierbein formalism). That is unique up to local SO(3,1).



$$e_m = e_m^\mu \partial_\mu \,. \tag{4.18}$$

Local SO(3,1) transformations act on the index m and general coordinate transformations on the index μ. One then writes

$$g_{\mu\nu} = e_\mu^m e_\nu^n \eta_{mn} \,, \tag{4.19}$$

where e_μ^m is the inverse of e_m^μ. The Christoffel symbols are given by

$$\Gamma_{\mu\nu}{}^{\varrho}(g) \mapsto \omega_\mu{}^m{}_n(e) \,, \tag{4.20}$$

where the right-hand side is called spin-connection. That **implements local SO(3,1) covariance**. The covariant derivative is now given by

$$\nabla_\mu \lambda = \left(\partial_\mu + \frac{1}{2}\omega_\mu{}^{mn}\Sigma_{mn}\right)\lambda \,, \quad \Sigma_{mn} \equiv \frac{1}{4}[\gamma_m, \gamma_n] = r(M_{mn}) \,. \tag{4.21}$$

Σ_{mn} is the generator for Lorentz transformations in the spinor representation. There is one subtlety for the gravitino. Its covariant derivative is given by

$$\nabla_\mu \psi_\nu = \partial_\mu \psi_\nu + \frac{1}{2}\omega_\mu{}^{mn}\Sigma_{mn}\psi_\nu - \Gamma_{\mu\nu}{}^{\varrho}(g)\psi_\varrho \,. \tag{4.22}$$

It is local Lorentz-invariant and also invariant under general coordinate transformations, which act on the vector-like index. $\nabla_{[\mu}\psi_{\nu]}$ implies $\Gamma_{[\mu\nu]}{}^{\varrho} = 0$. The curvature tensor is given by

$$R_{\mu\nu}{}^m{}_n(e) = \partial_\mu \omega_\nu{}^m{}_n + \omega_\mu{}^n{}_p \omega_\nu{}^p{}_m - (\mu \leftrightarrow \nu) \,. \tag{4.23}$$

Furthermore, the curvature scalar is

$$R(e) = R_{\mu\nu}{}^{mn} e_m^\nu e_n^\mu = R[g] \,, \tag{4.24}$$

and

$$\gamma_\mu = e_\mu^m \gamma_m \,, \quad \sqrt{g} = \sqrt{(\det e_\mu^m)^2} = \det(e_\mu^m = e \,. \tag{4.25}$$

Now, we can write up the Lagrangian of pure supergravity:

$$\mathcal{L}_{\text{pure SUGRA}} = -\frac{e}{2}M_{\text{pl}}^2 R(e) - \frac{e}{2}\overline{\psi}_\mu \gamma^{\mu\nu\varrho}\nabla_\nu \psi_\varrho + \mathcal{L}_{4-\text{fermion}} \,, \quad \mathcal{L}_{4-\text{fermion}} = \mathcal{O}\left(\frac{1}{M_{\text{pl}}^2}\overline{\psi}\psi\overline{\psi}\psi\right) \,, \tag{4.26}$$

$$\delta e_\mu^m = \frac{1}{2M_{\text{pl}}}\bar{\varepsilon}\gamma^m \psi_\mu \Rightarrow \delta g_{\mu\nu} = \frac{1}{M_{\text{pl}}}\bar{\varepsilon}\gamma_{(\mu}\gamma_{\nu)} \,, \tag{4.27}$$

$$\delta\psi_\mu = M_{\text{Pl}}\nabla_\mu \varepsilon + \text{3-fermion terms} \,, \quad \text{3-fermion terms} = \mathcal{O}\left(\frac{1}{M_{\text{pl}}}\psi\psi\varepsilon\right) \,. \tag{4.28}$$

The interaction described by the four-fermion-term is supressed by two powers of the Planck mass. Hence, it is not very important. SUSY invariance can be shown as follows:

$$\delta_\varepsilon\left(-\frac{e}{2}\overline{\psi}_\mu \gamma^{\mu\nu\varrho}\nabla_\nu \psi_\varrho\right) = -e\overline{\psi}_\mu \gamma^{\mu\nu\varrho}\nabla_{[\nu}\nabla_{\varrho]}\varepsilon \,, \quad \nabla_{[\nu}\nabla_{\varrho]} = \frac{1}{8}R_{\nu\varrho}{}^{mn}\gamma_{mn}\varepsilon \sim R\varepsilon\gamma\psi \sim R\delta_\varepsilon e_\mu^n = -\frac{\delta S_{\text{EH}}}{\delta e_\mu^m}\cdot\delta_\varepsilon e_\mu^m + \ldots \,. \tag{4.29}$$

The whole calculation is done in Freedman, de Wit [Lectures, NATO, ∼ 1985]. Unfortunately, it contains many typos. (The formulation via uperspace becomes very messy in SUGRA. That is why we are using the on-shell formulation here.)

4.2 Generalizations

i) Add a cosmological constant term $\Lambda < 0$ (AdS). A constant $\Lambda > 0$ (dS) is not possible in supergravity. For $\Lambda > 0$ one needs to add matter multiplets.

ii) Add matter multiplets.

The most general field content in $N = 1$, $D = 4$ supergravity is

$$[(e_\mu^m, \psi_\mu)\ \text{(SUGRA)}] \oplus [n_c \text{ chiral multiplets } (\phi^a, \chi^a),\ a = 1, \ldots, n_c] \oplus [n_v \text{vector multiplets } (A_\mu^I, \lambda^I),\ I = 1, \ldots, n_v] \,. \tag{4.30}$$

The Lagrangian is uniquely specified by the following data:

i) Kähler potential: It is a function of the scalar fields: $K(\phi^a, (\phi^a)^*)$ with $(\phi^a)^* \equiv \phi^{\bar{a}}$. That determines the kinetic terms of the chiral multiplets in a sense that it is of the form:

$$\mathcal{L}_{\text{kin}}[\phi^a, \chi^a] = -g_{a\bar{b}}(\phi, \phi^*) \left[\partial_\mu \phi^a \partial^\mu \phi^{\bar{b}} + \text{fermionic terms}\right] \,. \tag{4.31}$$

$g_{a\bar{b}}(\phi, \phi^*)$ is a metric on a scalar manifold (space of all fields). This metric is given in terms of the Kähler potential and is a nonlinear σ-model.

$$\boxed{g_{a\bar{b}} = \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^{\bar{b}}} \,.} \tag{4.32}$$

Hence, M_{scalar} is a "Kähler manifold". Kähler terms can destroy the renormalizability of the theory. However, this is not an issue here, because gravity is nevertheless not renormalizable.

ii) Superpotential $W(\phi^a)$ (holomorphic)

This leads to self-interaction of chiral multiplets (ϕ^a, χ^a) described by the F-term potential:

$$\boxed{V_F = \exp\left(\frac{K}{M_{\text{Pl}}^2}\right) \left[g^{a\bar{b}} \mathscr{D} W \mathscr{D}_{\bar{b}} W^* - \frac{3|W|^2}{M_{\text{pl}}^2}\right] \,, \quad \mathscr{D}_a W \equiv \partial_a W + \left(\frac{\partial_a K}{M_{\text{pl}}^2}\right) W \,.} \tag{4.33}$$

The derivative $\mathscr{D}$ is called the Kähler covariant derivative. Decoupling SUGRA (equivalent with taking the global SUSY limit) in the limit $M_{\text{pl}} \mapsto \infty$ one obtains:

$$\boxed{V_F = g^{a\bar{b}} \partial_a W \partial_{\bar{b}} W^* \geq 0 \,.} \tag{4.34}$$

iii) gauge kinetic function $f_{IJ}(\phi^a)$ (holomorphic)

$$\boxed{\mathcal{L}_{\text{kin}}[A_\mu^I, \lambda^I] = -\frac{1}{4} (\text{Re}(f_{IJ})) F_{\mu\nu}^I F \mu\nu, J + \text{fermionic terms} \,.} \tag{4.35}$$

One still has to define the gauge group G of the theory plus its action on (ϕ^a, χ^a). The gauge group has to act on the matter fields that sit in the chiral multiplets. Then, there is an additional contribution to the scalar potential, the so-called D-term potential:

$$V_D = \frac{1}{2} (\text{Re}(f_{IJ})) D^I D^J \,, \quad D^I \sim (\partial_a K) \delta_{\text{gauge},I} \phi^a \,. \tag{4.36}$$

We realize $V_D \geq 0$ even in supergravity. This does not hold for the F-term potential.

iv) possible Fayet-Iliopoulos constants ξ^I for Abelian factors of G

$$D^I \mapsto (D^I + \xi^I) \,, \tag{4.37}$$

which only works for those ξ^I which correspond to Abelian vector fields.

The important task in string theory is to derive the above date from **compactification data**.

iii) Extended supergravity: $N = 1 \mapsto N > 1$.

In 4D, for $N = 1$ you find $V = V_F + V_D$ with $V_D \neq 0$ only for gauge interactions. V_F is not related at all to gauge interactions. This splitting ends to exist in supergravity theories in the following sense. For $N > 1$, it holds that $V \neq 0$ if and only if there exist non-trivial **gauge interactions** ("gauged supergravity"). They are important for moduli stabilizations of AdS/CFT correspondence.

iv) Replace $D = 4$ by something $D \neq 4$.

How do we get spinor representations in arbitrary space-time dimensions? Look at the Clifford algebra:

$$\{\gamma_M, \gamma_N\} = 2\eta_{MN}\mathbb{1}_D , \tag{4.38}$$

with $(M, N) = 0, \ldots, D$. The Clifford algebra induces spinor representations of the corresponding Lorentz group. $\Sigma_{MN} \equiv 1/4[\gamma_M, \gamma_N]$ is a spinor representation of SO(1,$D-1$). But, Σ_{MN} is, in general, **reducible**. Therefore, one has to impose constraints on the spinors:

i) chirality (Weyl condition):

$$P_L\psi = 0 , \quad P_L = \frac{1}{2}\left(\mathbb{1} + \text{“}\gamma_5\text{”}\right) , \tag{4.39}$$

with the D-dimensional analogue of the γ_5 matrix: $\gamma_5 \sim \gamma_0 \ldots \gamma_{D-1}$. This is only possible for $D = 2n$, because otherwise γ_5 turns out to be proportional to the unit matrix.

ii) reality condition (Majorana):

$$\psi^* = B\psi , \quad \gamma_M^* = \eta B\gamma_M B^{-1} , \tag{4.40}$$

with a constant $\eta = \pm 1$. What η is, depends on the dimension. The general proof can be found in [Weinberg vol. 3, Ch. 32]. The Majorana condition should be self-consistent in the sense that $\phi^{**} = \psi$ and therefore, one needs $B^*B = \mathbb{1}$. However, it holds that $B^*B = \varepsilon\mathbb{1}$, where ϵ depends on the dimension D and η. The number of space-time components of spinors depends strongly on the dimension (look at the table). If scalars transform under the R-symmetry group, the R-symmetry group appears as a factor of the holonomy group of M_{scalar}, which is the group of all rotations one gets by parallel transporting. A large holonomy group strongly constraints a manifold. (In five dimensions: $N = 8$: USP(8) $\equiv$ SP(8, R) $\cap$ U(8), 42 real scalars $\in M_{\text{scalar}} \equiv E_{6(6)}$/USP(8))