Mitschrift zur Vorlesung: Gravitation and Cosmology

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Vorlesung Wintersemester 2006/2007

Letzte Aktualisierung und Verbesserung: February 27, 2010

Mitschrieb der Vorlesung Gravitation and Cosmologie
von Herrn Prof. Dr. Klinkhamer im Wintersemester 2006/2007
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

Contents

Chapter 1

Special Relativity

In this lecture we want to use natural units: $c = \hbar = 1$. Furthermore, let x^α (with $\alpha = 0, 1, 2, 3$) have the dimension length ($x^0 = ct = t$). Anyway, we make use of Einstein's summation convention.

1.1 Lorentztransformations and the concept of tensors

Einstein (1905) had two postulates:

1.) Principle of Relativity (holds between so-called inertial coordinate systems)

2.) Constancy of the velocity of light in vacuo

   The velocity of light is the same

These two postulates are grouped together in the Principle of Special Relativity. Specially, the Principle of Special Relativity says, that the laws of Nature are **invariant under Lorentztransformations**.
The **Lorentztransformation** between two spacetime cartesian coordinate systems S und S' is defined by the relation $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$ (1) with $\alpha = 1, 2, 3, 0$ and the **constants** $\Lambda^\alpha{}_\beta$ and a^α. Anyway, the property

$$\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta \eta_{\alpha\beta} = \eta_{\gamma\delta} = \mathrm{diag}(1, 1, 1, -1) \tag{1.1}$$

should hold. The first three components are the space component and the forth the time component. So the Lorentztransformation is a linear transformation between der coordinates of system S and S' with the above additional property. These matrices have the property, that they leave the **proper time** $\mathrm{d}\tau^2 = \mathrm{d}t^2 - |\mathrm{d}\mathbf{x}|^2$ **invariant**.

Proof:

It holds $\mathrm{d}x'^\alpha = \Lambda^\alpha{}_\delta \mathrm{d}x^\delta$.

$$\mathrm{d}\tau'^2 \equiv -\eta_{\alpha\beta}\,\mathrm{d}x'^\alpha\,\mathrm{d}x'^\beta = -\eta_{\alpha\beta}\Lambda^\alpha{}_\gamma \Lambda^\beta{}_\delta\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta = -\eta_{\gamma\delta}\,\mathrm{d}x^\gamma\,\mathrm{d}x^\delta = \mathrm{d}\tau^2 \tag{1.2}$$

This "explains" the experiment of Michselson and Morley of 1887. They found

$$\frac{c_\parallel - c_\perp}{c_\parallel + c_\perp} \lesssim 3 \cdot 10^{-10} \text{ whereas } \left|\frac{v_\text{earth}}{c}\right| \approx 10^{-4} \tag{1.3}$$

For a light wave with velocity $|\mathrm{d}\mathbf{x}/\mathrm{d}t| = 1$, it holds $\mathrm{d}\tau^2 = 0$ in the unprimed coordinate system (with the sun at rest). After a Lorentztransformation (to instantaneous moving frame of the earth) one finds $\mathrm{d}\tau'^2 = 0$ (because $\mathrm{d}\tau^2$ is invariant under Lorentztransformation), so also $|\mathrm{d}\mathbf{x}'/\mathrm{d}t'| = 1$. This is the mathematical embodyment, which describes the Michelson-Morley-experiment. For $\mathbf{v} = v\hat{e}_x$ (S' moves with velocity v with respect to system S) we have the following matrix:

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \text{ and } \Lambda^{-1} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma v & 0 & 0 & \gamma \end{pmatrix} \text{ with } \gamma \equiv \frac{1}{\sqrt{1 - v^2}} \tag{1.4}$$

$$\begin{pmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{pmatrix} = \Lambda \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \gamma(x_1 + vx_0) \\ x_2 \\ x_3 \\ \gamma(x_0 + vx_1) \end{pmatrix} \tag{1.5}$$

Furthermore, we want to show $\Lambda^\intercal \eta \Lambda = \eta$:

$$\begin{aligned} \Lambda^\intercal \eta \Lambda &= \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} = \\ &= \begin{pmatrix} \gamma & 0 & 0 & -\gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & -\gamma \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 & \gamma v \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma v & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \gamma^2(1-v^2) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \gamma^2(v^2-1) \end{pmatrix} = \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta \end{aligned} \tag{1.6}$$

1.1.1 Classification of transformations

1.) Poincare transformation (inhomogeneous Lorentz transformation): $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha$

2.) (homogeneous) Lorentz transformation: $x'^\alpha = \Lambda^\alpha{}_\beta x^\beta$ $(a^\alpha = 0)$

a.) proper Lorentz transformation: $\det \Lambda = 1$, $\Lambda^0{}_0 \geq 1$
These are transformations, that can continuously changed to the identity transformation. The number of generators is six. When we know the generators, we know the Lie group.

i.) pure rotations: $\Lambda^i{}_j = R_{ij}$ (3), $\Lambda^0{}_0 = 1$ and $\Lambda^i{}_0 = \Lambda^0{}_i = 0$
R is a unimodular, orthogonal 3×3 matrix. A rotation is characterized by three generators (e.g. rotation around three axes or three Eulerian angles). Invariance under rotations means, that the laws of nature are isotropic.

ii.) boosts:
We relate two coordinate systems with constant, uniform relative velocity $\mathbf{v}$ ($\widehat{v} \equiv \mathbf{v}/v$).

$$\Lambda^i{}_j = \delta_{ij} + \widehat{v}_i \widehat{v}_j (\gamma - 1) \tag{1.7}$$

$$\Lambda^j{}_0 = \Lambda^0{}_j = \gamma v_j \text{ with } \Lambda^0{}_0 = \gamma \equiv \frac{1}{\sqrt{1-v^2}} \tag{1.8}$$

The number of generators is three (three components of velocity $\mathbf{v}$ in system S).

b.) improper Lorentz transformation: $\det \Lambda = -1$

i.) space reflection: $\Lambda^0{}_0 \geq 1$

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.9}$$

ii.) time reversal: $\Lambda^0{}_0 \leq 1$

$$\mathcal{T} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.10}$$

1.1.2 Lorentz tensors

These are quantities, that transforms **homogeneously** under Lorentz transformations $x'^\alpha \mapsto \Lambda^\alpha{}_\beta x^\beta + a^\alpha$.

1.) Scalar: $S(x) \mapsto S'(x') = S(x)$
An example is the proper time $\mathrm{d}\tau^2 = -\eta_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta$.

2.) Contravariant four-vector: $V^\alpha(x) \mapsto V'^\alpha(x')^\alpha = \Lambda^\alpha{}_\beta V^\beta(x)$
An example of a contravariant four-vector is $\mathrm{d}x^\alpha$.

3.) Covariant four-vector: $W_\alpha(x) \mapsto W'_\alpha(x') = \Lambda_\alpha{}^\beta W_\beta$ with $\Lambda_\alpha{}^\beta \equiv \eta_{\alpha\gamma}\Lambda^\gamma{}_\delta\eta^{\delta\beta}$ and $\eta^{\beta\delta} \equiv \mathrm{diag}(1,1,1,-1)$
The index is downstairs, just to indicate, that it transforms differently. The matrix $\Lambda_\alpha{}^\beta$ is the inverse of the matrix $\Lambda^\alpha{}_\beta$. An example for a covariant four-vector is the partial differential $\partial/\partial x^\alpha$.

$V^\alpha W_\alpha$ is invariant. Indices can be raised and lowered with $\eta^{\alpha\beta}$ and $\eta_{\gamma\delta}$. New tensors (T) can be built from old tensors (R, S) in three ways.

1.) linear combination: $T^\alpha{}_\beta = cR^\alpha{}_\beta + dS^\alpha{}_\beta$

2.) direct product: $T^\alpha{}_\beta{}^\gamma{}_\delta = R^\alpha{}_\beta S^\gamma{}_\delta$

3.) contraction: $T = R^\alpha{}_\alpha$

There are three types of tensors, whose components are the **same** in **all** inertial coordinate systems.

1.) Minkowski tensor $\eta_{\alpha\beta}$
This only holds for **special** relativity.

2.) Levi-Civita (pseudo)-tensor $\varepsilon^{\alpha\beta\gamma\delta}$, which is completely asymmetric with $\varepsilon^{1230} = 1$.

3.) zero tensor $T_{\mu_1,\ldots,\mu_n} = 0$

With tensors it is easy to determine, whether or not an equation is lorentz invariant. Suppose in system C it holds $T^{\alpha\beta} = S^{\alpha\beta}$.

$$C \underset{\Lambda}{\longrightarrow} C': \qquad T'^{\alpha\beta} = \Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta T^{\gamma\delta} = \Lambda^\alpha{}_\gamma\Lambda^\beta{}_\delta S^{\gamma\delta} = S'^{\alpha\beta} \tag{1.11}$$

In particular, if $S^{\alpha\beta} = 0$, then equation $T^{\alpha\beta} = 0$ keeps the **same** form in **all** inertial coordinate systems.

1.2 Particle dynamics

Suppose, that in a rest frame the force $\mathbf{F}$ is known and that Newtonian dynamics hold. A relativistic equation for a particle with rest mass m and coordinate $x^\alpha(\tau)$ is given by:

$$m\frac{\mathrm{d}^2x^\alpha}{\mathrm{d}\tau^2} = f^\alpha \tag{1.12}$$

Here, f^α is the four-vector, which in the particle rest frame ($\mathrm{d}\tau = \mathrm{d}t$) is

$$f^\alpha_{\mathrm{rest}} = (\mathbf{F}, 0) \tag{1.13}$$

We have to check now, that equation (1.12) is really a tensor equation. In an arbitrary frame with instantaneous velocity $\mathbf{v}$ we have

$$f^\alpha = \Lambda^\alpha{}_\beta(\mathbf{v})f^\beta_{\mathrm{rest}} = (\mathbf{F} + (\gamma-1)\widehat{v}(\widehat{v}\cdot\mathbf{F}), \gamma\mathbf{v}\cdot\mathbf{F}) \tag{1.14}$$

Define **energy-momentum-four-vector**

$$p^\alpha \equiv m\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \tag{1.15}$$

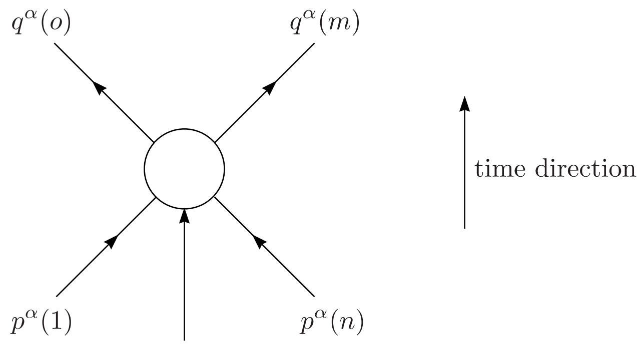
so that equation (1.12) reeds $\mathrm{d}p^\alpha/\mathrm{d}\tau = f^\alpha$ (5'). Using

$$\mathrm{d}\tau = \sqrt{\mathrm{d}t^2 - \mathrm{d}\mathbf{x}^2} = \sqrt{1-v^2}\,\mathrm{d}t \tag{1.16}$$

we get time dilatation $\mathrm{d}t = \mathrm{d}\tau/\sqrt{1-v^2} \geq \mathrm{d}\tau$ and can work the following out:

$$p^\alpha \equiv (\mathbf{p}, E) = (\gamma m\mathbf{v}, m\gamma) \tag{1.17}$$

If non-relativistic forces conserve energy and momentum separately, then relativistic energy-momentum is conserved.



$$\sum_{i=1}^{m} q^\alpha(i) - \sum_{j=1}^{n} p^\alpha(j) = 0 \tag{1.18}$$

Let us furthermore consider the mass-shell relation, which ist given by $E^2 = |\mathbf{p}|^2 + m^2$ (10). With $\mathbf{p} = \gamma m \mathbf{v}$ and $E = \gamma m$ ($\gamma = (1-v^2)^{-\frac{1}{2}}$) one obtains:

$$E^2 - |\mathbf{p}|^2 - m^2 = \gamma^2 m^2 - \gamma^2 m^2 v^2 - m^2 = \gamma^2 m^2 (1 - v^2) - m^2 = m^2 - m^2 = 0 \tag{1.19}$$

1.3 Electromagnetism

Charged particles, which have positions $\mathbf{x}_n(t)$ and charges e_n, generate a **current four-vector** ($x_n^0(t) = t$), which is the source for the fields.

$$J^\alpha(x) \equiv (\mathbf{J}(\mathbf{x},t), \varrho(\mathbf{x},t)) = \sum_n e_n \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t)) \frac{\mathrm{d}x_n^\alpha}{\mathrm{d}t} \tag{1.20}$$

Is $J^\alpha(x)$ a four-vector?

$$J^\alpha = \int \mathrm{d}t' \sum_n \varphi_n \delta^{(4)}(x - x_n(t')) \frac{\mathrm{d}x_n^\alpha(t')}{\mathrm{d}t'} = \sum_n \int \mathrm{d}t' \, \varphi_n \delta^{(4)}(x - x_n(t')) \frac{\mathrm{d}x_n^\alpha(t')}{\mathrm{d}t'} \tag{1.21}$$

Now we replace $\mathrm{d}t'$ by the proper time $\mathrm{d}\tau$ for each particle independently.

$$J^\alpha = \sum_n \int \mathrm{d}\tau \, e_n \delta^{(4)}(x - x_n(\tau)) \frac{\mathrm{d}x_n^\alpha(\tau)}{\mathrm{d}\tau} \tag{1.22}$$

$\mathrm{d}\tau$ and e_n are scalars and $\delta^{(4)}(x - x_n(\tau))$ is a pseudoscalar (like a scalar for proper Lorentz transformations with $\det(\Lambda) = 1$). Since $\mathrm{d}x_n^\alpha(\tau)/\mathrm{d}\tau$ is a four-vector, also J^α is a four-vector. **Current conservation** follows from the Lorentz invariant equation $\partial/\partial x^\alpha J^\alpha = 0$ (12). The Maxwell equations have the following form:

$$\frac{\partial}{\partial x^\alpha} F^{\alpha\beta} = -J^\beta \text{ and } \varepsilon^{\alpha\beta\gamma\delta} \frac{\partial}{\partial x^\beta} F_{\gamma\delta} = 0 \tag{1.23}$$

$F^{\alpha\beta}$ is the antisymmetric field strength tensor with $F^{12} = B_3$ and $F^{0m} = E_m$, so it holds $F^{\alpha\beta} = -F^{\beta\alpha}$. This tensor has $3 + 2 + 1 = 3 + 3$ independent components. So, that is why the field $\mathbf{E}$ and $\mathbf{B}$ (with $3 + 3 = 6$ components) merge together. The electromagnetic force four-vector on particles with charge is

$$f^\alpha = eF^\alpha{}_\beta \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \tag{1.24}$$

In the rest frame it holds $f^\alpha_{\text{rest}} = (e\mathbf{E}, 0)$. $e\mathbf{E}$ is the Coulomb force.

1.4 Energy momentum tensor

Let us consider density and currents not of electric charge e_n, but of energy momentum $p_n^\alpha(t)$.

a.) density: $T^{\alpha 0}(\mathbf{x},t) \equiv \sum_n p_n^\alpha(t) \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t))$

b.) current: $T^{\alpha i}(\mathbf{x},t) \equiv \sum_n p_n^\alpha(t) \frac{\mathrm{d}x^i}{\mathrm{d}t} \delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t))$

Together, they form a **symmetric energy-momentum tensor** $T^{\alpha\beta} = T^{\beta\alpha}$.

$$T^{\alpha\beta}_{\text{particle}}(x) \equiv \sum_n \int \mathrm{d}\tau\, p_n^\alpha \frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} \delta^{(4)}(x - x_n(\tau)) \tag{1.25}$$

For proper Lorentz transformations, this is a genuine tensor. Recall $p^\alpha = m\mathrm{d}x^\alpha/\mathrm{d}\tau$. Using the relativistic force law $\mathrm{d}p^\alpha/\mathrm{d}\tau = f^\alpha$ we get:

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = \sum_n \underbrace{\frac{\mathrm{d}\tau}{\mathrm{d}t} f_n^\alpha(t)\delta^{(3)}(\mathbf{x} - \mathbf{x}_n(t))}_{\text{“ density of force”}} \tag{1.26}$$

For **free** particles or **localized** interactions, we obtain the conservation laws

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta} = 0 \tag{1.27}$$

Equation (1.27) is not correct for **long range** interactions. It is possible to introduce **force fields**, so that **total** energy momentum tensor **is** conserved.

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta}_{\text{total}} = \frac{\partial}{\partial x^\beta} (T^{\alpha\beta}_{\text{particle}} + T^{\alpha\beta}_{\text{field}}) = 0 \tag{1.28}$$

Examples are the long ranged electromagnetic forces.

$$T^{\alpha\beta}_{\text{emfields}} \equiv F^{\alpha\gamma}\eta_{\gamma\delta}F^{\beta\delta} - \frac{1}{4}\eta^{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta} \qquad (26) \tag{1.29}$$

In components, we have:

$$T^{00}_{\text{em}} = \frac{1}{2}(|\mathbf{E}|^2 + |\mathbf{B}^2|) \text{ and } T^{i0}_{\text{em}} = (\mathbf{E} \times \mathbf{B})^i \tag{1.30}$$

T^{00}_{em} ist the density of the field energy. T^{i0}_{em} is the density of field momentum, where the Poynting vector is precicely $\mathbf{E} \times \mathbf{B}$.

Chapter 2

General relativity

2.1 Gravitational effects

2.1.1 Principle of Equivalence

In an arbitrary gravitational field, it is possible to choose locally inertial coordinate frames, so that in a sufficiently small region of space time all laws of nature take the same form as in the absence of gravity. That is, the equations of special relativity hold. This **postulate** is based on experimental results, for example $m_{\text{inert}} = m_{\text{grav}}$.
We consider a system with $N+1$ nonrelativistic particles, in which one particle interacts with the other N. (The forces $\mathbf{F}_{ij}$ are of course known.) In a static, homogeneous gravitational field with $\mathbf{g} = \text{const.}$ we have:

$$m_{\text{inert}} \frac{\mathrm{d}^2 \mathbf{x}_p}{\mathrm{d}t^2} = m_{\text{grav}} \mathbf{g} + \sum_{n=1}^{N} \mathbf{F}(\mathbf{x}_p - \mathbf{x}_n) \tag{2.1}$$

We know want to make a non-Galilean coordinate transformation, namely $\mathbf{x}' = \mathbf{x} - 1/2\mathbf{g}t^2$ with $t' = t$.

$$m_{\text{inert}} \frac{\mathrm{d}^2 \mathbf{x}'_p}{\mathrm{d}t'^2} = \sum_{n=1}^{N} \mathbf{F}(\mathbf{x}'_p - \mathbf{x}'_n) + (m_{\text{grav}} - m_{\text{intert}})\mathbf{g} \tag{2.2}$$

With $m_{\text{inert}} = m_{\text{grav}}$, observer O' sees the **same** physics as O, but **without** gravity. We now want to derive the equation of motion of a particle in an **external** gravitational field. In the freely falling coordinate system ξ^α, the equation of motion according the Einstein equivalence principle is the straight line

$$\frac{\mathrm{d}^2 \xi^\alpha}{\mathrm{d}\tau^2} = 0 \tag{2.3}$$

with proper time defined by

$$\mathrm{d}\tau^2 = -\eta_{\alpha\beta} \mathrm{d}\xi^\alpha \, \mathrm{d}\xi^\beta \tag{2.4}$$

In an arbitrary coordinate system $x^\mu = x'^\mu(\xi)$, $\xi^\alpha = \xi^\alpha(x)$.

$$0 = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\right) = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial^\mu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \tag{2.5}$$

Contract with $\partial x^\lambda / \partial \xi^\alpha$:

$$\boxed{0 = \frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}} \qquad (2.3)\ (2.6)$$

with the **affine connection**

$$\boxed{\Gamma^\lambda{}_{\mu\nu} \equiv \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}} \qquad (2.4)\ (2.7)$$

In the same way, $\mathrm{d}\tau^2 = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$ (2.5) with the **metric tensor**

$$g_{\mu\nu} \equiv \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} \qquad (2.7)\ (2.8)$$

2.2 Analogy between Yang-Mills theory and gravity

There an analogy between the fiber bundle connection $A^a_\mu(x)\,\mathrm{d}^\mu$ and the affine connection ${\Gamma^\lambda}_{\mu\nu}(x)$. **But** the affine connection ${\Gamma^\lambda}_{\mu\nu}$ is built up of something more fundamental ${\Gamma^\lambda}_{\mu\nu} = {\Gamma^\lambda}_{\mu\nu}[g]$. (Sometimes, the $g_{\mu\nu}$ are called "pre-potentials".)

$$\frac{\partial}{\partial x^\lambda} g_{\mu\nu} = \frac{\partial}{\partial x_\lambda}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}\right) = \frac{\partial^2 \xi^\alpha}{\partial x^\lambda \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\lambda \partial x^\nu}\eta_{\alpha\beta} =$$
$$= {\Gamma^\kappa}_{\lambda\mu}\frac{\partial \xi^\alpha}{\partial x^\kappa}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + {\Gamma^\kappa}_{\lambda\nu}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\kappa}\eta_{\alpha\beta} = {\Gamma^\kappa}_{\lambda\mu} g_{\kappa\nu} + {\Gamma^\kappa}_{\lambda\nu} g_{\kappa\mu} \tag{2.9}$$

$$\frac{\partial}{\partial x^\lambda} g_{\mu\nu} + \frac{\partial}{\partial x^\mu} g_{\lambda\nu} - \frac{\partial}{\partial x^\nu} g_{\mu\lambda} = ({\Gamma^\kappa}_{\lambda\mu} g_{\kappa\nu} + {\Gamma^\kappa}_{\lambda\nu} g_{\kappa\mu}) + ({\Gamma^\kappa}_{\mu\lambda} g_{\kappa\nu} + {\Gamma^\kappa}_{\mu\nu} g_{\kappa\lambda}) - ({\Gamma^\kappa}\quad g_{\kappa\lambda} + {\Gamma^\kappa}_{\nu\lambda} g_{\kappa\mu}) =$$
$$= 2{\Gamma^\kappa}_{\lambda\mu} g_{\kappa\nu} \tag{2.10}$$

Introduce the **inverse** metric $g^{\mu\nu}$ with the relation $g^{\mu\nu} g_{\nu\varrho} = {\delta^\mu}_\varrho$. Contract with $g^{\nu\sigma}$:

$${\Gamma^\sigma}_{\lambda\mu} = \frac{1}{2} g^{\sigma\nu}\left(\frac{\partial}{\partial x^\lambda} g_{\mu\nu} + \frac{\partial}{\partial x^\mu} g_{\nu\lambda} - \frac{\partial}{\partial x^\nu} g_{\lambda\mu}\right) \tag{2.8) (2.11}$$

If $g_{\mu\nu} = \eta_{\mu\nu}$ holds, there is no gravitation because of ${\Gamma^\sigma}_{\lambda\mu} = 0$. We also can get the equation with a variational principle. Let $x^\mu(s)$ be the path of a particle. Then the proper time between points A und B is given by:

$$\tau_{\mathrm{BA}} = \int_{s_A}^{s_B} \frac{\mathrm{d}\tau}{\mathrm{d}s}\,\mathrm{d}s = \int \sqrt{-g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}}\,\mathrm{d}s \tag{2.9) (2.12}$$

We want to consider variations $x^\mu(s) \mapsto x^\mu(s) + \delta x^\mu(s)$.

$$\mathrm{d}\tau_{\mathrm{BA}} = -\int_{\tau_A}^{\tau_B} \mathrm{d}\tau \left(\frac{\mathrm{d}^2 x^\nu}{\mathrm{d}\tau^2} + {\Gamma^\nu}_{\mu\sigma}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) g_{\lambda\nu}\delta x^\lambda \tag{2.13}$$

Hence, because of $\mathrm{d}\tau_{\mathrm{AB}} = 0$, the equation of motion (2.3) holds. (2.3) is called the **geodetic** equation.
For a non relativistic particle ($\mathrm{d}x/\mathrm{d}\tau \ll \mathrm{d}t/\mathrm{d}\tau$), the Newton equation of motion holds, namely

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}\phi_N \tag{2.14}$$

with $g_{00} = -(1+2\phi_N)$ upon identification. We consider a non relativistic particle in a weak static gravitational field. We get four equations of motion:

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}^2\tau^2} + {\Gamma^\mu}_{\sigma\sigma}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 0 \tag{2.15}$$

Because the gravitational field is static, all time derivatives vanish and we get:

$${\Gamma^\mu}_{00} = \frac{1}{2} g^{\mu\nu}(\partial_0 g_{0\nu} + \partial_0 g_{\nu 0} - \partial_\nu g_{00}) = -\frac{1}{2} g^{\mu\nu}\frac{\partial g_{00}}{\partial x^\nu} \tag{2.16}$$

We can consider small deviations $h_{\alpha\beta}$ from the flat metric $\eta_{\alpha\beta}$, that is $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ with $|h_{\alpha\beta}| \ll 1$. Furthermore $g^{\alpha\beta} = \eta_{\alpha\beta} - h_{\alpha\beta}$ holds. For $n = 1, 2, 3$ we obtain:

$$\frac{\mathrm{d}^2 x^m}{\mathrm{d}\tau^2} = \frac{1}{2}\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 \frac{\partial}{\partial x^m} h_{00},\; \frac{\mathrm{d}^2 t}{\mathrm{d}\tau^2} = 0 \Rightarrow \frac{\mathrm{d}t}{\mathrm{d}\tau} = \mathrm{const.} \tag{2.17}$$

Three equations:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = \frac{1}{2}\boldsymbol{\nabla} h_{00} \tag{2.18}$$

We recall the Newton equation of motion:

$$\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = -\boldsymbol{\nabla}\phi_N \Rightarrow g_{00} = \eta_{00} + h_{00} = -1 - 2\phi_N \tag{2.19}$$

For a Newton point mass $\phi_N = -GM/r$ holds (with $\phi(\infty) = 0$ and at surface of the sun $\phi_N = -2 \cdot 10^{-6}$). The prediction from the equivalence principle is the gravitational red shift. We consider a standard clock with arbitrary velocity in an arbitrary gravitational field. In a local inertial coordinate system $\{\xi^\mu\}$ the space-time interval between ticks $\sqrt{-\eta_{\alpha\beta}\xi^\alpha\xi^\beta}$ equals the time interval Δt for a clock at rest and absence of gravitational fields. In an arbitrary coordinate system $x^\mu = x^\mu(\xi)$, we get:

$$\Delta t = \sqrt{-g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu} \tag{2.20}$$

For a clock a rest ($\mathrm{d}x^\mu = 0$), the interval between the ticks is

$$\mathrm{d}t = \frac{\Delta t}{\sqrt{-g_{00}}} \qquad (2.12) \tag{2.21}$$

This can not be observed: all clock run slower (faster) including the one for the standard time.
In a **stationary** gravitational field a **different** locations ① and ②, we consider the **same** atomic transition, so that we have $\Delta t_1 = \Delta t_2 := \Delta t$. Observer ① measures $\mathrm{d}t_2$ and compares with his Δt interval $\mathrm{d}t_1$ from his **own** source.

$$\frac{\mathrm{d}t_1}{\mathrm{d}t_2} = \frac{\frac{\Delta t}{\sqrt{-g_{00}(x_1)}}}{\frac{\Delta t}{\sqrt{-g_{00}(x_2)}}} = \sqrt{\frac{g_{00}(x_2)}{g_{00}(x_1)}} \qquad (2.13a) \tag{2.22}$$

Write $v_2 \equiv v + \Delta v$ and $v_1 \equiv v$. Then, for weak fields $g_{00} = -1 - 2\phi_N$, it holds:

$$\frac{\Delta v}{v} = \phi(x_2) - \phi(x_1) \equiv \Delta\phi \tag{2.23}$$

Light, which travels out of a potential wall at the sun surface, is red shifted, when seen on earth.

2.3 Tensors and curvature

The equivalence principle says, that a physical equation holds in a general gravitational field, if

1.) the equation is **generally covariant** (form is preserved under general coordinate transformations $x^\mu \mapsto x'^\mu$) or

2.) the equation holds in the **absence** of gravitation (e.g. for $g_{\mu\nu} = \eta_{\mu\nu}$ and $\Gamma^\lambda{}_{\mu\nu} = 0$ locally), that is, special relativity holds.

Such physical equations can be built out of **tensor** objects, which transform **homogeneously** under general coordinate transformations $x'^\mu = x'\mu(x)$.

$$T'^{\mu\ldots}{}_{\nu\ldots} = \frac{\partial x'^\mu}{\partial x^\kappa}\cdots\frac{\partial x^\lambda}{\partial x'^\nu}\cdots T^{\kappa\ldots}{}_{\lambda\ldots} \tag{2.24}$$

2.4 Riemannian curvature tensor, Ricci tensor, and Ricci scalar

We define the Riemannian curvature tensor, the Ricci tensor and the Ricci scalar as follows:

$$R^\lambda{}_{\mu\nu\varrho} \equiv \frac{\partial}{\partial x^\varrho}\Gamma^\lambda{}_{\mu\nu} - \frac{\partial}{\partial x^\nu}\Gamma^\lambda{}_{\mu\varrho} + \Gamma^\sigma{}_{\mu\nu}\Gamma^\lambda{}_{\varrho\sigma} - \Gamma^\sigma{}_{\mu\varrho}\Gamma^\lambda{}_{\nu\sigma} \tag{2.25}$$

$$R_{\mu\varrho} \equiv R^\lambda{}_{\mu\lambda\varrho},\; R \equiv g^{\mu\varrho}R_{\mu\varrho} \tag{2.26}$$

We want to write the Riemannian curvature tensor with the metric $g_{\mu\nu}$:

$$R_{\lambda\mu\nu\varrho} = \frac{1}{2}\left[\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\varrho} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\varrho} - \frac{\partial^2 g_{\lambda\varrho}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\varrho}}{\partial x^\lambda \partial x^\nu} + g_{\kappa\sigma}\left(\Gamma^\kappa{}_{\nu\lambda}\Gamma^\sigma{}_{\mu\varrho} - \Gamma^\kappa{}_{\varrho\lambda}\Gamma^\sigma{}_{\mu\nu}\right)\right] \tag{2.27}$$

This tensor has many symmetries in it:

a.) symmetry: $R_{\lambda\mu\nu\varrho} = R_{\nu\varrho\lambda\mu}$

b.) antisymmetry: $R_{\lambda\mu\nu\varrho} = -R_{\mu\lambda\nu\varrho} = -R_{\lambda\mu\varrho\nu}$

c.) cyclicity: $R_{\lambda\mu\nu\varrho} + R_{\lambda\varrho\mu\nu} + R_{\lambda\nu\varrho\mu}$

These properties can be checked by putting the Riemannian tensor in. From this follows, that the Ricci tensor is **symmetric**.

$$R_{\mu\varrho} \equiv g^{\lambda\kappa} R_{\kappa\mu\lambda\varrho} = g^{\kappa\lambda} R_{\lambda\varrho\kappa\mu} = R_{\varrho\mu} \tag{2.28}$$

$$R_{\mu\varrho} = R^{\nu}{}_{\mu\nu\varrho} = g^{\lambda\nu} R_{\lambda\mu\nu\varrho} = -g^{\lambda\nu} R_{\mu\lambda\nu\varrho} = -g^{\lambda\nu} R_{\nu\varrho\mu\lambda} = -R^{\lambda}{}_{\varrho\mu\lambda} \tag{2.29}$$

The same you can show, that the Ricci scalar is unique up to signs.

2.4.1 A potential pseudo scalar

$$\frac{1}{\sqrt{g}} \varepsilon^{\lambda\mu\nu\varrho} R_{\lambda\mu\nu\varrho} = 0 \text{ with } g = -\det(g_{\mu\nu}) \tag{2.30}$$

This is zero because of the cyclicity of the Riemannian tensor and the antisymmetrie of the four dimensional Levi-Civita tensor. We want to consider now a pseudo tensor τ, which is called density:

$$\tau' = \left|\frac{\partial x'}{\partial x}\right|^w \left(\frac{\partial x}{\partial x'}\right) \dots \left(\frac{\partial x'}{\partial x}\right) \dots \tau \tag{2.31}$$

$|\partial x'/\partial x|$ ist the Jacobian determinant and w the so-called weight. So tensors are densities of weight $w = 0$. Let us look at some examples:

1.) Metric determinant:

$$g' = \left|\frac{\partial x}{\partial x'}\right|^2 g \Rightarrow w(g) = -2 \tag{2.32}$$

2.) The Levi-Civita symbol is a density of weight $w(\varepsilon) = -1$.

So (2.30) has weight zero, so it's a tensor, as a matter of fact a pseudo tensor, because under reflections it changes sign.
Until now we have been discussing identities via **algebra**, but there can also be **differential** identities, so-called **Bianchi identity**. Identities hold generally, as the following (where $;\nu$ means the covariant derivative with respect to ν):

$$R_{\lambda\mu\nu\varrho;\sigma} + R_{\lambda\mu\sigma\nu;\varrho} + R_{\lambda\mu\varrho\sigma;\nu} = 0 \tag{2.33}$$

Contract twice (first λ_ν, then $\mu\varrho$):

$$\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0 \tag{2.34}$$

From the Bianchi identity we get:

$$R_{\lambda\mu\nu\varrho;\sigma} - R_{\lambda\mu\nu\sigma;\varrho} + R_{\lambda\mu\varrho\sigma;\nu} = 0 \tag{2.35}$$

We contract λ and ν, which results in

$$R_{\mu\varrho;\sigma} - R_{\mu\sigma;\varrho} + R^{\lambda}{}_{\mu\varrho\sigma;\lambda} = 0 \Rightarrow R_{;\sigma} - R^{\mu}{}_{;\mu} - R^{\lambda}{}_{\sigma;\lambda} = 0 \tag{2.36}$$

$$g^{\mu\varrho} R_{\mu\varrho;\sigma} = (g^{\mu\varrho} R_{\mu\varrho})_{;\sigma} = R_{;\sigma} \tag{2.37}$$

The covariant derivative acts precisely as the normal derivative. But only the covariant derivative of the metric is zero and then we can pull the metric tensor in.

$$R_{;\sigma} - 2R^{\mu}{}_{\sigma;\mu} = 0 \Rightarrow \left(R^{\mu}{}_{\sigma} - \frac{1}{2}\delta^{\mu}{}_{\sigma} R\right)_{;\mu} = 0 \tag{2.38}$$

Finally, we contract this with the inverse metric:

$$\left(g^{\nu\sigma} R^{\mu}{}_{\sigma} - \frac{1}{2} g^{\mu\nu} R\right)_{;\mu} = 0 \tag{2.39}$$

2.5 Parallel transport

We want to consider a **vector** defined over a **curve** $x^\mu(\tau)$. One example is the spin vector $S_\mu(\tau)$ of a particle. The covariant transformation is

$$S'_\mu(\tau) = \frac{\partial x^\nu}{\partial x'^\mu} S_\nu(\tau) \tag{2.40}$$

$\mathrm{d}/\mathrm{d}\tau$ gives for the right-hand-side of this equation:

$$\frac{\partial x^\nu}{\partial x'^\mu}\frac{\mathrm{d}}{\mathrm{d}\tau}S_\nu + \frac{\mathrm{d}x'^\kappa}{\mathrm{d}\tau}\frac{\partial^2 x^\nu}{\partial x'^\kappa x'^\mu}S_\nu \tag{2.41}$$

So the spin vector transforms like this along the path $x^\mu(\tau)$. Then, we can define the **covariant derivative** $\mathcal{D}$ along the path $x^\mu(\tau)$:

$$\frac{\mathcal{D}S_\mu}{\mathcal{D}\tau} \equiv \frac{\mathrm{d}S_\mu}{\mathrm{d}\tau} - \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}S_\lambda \tag{2.42}$$

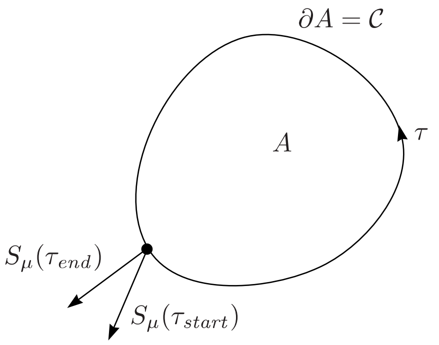
In the absence of other forces (non-gravitational forces), we have in the local inertial coordinate system ($\Gamma^\lambda{}_{\mu\nu} = 0$ and $\mathrm{d}S_\mu/\mathrm{d}\tau = 0$), we obtain

$$\frac{\mathcal{D}S_\mu}{\mathcal{D}\tau} = 0 \tag{2.43}$$

Since this equation has a covariant form, it also holds in an **arbitrary** coordinate system. We say, that **any** vector S_μ is **parallel transported**, if it obeys the equation

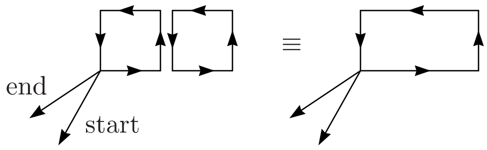
$$\frac{\mathrm{d}S_\mu}{\mathrm{d}\tau} = \Gamma^\lambda{}_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}S_\lambda \tag{2.44}$$

What happens, if S_μ is parallel transported along a **closed** curve $\mathcal{C}$: $\tau \in [\tau_{\text{start}}, \tau_{\text{end}}]$ and $x^\mu(\tau_{\text{start}}) = x^\mu(\tau_{\text{end}}) \equiv x^\mu$. Be $\Delta_{X,\mathcal{C}}S_\mu \equiv \Delta S_\mu \equiv S_\mu(\tau_{\text{end}}) - S_\mu(\tau_{\text{start}})$.



Furthermore, let $\mathcal{C}$ border an area A. What we now want to do, is, to break up A in very many (N) very small "tiles" δa:

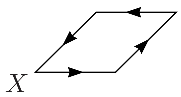
$$A = \sum_{n=1}^{N}(\delta a)_n \tag{2.45}$$



So we have

$$\Delta_{X,\mathcal{C}}S_\mu = \sum_{n=1}^{N}\Delta_n S_\mu \tag{2.46}$$

We need the change of S_μ for a parallel transport along an infinitesimal loops $\delta\mathcal{C}$.



Taylor expand ${\Gamma^\lambda}_{\mu\nu}(x)$ around X^μ.

$$\Delta_{X,\delta c} S_\mu = \frac{1}{2} {R^\lambda}_{\mu\nu\varrho}(X) S_\lambda(X) \oint_{\delta c} x^\varrho \, \mathrm{d}x^\nu \tag{2.47}$$

More precisely:

$$\oint \equiv \int_{\tau_\text{start}}^{\tau_\text{end}} \mathrm{d}\tau \, x^\varrho(\tau) \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \, \mathrm{d}\tau \tag{2.48}$$

If and only if the Riemann tensor vanishes at the space-time point X, for any arbitrary small loop δc the value $\Delta_{X,\delta c} S_\mu$ is zero. This is independent of the coordinates; it describes the geometrical structure of space-time. A non-vanishing Riemann tensor indicates a **genuine** gravitational field, instead of merely **exotic** coordinates.

Theorem:

For a metric $g_{\mu\nu}(x)$ to be equivalent to the constant Minkowski metric $\eta_{\mu\nu}$ (for example there being **global** coordinates $\xi^\alpha(x)$, so that $g^{\mu\nu}(\partial\xi^\alpha/\partial x^\mu)(\partial\xi^\beta/\partial x^\nu) = \eta^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$), the necessary and sufficient conditions are

1.) ${R^\lambda}_{\mu\nu\kappa}[g_{\mu\nu}(x)] = 0$ everywhere

2.) at some point X^μ, the metric $g_{\mu\nu}(X)$ has three positive and one negative eigenvalues.

2.6 Recap of the energy-momentum tensor

In an **isolated** system, in special relativity we have conservation equations $\partial T^{\alpha\beta}/\partial x^\alpha = 0$. Introduce a contravariant tensor $T^{\mu\nu}$ (in local inertial coordinate system reducing to a special relativity object). In a given gravitational field we have the following equation:

$$T^{\mu\nu}{}_{;\mu} = 0 \tag{2.49}$$

With some effort, one can rewrite this equation as

$$\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^\mu} (\sqrt{g} T^{\mu\nu}) = {\Gamma^\nu}_{\varrho\sigma} T^{\varrho\sigma} \text{ with } g \equiv -\det(g_{\mu\nu}) \tag{2.50}$$

For special relativity, we have point particles:

$$T^{\alpha\beta}(x) = \sum_n m_n \int \frac{\mathrm{d}x_n^\alpha}{\mathrm{d}\tau} \mathrm{d}x^\beta \delta^{(4)}(x - x_n) \tag{2.51}$$

Contravariant tensor in general relativity:

$$T^{\mu\nu}(x) = \frac{1}{\sqrt{g}} \sum_n m_n \int \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau} \mathrm{d}x_n^\nu \delta^{(4)}(x - x_n) \tag{2.52}$$

$1/\sqrt{g}$ are scalar densities, m_n are scalars, $\delta^{(4)}(x - x_n)$ is a scalar density and $(\mathrm{d}x_n^\mu/\mathrm{d}\tau)\mathrm{d}x_n^\nu$ is a tensor. The scalar volume element ist $\sqrt{g}\,\mathrm{d}^4x$ and the scalar δ-function $1/\sqrt{g}\delta^{(4)}$. By integration we get:

$$\int T^{\mu 0} \sqrt{g} \, \mathrm{d}^3x = \sum_n m_n \frac{\mathrm{d}x_n^\mu}{\mathrm{d}\tau} \tag{2.53}$$

This suggests a “total energy-momentum”.

$$\text{“}P^\mu\text{”} \equiv \int T^{\mu\nu} \sqrt{g} \, \mathrm{d}^3x \tag{2.54}$$

But in general, this “P^μ” is not a contravariant vector and worse, it is also **not** conserved:

$$\left(\frac{\partial}{\partial x^\nu} \sqrt{g} T^{\mu\nu} \right) = \sqrt{g} {\Gamma^\mu}_{\varrho\sigma} T^{\varrho\sigma} \neq 0 \tag{2.55}$$

2.7 Gravitational field equations

2.7.1 Einstein equations

How are gravitational fields generated by matter distribution? Since gravitational fields carry energy-momentum, we obtain **nonlinear** partial differential equations. On the other hand, the Maxwell equations are linear, because electromagnetic fields carry no electric charge. The "derivation" of the Einstein equations is guided by two requirements:

1.) Principle of equivalence (in local inertial coordinate system at point X: $g_{\alpha\beta}(X) = \eta_{\alpha\beta}$ and first derivatives vanishing, what means, that the Christoffel symbols are zero)

2.) Newtonian limit

We will start from the Poisson equation for a scalar potential ϕ:

$$\boldsymbol{\nabla}^2\phi = 4\pi G\varrho \text{ with } g_{00} \simeq -1 - 2\phi \text{ and } T_{00} = \varrho \tag{2.56}$$

With these definitions we can write the Poisson equation as

$$\boldsymbol{\nabla}^2 g_{00} = -8\pi G T_{00} \tag{2.57}$$

which holds for weak, static gravitational fields.

Let us now make the *Ansatz*

$$G_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{2.58}$$

with the energy-momentum **tensor** $T_{\mu\nu}$ and a tensor $G_{\mu\nu}$, which is linear in the second derivative of the metric $g_{\mu\nu}$ and quadratic in the first derivative:

$$G_{\mu\nu} \approx \frac{\partial g_{\mu\nu}}{\partial x}\frac{\partial g}{\partial x} + \frac{\partial^2 g}{\partial x^2} \text{ with } [G_{\mu\nu}] = \text{length}^2 \tag{2.59}$$

1.) $T_{\mu\nu}$ is symmetric. That means, that $G_{\mu\nu}$ is also symmetric: $G_{\mu\nu} = G_{\nu\mu}$.

2.) $T_{\mu\nu}$ is conserved, so $G^{\mu}{}_{\nu;\mu} = 0$ must hold.

3.) $G_{00} \simeq \boldsymbol{\nabla}^2 g_{00}$ for weak, static fields

Only the Riemannian tensor is available and its contractions $R_{\mu\nu}$ and R, so we write:

$$G_{\mu\nu} = C_1 R_{\mu\nu} + C_2 R g_{\mu\nu} \tag{2.60}$$

The Bianchi identity gives the relation $C_2/C_1 = -1/2$. Then we get:

$$G_{\mu\nu} = C\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = -8\pi G T_{\mu\nu} \tag{2.61}$$

We consider a nonrelativistic system with $|T_{ij}| \ll |T_{00}|$. (The space components are much less then the rest mass of the particles.) So we get $R_{ij} \sim 1/2 g_{ij} R$. Because of $g_{\mu\nu} \sim \eta_{\mu\nu}$ it follows $R \sim R_{ii} - R_{00}$. With $R_{ij} \sim 1/2 g_{ij} R$ we obtain further:

$$R \sim \frac{3}{2}R - R_{00} \Rightarrow R \sim 2R_{00} \Rightarrow G_{00} \sim 2CR_{00} \equiv 2C(R_{0l0l} - R_{0000}) \tag{2.62}$$

For static, weak gravitational fields in the local inertial coordinate system the Riemannian tensor has the following form:

$$R_{\lambda\mu\nu\varrho} \sim \frac{1}{2}\left(\frac{\partial^2 g_{\lambda\nu}}{\partial x^\mu \partial x^\varrho} - \frac{\partial^2 g_{\mu\nu}}{\partial x^\lambda \partial x^\varrho} - \frac{\partial^2 g_{\lambda\varrho}}{\partial x^\mu \partial x^\nu} + \frac{\partial^2 g_{\mu\varrho}}{\partial x^\lambda \partial x^\nu}\right) \Rightarrow R_{0000} \sim 0 \tag{2.63}$$

$$R_{l0n0} \sim \frac{1}{2}\frac{\partial^2 g_{00}}{\partial x^l \partial x^n} \tag{2.64}$$

So, $G_{00} \sim C\boldsymbol{\nabla}^2 g_{00}$ holds for this special case. In order to recover the Newtonian limit, we have to set $C = 1$. Now, we can write up the Einstein field equations:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G T_{\mu\nu} \tag{2.65}$$

1.) In vacuum (outside every "source"), one haves $R_{\mu\nu} - 1/2 g_{\mu\nu} R = 0$. By contraction with $g^{\mu\nu}$, we obtain $R - 2R = 0$, so $R = 0$. The Einstein field equation reduces to $R_{\mu\nu}|_{\text{vac}} = 0$. One $1+1$ or $2+1$ dimensions, that would mean $R_{\lambda\mu\nu\varrho} = 0$. However, in $3+1$ dimensions (ore more) we do have genuine gravitational fields. Contraction with $g^{\mu\nu}$ on (2.65) delivers:

$$R_2 R = -8\pi G T^{\mu}{}_{\mu} \Rightarrow R = 8\pi G T^{\lambda}{}_{\lambda} \tag{2.66}$$

$$\Rightarrow R_{\mu\nu} = -8\pi G \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T^{\lambda}{}_{\lambda} \right) \tag{2.67}$$

2.) We consider another *Ansatz* with also a possible $g_{\mu\nu}$ term in $G_{\mu\nu}$:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{2.68}$$

Λ is called the cosmological constant.

2.7.2 "Gauge fixing" (for Maxwell and Einstein)

As some kind of warm-up, we want to consider the Maxwell equations for a vector potential A_α:

$$\Box A_\alpha - \partial_\alpha \partial_\beta A^\beta = -J_\alpha \text{ with } \Box = \eta^{\alpha\beta} \partial_\alpha \partial_\beta \equiv \boldsymbol{\nabla}^2 - \partial_t^2 \tag{2.69}$$

These are four independent equations. But A_α is **not** determined **uniquely** by (2.69), because the following **identity** holds for any A_α without using the Maxwell equations:

$$\partial_\alpha (\Box A^\alpha - \partial^\alpha \partial_\beta A^\beta) = 0 \tag{2.70}$$

With current conservation ($\partial_\alpha J^\alpha = 0$) the effective number of equations is $4 - 1 = 3$. Hence, one degree of freedom in A_α remains undetermined and this corresponds to the gauge invariance of the theory, namely

$$A'_\alpha(x) = A_\alpha(x) + \partial_\alpha \Lambda(x) \tag{2.71}$$

for arbitrary $\Lambda(x)$. In other words, both $\overline{A_\alpha}(x)$ and $\overline{A_\alpha}(x) + \partial_\alpha \overline{\Lambda}(x)$ can be solutions of (2.37), because $\Box(\partial_\Lambda) - \partial_\alpha \partial_\beta (\partial^\beta \Lambda)$ vanishes identically. So, with one additional condition, we have four equations for four unknowns, namely the so-called gauge fixing condition. For example, the **Lorentz gauge** says $\partial_\alpha A'^\alpha = 0$ (2.39), which is reached by a particular gauge transformation $A'_\alpha = A_\alpha + \partial_\alpha \Lambda_L$ with $\Box \Lambda_L = -\partial_\alpha A^\alpha$. Then, the equations are (2.39) and $\Box A'_\alpha = -J_\alpha$ (2.37').

Now let us come to general relativity. There are ten Einstein equations and four Bianchi identities. So, effectively you have six equations. The metric $g_{\mu\nu}(x)$ is determined by Einstein equations only **up to arbitrary coordinate transformations** x'^μ. There are four degrees of freedom undetermined. In order to fix the gauge, **four** coordinate conditions are needed.

2.7.3 Example: Harmonic coordinate conditions

One gauge, which has certain advantages, is the harmonic gauge. The so-called **harmonic coordinate conditions** are given by $\Gamma'^\lambda \equiv \Gamma'^\lambda{}_{\mu\nu} g'^{\mu\nu} = 0$. A function f is called harmonic, if $\Box f \equiv (g^{\mu\nu} f_{;\mu})_{;\nu} = 0$, or equivalently

$$\Box f = g^{\mu\nu} \partial_\mu \partial_\nu f - \Gamma^\lambda \partial_\lambda f = 0 \tag{2.72}$$

If $\Gamma^\lambda = 0$, then $\Box x^\mu = 0$ holds, so the coordinates are harmonic functions. Without gravitational fields, the harmonic coordinate system can be the Minkowski coordinate system with $g_{\mu\nu}(x) = \eta_{\mu\nu}$ ($\Gamma^\lambda = 0$). For **weak** gravitational fields, the harmonic coordinate system is nearly Minkowskian.

2.8 Classic tests for general relativity

There are three classic tests for this theory:

1.) deflection of light by the sun

solar eclipse: look six months earlier and measure angle

General relativity predicts $\Delta\varphi = 4GM_\odot / R_\odot = 1,75\,\text{arcsec}$.

Eclipse May 29th, 1919	$1,91'' \pm 0,12''$
	$1,61'' \pm 0,31''$
Quasar 3C279	$1,77''$

2.) Advance of the perihilium of Mercury

According to general relativity, Einstein found:

$$\Delta\varphi_{\text{Mercury}} = \frac{6\pi G M_{\odot}}{(1-e^2)a} = 43,04\,\frac{\text{arcsec}}{\text{century}} \tag{2.73}$$

It moves in the direction of the motion. Astronomy gives the value $43,11 \pm 0,45$.

3.) Radar time delay:

General relativity predicts $\gamma \equiv 1$. In the Viking mission the value $\gamma = 1,006 \pm 0,002$ was measured.

2.9 The Schwarzschild solution

We want to consider a static isotropic gravitational field. For appropriate coordinates $\mathbf{x}$ and $x^0 \equiv t$, we have $\mathrm{d}\tau^2 = -g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$, which is independent of t and only dependent on rotational invariants $\mathbf{x}\cdot\mathbf{x}$, $\mathbf{x}\cdot\mathrm{d}\mathbf{x}$ and $\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x}$. We use spherical coordinates for $\mathbf{x}$, namely r, ϑ, and φ. With redefinitions of t and r, the proper time can be brought in the **standard** form:

$$\mathrm{d}\tau^2 = B(r)\,\mathrm{d}t^2 - A(r)\,\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{2.74}$$

Hence, the metric (-,+,+,+) has components

$$g_{\mu\nu} = \mathrm{diag}(-B(r), A(r), r^2, r^2\sin^2\vartheta) \tag{2.75}$$

The inverse metric is given by

$$g^{\mu\nu} = \mathrm{diag}\left(-\frac{1}{B(r)}, \frac{1}{A(r)}, \frac{1}{r^2}, \frac{1}{r^2\sin^2\vartheta}\right) \tag{2.76}$$

It is straightforward, to calculate the Christoffel symbols and the Ricci tensor. We want to determine the vacuum solution generated by a **point mass *M***. We use (2.75) as *Ansatz* for the field equations. They are $R_{\mu\nu} = 0$ (2.43a) with boundary conditions $\lim_{r\mapsto\infty} A(r) = 1$ and $\lim_{r\mapsto\infty} B = 1$ (2.43b). From the rr- and tt-equations one gets

$$0 = \frac{R_{rr}}{A(r)} + \frac{R_{tt}}{B(r)} = -\frac{1}{rA(r)}\left(\frac{A'(r)}{A(r)} + \frac{B'(r)}{B(r)}\right) \Rightarrow A'(r)B + B'(r)A = 0 \Rightarrow A\cdot B = 1 \tag{2.77}$$

From the $\vartheta\vartheta$-equation (using $A = 1/B$) one gets:

$$0 = R_{\vartheta\vartheta} = -1 + rB'(r) + B(r) = -1 + (rB)' \Rightarrow \frac{\mathrm{d}}{\mathrm{d}r}(rB) = 1 \tag{2.78}$$

So the solution is $rB = r + \text{const.}$ or $B = 1 + \text{const.}/r$, respectively. The Newtonian approximation for $r \mapsto \infty$ is given by $g_{tt} \equiv -B$:

$$-1 - 2\phi_N = -1 + \frac{2GM}{r} \tag{2.79}$$

Therefore, the constant is $-2GM$ and the Schwarzschild solution (1916) is given by

$$\mathrm{d}\tau^2 \equiv -g^{\text{Schw}}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \tag{2.80}$$

Note, that the metric in this form is singular at the **Schwarzschild radius**

$$R_{\text{Schw}} \equiv 2GM = 3\,\text{km}\left(\frac{M}{M_{\odot}}\right) \text{ where } M_{\odot} = 2\cdot 10^{33}\,\text{g} \tag{2.81}$$

The sun has actually $R_{\odot} = 7\cdot 10^5\,\text{km}$. The Schwarzschild metric is only relevant for $r > R_{\odot}$. Birkhoff's theorem says, that outside a spherical body the metric is the same as for a point mass equal to the total mass. For a **very** compact object, this Schwarzschild singularity seems to be a problem. However, this is not so! Local physical quantities are well behaved at $r = R_{\text{Schw}}$.

$$R_{\vartheta\vartheta} = -1 + \frac{1}{A(r)} + \frac{r}{2A(r)}\left(\frac{B'(r)}{B(r)} - \frac{A'(r)}{A(r)}\right) \text{ with } AB = 1 \tag{2.82}$$

This becomes

$$R_{\vartheta\vartheta} = -1 + \frac{1}{A(r)} + \frac{r}{A(r)}\left(\frac{B'(r)}{B(r)}\right) = -1 + \left(1 - \frac{2GM}{r}\right) + r\left(1 - \frac{2GM}{r}\right)\frac{\frac{2GM}{r^2}}{1 - \frac{2GM}{r}} = -1 + 1 = 0 \quad (2.83)$$

Hence, the apparent singularity of the metric (2.80) at $r = R_{\text{Schw}}$ is just a **coordinate singularity** (can be transformed away by an appropriate coordinate transformation). On the other hand, the singularity at $r = 0$ is **real**. This singular point has infinite density and curvature.

1.) Let us consider the Schwarzschild metric for $r \mapsto \infty$, where we are only interested in the radial behaviour: $\mathrm{d}\tau^2 \mapsto \mathrm{d}t^2 - \mathrm{d}r^2$. This is just the Minkowski metric. For $r > R_{\text{Schw}}$ we have $\mathrm{d}\tau^2 = (\text{pos})\,\mathrm{d}t^2 + (\text{neg})\,\mathrm{d}r^2$ and for $r < R_{\text{Schw}}$ it holds $\mathrm{d}\tau^2 = (\text{neg})\,\mathrm{d}t^2 + (\text{pos})\,\mathrm{d}r^2$. At $r = R_{\text{Schw}}$, the light-cons **tip over**. The light, that is emmited inside is forever trapped inside a sphere with the Schwarzschild radius; that sphere is called the event horizon.

2.) Black holes are not totally black due to quantum effects. With $G = c = \hbar = k_B = 1$ (natural units) the Schwarzschild radius is given by $R_{\text{Schw}} = 2M$ and the Hawking temperature is $T_H = \kappa/2\pi$. The surface gravity of a Schwarzschild black hole is $\kappa = 1/(4M)$. So we find $T_H^{(\text{Schw})} = 1/(8\pi M)$. In SI units we obtain

$$T_H = \frac{k_B^{-1}\hbar c^3}{8\pi GM} = \frac{k^{-1}\hbar c}{4\pi R_{\text{Schw}}} = 10^{-7}\,\text{K}\left(\frac{M_\odot}{M}\right) \quad (2.84)$$

2.10 Gravitational waves

These are extremely weak effects. For an atomic transition ($\nu = \Delta E$, $\hbar = c = 1$), the ratio of emission rates is given by

$$\frac{\Gamma_{\text{grav}}}{\Gamma_{\text{em}}} \sim \frac{G(\Delta E)^2}{e^2} \sim \frac{1}{\alpha}\left(\frac{\Delta E}{M_{\text{Planck}}}\right)^2 \text{ with } M_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} = 1,22 \cdot 10^{19}\,\text{GeV} \quad (2.85)$$

$$\frac{\Gamma_{\text{grav}}}{\Gamma_{\text{em}}} \sim 10^{-54}\left(\frac{\Delta E}{\text{eV}}\right)^2 \quad (2.86)$$

Still, there could be **coherent** gravitational waves from **macroscopic** sources. Because die Einstein field equations are nonlinear, handling with gravitational waves is much more difficult than with electromagnetic waves. So we make use of **the weak field approximation**: $g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ with $|h_{\mu\nu}| \ll 1$. With this *Ansatz* the Christoffel symbols are of the form

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}\eta^{\lambda\varrho}(\partial_\mu h_{\varrho\nu} + \partial_\nu h_{\varrho\mu} - \partial_\varrho h_{\mu\nu}) + \mathcal{O}(h^2) \quad (2.87)$$

From this, we can compute the Ricci tensor:

$$R_{\mu\nu} = \partial_\nu \Gamma^\lambda{}_{\lambda\mu} - \partial_\lambda \Gamma^\lambda{}_{\mu\nu} + \mathcal{O}(h^2) \quad (2.88)$$

Because everything shall be linear in $h_{\mu\nu}$, indices can be raised and lowered just with the Minkowski metric. The Einstein field equations are given by:

$$\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \partial_\mu\partial_\nu h^\lambda{}_\lambda = -16\pi G S_{\mu\nu} \text{ with } \Box \equiv \boldsymbol{\nabla}^2 - \partial_t^2 \quad (2.89)$$

$S_{\mu\nu}$ is our source term. This is given by:

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\lambda{}_\lambda \quad (2.90)$$

Gravitational forces are assumed to be unimportant, so we have momentum conservation $\partial_\mu T^\mu{}_\nu = 0$. We also have the freedom, how we choose the coordinates. With the coordinate transformation $x'^\mu = x^\mu + \varepsilon^\mu(x)$, where $\varepsilon^\mu(x)$ is $\sim O(h)$, we get

$$h'^{\mu\nu} = h^{\mu\nu} - \partial_\lambda\varepsilon^\mu\eta^{\lambda\nu} - \partial_\lambda\varepsilon^\nu\eta^{\lambda\mu} \quad (2.91)$$

This has the form of a gauge transformation. Now, we want to use the harmonic gauge condition $g^{\mu\nu}\Gamma^\lambda{}_{\mu\nu} = 0$, so we get $\partial_\mu h^\mu{}_\nu = 1/2\partial_\nu h^\mu{}_\mu$ (2.51). Then, (2.89) becomes very simple, namely $\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$. For the Green's function $\Box G(x, x') = -4\pi\delta^{(4)}(x - x')$ holds.

$$G^\pm(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta\left(t' - [t \mp |\mathbf{x}' - \mathbf{x}|]\right)}{|\mathbf{x}' - \mathbf{x}|} \quad (2.92)$$

We take the retarded Green's function, in order to solve the above equation (2.52).

$$h_{\mu\nu}(\mathbf{x},t) = 4G\int \mathrm{d}^3x' \, \frac{S_{\mu\nu}\left(\mathbf{x}', t - |\mathbf{x}' - \mathbf{x}|\right)}{|\mathbf{x}' - \mathbf{x}|} \tag{2.93}$$

Hence, the gravitational effects move away from the source with speed of light in vacuo in Minkowski space ($c = 1$). For plane waves we make an *Ansatz* of the form

$$h_{\mu\nu}(x) = e_{\mu\nu}\exp(\mathrm{i}k_\lambda x^\lambda) + e^*_{\mu\nu}\exp(-\mathrm{i}k_\lambda x^\lambda) \tag{2.94}$$

$e_{\mu\nu}$ is the polarization tensor, which is symmetric ($e_{\mu\nu} = e_{\nu\mu}$). In vacuo, $S_{\mu\nu} = 0$ and $k_\lambda k^\lambda = 0$ (2.55) holds. From the usage of the harmonic gauge, we furthermore obtain $k_\mu e^\mu{}_\nu = 1/2 k_\nu e^\mu{}_\mu$ (2.56). The ten components of $e_{\mu\nu}$ are reduced by four because of the harmonic gauge and by four from the **remaining** gauge freedom, leaving us with **two physical polarizations**. Rotation by $\Delta\varphi$ and $\mathbf{k}$, changes the wave function by a factor $\exp(\pm 2\mathrm{i}\Delta\varphi)$, so the helicity is $h = 2$. If one make a particular transformation, namely

$$x'^\mu \mapsto x^\mu + \mathrm{i}\varepsilon^\mu\exp(\mathrm{i}kx) - \mathrm{i}(\varepsilon^*)^\mu\exp(-\mathrm{i}kx) \tag{2.95}$$

So the polarization tensor transforms like

$$e^{\mu\nu} \mapsto e^{\mu\nu} + k^\mu\varepsilon^\nu + k^\nu\varepsilon^\mu \equiv e'^{\mu\nu} \tag{2.96}$$

This transformation is compatible with the Lorentz gauge, so it leaves the condition (2.56) invariant. There exist two polarizations $+$ and $\times$. We are interested in the difference of the lengths in our interferometer: $\Delta L = L_1 - L_2$.

$$\frac{\Delta L}{L} = F_+ h_+(t) + F_\times h_\times(t) \tag{2.97}$$

F_+ and $F_\times$ are numbers of order 1, depending on the orientation both of source and detector. ΔL can be measured to $10^{-18}\,\mathrm{m} = 10^{-8}\,\text{Å}$. Typically, the interferometers, which are being built now, have a length of several kilometers.

2.11 The formulation in terms of an action

There are two advantages of the action:

1.) **compact** formulation of the dynamics

2.) role of **symmetries transparent**

We want to postulate the total action I (scalar under general coordinate transformations):

$$I = I_M + I_G \text{ with } I_G = -\frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{|g(x)|}R(x) \text{ with } g = -\det(g_{\mu\nu}) \tag{2.98}$$

Now, we consider an infinitesimal variation $g_{\mu\nu}(x) \mapsto g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$, where $g_{\mu\nu}(x)$ goes to zero for $x \mapsto \infty$. We keep the matter fields fixed and we have generically $\delta I_M \neq 0$. Then, **define**

$$\delta I_M \equiv \frac{1}{2}\int \mathrm{d}^4x\,\sqrt{g(x)}T^{\mu\nu}(x)\delta g_{\mu\nu}(x) \tag{2.99}$$

This $T^{\mu\nu}$ is precisely our energy-momentum tensor from before. After "some work" one finds the following:

$$\delta I_G = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{g(x)}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\delta g_{\mu\nu} \tag{2.100}$$

Because of the stationarity of action ($\delta I = 0$) for **arbitrary** $\delta g_{\mu\nu}(x)$ gives **local** equations:

$$R^{\mu\nu}(x) - \frac{1}{2}g^{\mu\nu}(x)R(x) + 8\pi G T^{\mu\nu}(x) = 0 \tag{2.101}$$

Nota bene, since **both** I_M and I_G are scalars, you can obtain immediately two results:

1.) energy-momentum conservation: $T^\mu{}_{\nu;\mu} = 0$

2.) Bianchi identity: $(R^{\mu}{}_{\nu} - 1/2\delta^{\mu}{}_{\nu}R)_{;\mu} = 0$

Firstly, we want to derive the following lemmas:

$$\delta\left(\sqrt{g}g^{\mu\nu}R_{\mu\nu}\right) = \delta(\sqrt{g})g^{\mu\nu}R_{\mu\nu} + \sqrt{g}\delta g^{\mu\nu}R_{\mu\nu} + \sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} \tag{2.102}$$

i.) general matrix A, $a \equiv \det(A)$:

$$\begin{aligned}\frac{1}{a}\delta a = \delta\ln(a) = \delta\ln(\det(A)) = \ln\det(A+\delta A) - \ln\det(A) = \ln\left(\frac{\det(A+\delta A)}{\det(A)}\right) = \\ = \ln\det[A^{-1}(A+\delta A)] = \ln\det(1 + A^{-1}\delta A) = \\ = \ln(1 + \mathrm{Tr}(A^{-1}\delta A)) = \mathrm{Tr}(A^{-1})\delta A\end{aligned} \tag{2.103}$$

So we obtain:

$$\boxed{\frac{1}{g}\delta g = g^{\mu\nu}\delta g_{\nu\mu}} \tag{2.104}$$

ii.) Using $g^{\kappa\lambda}g_{\lambda\mu} = \delta^{\kappa}{}_{\mu}$ we get

$$(\delta g^{\kappa\lambda}g_{\lambda\mu} + g^{\kappa\lambda}\delta g_{\lambda\mu})g^{\mu\varrho} = 0 \Rightarrow \boxed{\delta g^{\kappa\varrho} = -g^{\kappa\lambda}g^{\varrho\mu}\delta g_{\lambda\mu}} \tag{2.105}$$

iii.) $\int \frac{\partial}{\partial x}\left(\sqrt{g}g^{\mu\nu}\delta\Gamma^{\lambda}{}_{\mu\nu}\right) = 0$

iv.) $\delta\sqrt{g} = \frac{1}{2\sqrt{g}}\delta g = \frac{1}{2}\sqrt{g}g^{\mu\nu}\delta g_{\mu\nu}$

So, we want to write out (iv) and get:

$$\delta\sqrt{g} = \sqrt{g}\left(\frac{1}{2}g^{\mu\nu}R - R^{\mu\nu}\right)\delta g_{\mu\nu} \tag{2.106}$$

2.11.1 Tetrads and spinors

We are going to decompose the metric in three parts, namely $g_{\mu\nu}(x) = e^{a}{}_{\mu}(x)\eta_{ab}e^{b}{}_{\nu}(x)$. $e^{a}{}_{\mu}$ is called Tetrad (Vierbein, moving frames). A tetrad consists of **four** covariant vector fields. We call $E_{a}{}^{\mu}$ the inverse tetrad and these are **four** contravariant vectors.

$$e^{a}{}_{\mu}E_{b}{}^{\mu} = \delta^{a}{}_{b} \text{ and } E_{a}{}^{\mu}e^{a}{}_{\nu} = \delta^{\mu}{}_{\nu} \tag{2.107}$$

We denote the indices as follows: a be the Lorentz index and μ the Einstein index. The tetrad tells us, how the local inertial coordinate system is chosen at a special point. The matter action, in particular, must be both an Einstein scalar and a Lorentz scalar.
The tetrad formalism allows us, to incorporate **spinors**, which could not be dealt with by our simple recipe: The idea is, to generalize Lorentz tensors to Einstein tensors. We also have to replace the Lorentz metric $\eta_{\alpha\beta}$ by the Einstein metric $g_{\mu\nu}(x)$ and the partial derivatives by covariant derivatives $D_\mu \sim \partial_\mu + \Gamma$. But the problem is, that spinors are completely Lorentzian. Nevertheless, there must be a way, how to include spinors in general relativity. One has to make sure, that one gets rid of all Einstein indices. Simply use Dirac spinors together with Lorentz-index expression. The subtlety is the Lorentz covariant derivative, which involves the so-called spin connection $\omega_{ab}(x)$.

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