Mitschrieb zur Vorlesung: Theoretical Cosmology (Gravitation and Cosmology II)

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Mitschrieb der Vorlesung Theoretical Cosmology (Gravitation and Cosmology II)
von Herrn Prof. Dr. Klinkhamer im Sommersemester 2007
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
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Kapitel 1

Cosmology

1.1 Robertson-Walker-Metric

Our **working hypothesis** is, that we have spatial homogenity **and** isotropy (of the „smeared-out cosmic fluid"). For „comoving coordinates" the metric is (Robertson 1935, Walker 1936)

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a(t)^2 \left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2) \right] \tag{1.1}$$

$a(t)$ is called the cosmic scale factor and t „cosmic time". k is a constant, which has been renormalized to take the values $\{-1, 0, 1\}$. We use the notation $\mathrm{d}s^2 \equiv -\mathrm{d}\tau^2 = g_{\mu\nu}(x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$. The idea is, that the cosmic time is related to the evolution of a scalar quantity. An example would be the CMB photon temperature T_γ, so it holds for example $t = t(T_\gamma)$. r, ϑ and φ are **constant** for a comoving galaxy. The 3-curvature scalar is given by ${}^3K(t) = k/a^2(t)$. This is important, to characterize the different kinds of universes we have.

	3-space	
	curvature ${}^3K$	volume ${}^3V$
$k=-1$	negative	infinite (or finite)
$k=0$	zero (flat)	infinite (or finite!)
$k=1$	positive (circumference ${}^3L(t) = 2\pi a(t)$)	finite

For comoving galaxies there is a current four-vector $J^\mu_{\mathrm{gal}} = n_{\mathrm{gal}}(t)U^\mu$. In **these** coordinates U^μ is given by $U^\mu = (1,0,0,0)$. Then, average cosmic matter has the energy-momentum tensor

$$T_{\mu\nu} = (\varrho(t) + p(t))\,U_\mu U_\nu + p(t)g_{\mu\nu}(t) \tag{1.2}$$

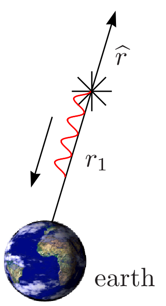
$\varrho(t)$ is the energy density and $p(t)$ stands for the density. Recall a perfect fluid in Minkowski ($g_{00} = -1$): $T^{\mathrm{Mink}}_{\mu\nu} = \mathrm{diag}(\varrho, p, p, p)$. From $J^\mu{}_{;\mu}$ and $T^{\mu\nu}{}_{;\nu} = 0$ one finds

$$n_{\mathrm{gal}}(t)a^3(t) = \mathrm{const.} \tag{1.3}$$

by particle number conservation and

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varrho a^3(t)) + p\frac{\mathrm{d}}{\mathrm{d}t}a^3 = 0 \tag{1.4}$$

(adiabatic expansion). We consider light from a typical distant galaxy with comoving coordinates (r, ϑ, φ). Our position is $r = 0$.



$$0 = \mathrm{d}t^2 - a^2(t)\frac{\mathrm{d}r^2}{1-kr^2} \tag{1.5}$$

The first wave crest is emitted at cosmic time t_1 and observed at t_0 has

$$\int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = -\int_{r_1}^{0} \frac{\mathrm{d}r}{\sqrt{1-kr^2}} = I(r_1) \tag{1.6}$$

Now, we are looking at a second wave crest, which is emitted after one oscillation:

$$\int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{\mathrm{d}t}{a(t)} = I(r_1) \tag{1.7}$$

The space coordinates stay the same, because both source and observer are comoving. So we subtract these two equations:

$$0 = \int_{t_1+\Delta t_1}^{t_0+\Delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_1}^{t_0} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{t_0+\Delta t_0} \frac{\mathrm{d}t}{a(t)} - \int_{t_1}^{t_1+\Delta t_1} \frac{\mathrm{d}t}{a(t)} \tag{1.8}$$

With $|\Delta t| \ll |t|$ we obtain:

$$0 = \frac{\Delta t_0}{a(t_0)} - \frac{\Delta t_1}{a(t_1)} \Rightarrow \frac{\Delta t_1}{a(t_1)} \sim \frac{\Delta t_0}{a(t_0)} \Rightarrow \frac{\nu_0}{\nu_1} \equiv \frac{\Delta t_1}{\Delta t_0} = \frac{a(t_1)}{a(t_0)} \tag{1.9}$$

For an **expanding** universe the observed frequency is less than the emitted frequency (**redshifts**). For not too far galaxies we have (with $\lambda = c/\nu$):

$$z \equiv \frac{\lambda_0 - \lambda_1}{\lambda_1} = \frac{a(t_0)}{a(t_1)} - 1 \approx \frac{a_0}{a_0 + \dot{a}_0(t_1 - t_0)} - 1 \approx 1 - \left(\frac{\dot{a}}{a}\right)_{t_0}(t_1 - t_0) - 1 = -\left(\frac{\dot{a}}{a}\right)_{t_0}(t_1 - t_0) \approx \left(\frac{\dot{a}}{ca}\right)_{t_0} r_1 \tag{1.10}$$

We find, that the redshift is proportional to the distance between source and observer. This agrees with the **observed** Hubble law (1929) $cz \sim H_0 d$, where $H_0 \approx 75\,\frac{\mathrm{km}}{\mathrm{s\cdot Mpc}} \approx (13 \cdot 10^9\,\mathrm{yr})^{-1}$ is the Hubble constant.

1.2 Hot Big Band Model

Let us write the Robertson-Walker-metric a little more compactly:

$$g_{tt} = -1,\; g_{nt} = 0,\; g_{mn} = a^2(t)\widetilde{g}_{mn}(x^1, x^2, x^3) \text{ with } m, n \in \{1, 2, 3\} \tag{1.11}$$

We use this metric as an *Ansatz* for solving the Einstein field equations:

$$R_{\mu\nu} = -8\pi G S_{\mu\nu} \text{ with } S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \tag{1.12}$$

Putting the *Ansatz* in, we get the Ricci tensor:

$$R_{tt} = 3\frac{\ddot{a}}{a},\; R_{mt} = 0,\; R_{mn} = -(a\ddot{a} + 2\dot{a}^2 + 2k)\widetilde{g}_{mn} \tag{1.13}$$

If we put the energy momentum tensor (1.2) in, we get for the source term:

$$S_{tt} = \frac{1}{2}(\varrho + 3p),\; S_{mt} = 0,\; S_{mn} = \frac{1}{2}(\varrho - p)a^2\widetilde{g}_{mn} \tag{1.14}$$

$$T = g^{\mu\nu}T_{\mu\nu} = (p + \varrho)g^{00} + 4p = 3p - \varrho \tag{1.15}$$

$$S_{\mu\nu} = (p + \varrho)U_\mu U_\nu + \left(p - \frac{3}{2}p + \frac{1}{2}\varrho\right) g_{\mu\nu} \tag{1.16}$$

So, the Einstein equations (10 partial differential equations) reduce with the *Ansatz* to two ordinary differential equations:

$$3\ddot{a} = -4\pi G(\varrho + 3p)a \tag{1.17a}$$

$$a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\varrho - p)a^2 \tag{1.17b}$$

Eliminating $\ddot{a}$, we get the following equation (Friedman equation):

$$\dot{a}^2 + k = \frac{8\pi G}{3}\varrho a^2 \tag{1.18}$$

(1.4) leads to:

$$\frac{\mathrm{d}t}{\mathrm{d}a}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\varrho a^3) + p\frac{\mathrm{d}}{\mathrm{d}t}a^3\right) = 0 \Rightarrow \frac{\mathrm{d}}{\mathrm{d}a}(\varrho a^3) = -3pa^2 \tag{1.19}$$

Additionally, we have a third equation, namely the equation of state:

$$p = p(\varrho) \tag{1.20}$$

We want to consider two simple cases:

1) non relativistic matter ($p = 0$):

   For this case, it holds

$$\varrho_{\text{nonrel}} \sim a^{-3} \tag{1.21}$$

2) relativistic particles ($p = 1/3\varrho$):

$$\frac{\mathrm{d}}{\mathrm{d}a}(\varrho a^3) = -\varrho a^2 \Rightarrow \varrho_{\text{rel}} \sim a^{-4} \tag{1.22}$$

Even, if now $(\varrho_{\text{rel}})_0 \ll (\varrho_{\text{nonrel}})_0$, there **would** be a time, where $\varrho_{\text{rel}} \gg \varrho_{\text{nonrel}}$. Hence, this model is called **Hot** Big Bang. Let us now make two remarks:

1) Look at reduced Einstein equation:

$$3\ddot{a} = -4\pi G(\varrho + 3p)a \tag{1.23}$$

   With $a = a(t)$ **and** $\varrho + 3p$ positive, you find $\ddot{a} < 0$, so that the curve $a(t)$ is **concave**. Furthermore, with present expansion rate $H_0 \equiv (\dot{a}/a)_{t_0} > 0$, then there **must** be a time ($t = 0$), when $a = 0$, namely the Big Bang.

   Also, we have an **upper limit** on the age of **the universe**, namely $t_0 \leq H_0^{-1} \approx 13 \cdot 10^9\,\text{yr}$.

2) (1.18) evaluated now ($t = t_0$):

$$\varrho_0 = \frac{3}{8\pi G}\left(H_0^2 + \frac{k}{a_0^2}\right) = \varrho_{0,c} + \frac{3k}{8\pi G a_0^2} \tag{1.24}$$

   If $\varrho_0 > \varrho_{0,c}$, then it holds $k = 1$ and the universe if **finite**. If $\varrho_0 \leq \varrho_{0,c}$, then it is $k = 0, -1$ and the universe may be infinite. The critical density is defined as follows:

$$\varrho_{0,c} \equiv \frac{3H_0^2}{8\pi G} = 1,1 \cdot 10^{-29}\,\frac{\text{g}}{\text{cm}^3}\left(\frac{H_0}{75\,\frac{\text{km}}{\text{s}\cdot\text{Mpc}}}\right)^2 \tag{1.25}$$

   The main focus of astrophysics in the last fifty years was, to measure this critical density. So astronomy wants to determine the ratio $\Omega \equiv \varrho_0/\varrho_{0,c}$ (1.25). Roughly speaking, we can say $\Omega_{\text{visible matter}} = \mathcal{O}(1\%)$ and $\Omega_{\text{dark matter}} = \mathcal{O}(20\%)$.

Now determine $a = a(t)$:

1) Matter domination ($p = 0$): $\varrho \sim a^{-3}$

   Take for simplicity $k = 0$. From (1.18) we get $\dot{a}^2 \sim \varrho a^2 \sim a^{-1}$ and so $\dot{a} \sim a^{-\frac{1}{2}}$. So, one gets $a(t) \sim t^{\frac{2}{3}}$ for $p = 0 = k$. For $k = \pm 1$, explicit solutions can be found.

   For $p = 0$ and $k = 0$ we get:

$$a = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3}},\ \dot{a} = a_0 t_0^{-\frac{3}{2}}\frac{2}{3}t^{-\frac{1}{3}} \tag{1.26}$$

$$\frac{\dot{a}}{a} = \frac{2}{3}t^{-1} \Rightarrow t = \frac{2}{3}\frac{1}{\frac{\dot{a}}{a}} \Rightarrow t_0 = \frac{2}{3}\frac{1}{H_0} \approx 8,7 \cdot 10^9\,\text{yr} \tag{1.27}$$

2) Radiation domination ($p = \varrho/3$):

We neglect the k-term in (1.18) (will be justified a posteriori).

$$\dot{a}^2 = \frac{8\pi G}{3}\varrho a^2 \sim a^{-2} \Rightarrow \dot{a} \sim \frac{1}{a} \tag{1.28}$$

Here, we get the solution $a(t) \sim t^{\frac{1}{2}}$ ($p = \varrho/3$). Recall the Stefan Boltzmann law $\varrho \sim T^4$. Then, we obtain $T \sim a^{-1} \sim t^{-\frac{1}{2}}$. Let's now verify the neglect of this k-term:

$$\dot{a}^2 \sim \frac{1}{t},\ \varrho a^2 \sim \frac{a^2}{a^4} = \frac{1}{a^2} \sim \frac{1}{t},\ k \sim t^0 \tag{1.29}$$

So, in the early universe it does not matter, whether k is -1, 0 or 1.

1.3 The cosmological constant

Perhaps expansion now is driven not only by nonrelativistic matter but also by some kind of **vacuum energy**, namely the so-called „cosmological constant“ Λ. In 1917 Einstein considered a generalization of his field equations: $[\Lambda] = \text{length}^{-2}$.

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu} \tag{1.30}$$

Let us consider the energy momentum tensor $T_{\mu\nu}$ of a perfect fluid. So we can write this as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G\widetilde{T}_{\mu\nu} \text{ with } \widetilde{T}_{\mu\nu} \equiv T_{\mu\nu} - \frac{\Lambda}{8\pi G}g_{\mu\nu} = \widetilde{p}g_{\mu\nu} + (\widetilde{p} + \widetilde{\varrho})U_\mu U_\nu \tag{1.31}$$

So we find:

$$\widetilde{\varrho} = p - \frac{\Lambda}{8\pi G},\ \widetilde{\varrho} \equiv \varrho + \frac{\Lambda}{8\pi G} \tag{1.32}$$

So it acts in the same way as a perfect fluid, but with an equation of state, so that the pressure is the negative of the density. $\Lambda/(8\pi G)$ is the vacuum energy density. With $\widetilde{p}$ and $\widetilde{\varrho}$ all previous results remain valid. Consider a the case of a pressureless case ($p = 0$):

$$\frac{\mathrm{d}}{\mathrm{d}a}(\widetilde{\varrho}a^3) = -3\widetilde{p}\dot{a}^2 \xrightarrow[p=0]{} \frac{\mathrm{d}}{\mathrm{d}a}\left(\varrho a^3 + \frac{\Lambda}{8\pi G}a^3\right) = 3a^2\frac{\Lambda}{8\pi G} \Rightarrow \varrho a^3 = \text{const.} \tag{1.33}$$

We write this in the following form:

$$\varrho a^3 \equiv \frac{\alpha}{4\pi G\sqrt{|\Lambda|}} \text{ with } \alpha \in [0, \infty) \tag{1.34}$$

So we obtain for our Friedman equation:

$$\dot{a}^2 = -k + \frac{8\pi G}{3}\widetilde{\varrho}a^3 = \frac{1}{a}\left[-ka + \frac{8\pi G}{3}\frac{\alpha}{4\pi G\sqrt{|\Lambda|}} + \frac{8\pi G}{3}\frac{\Lambda}{8\pi G}a^3\right] \tag{1.35}$$

$$\Rightarrow \dot{a}^2 = \frac{1}{a}\left[\frac{\Lambda}{3}a^3 - ka + \frac{2}{3}\frac{\alpha}{\sqrt{|\Lambda|}}\right] \tag{1.36}$$

For $\Lambda > 0$, $k = \alpha = 0$ the universe is called de Sitter universe. For such a universe $a(t) \sim \exp(ct\sqrt{|\Lambda|})$ holds; so it expands exponentially. Another possible universe is the Lemaître universe, for which $\Lambda > 0$, $k = 1$ and $\alpha > 1$ holds. For the Einstein static universe $\Lambda > 0$, $k = 1$, $p = 0$ and $\varrho = \Lambda/(4\pi G)$ holds.

Kapitel 2

Homogeneous Hot Big Bang Model

2.1 Recombination and Cosmic Microwave Background (CMB)

We make a sketch of the universe **far** out (for example **long** way back in time).
The recombination temperature is set by the atomic scale (eV). More precisely, you find $k_B T_{\text{rec},\gamma} \approx 0,2\,\text{GeV}$, $T_\gamma^{\text{rec}} \approx 3000\,\text{K}$ (4.1). For a radiation-dominated universe, $a \sim \sqrt{t}$ holds and with the number of relativistic particles we get $t\,[\text{s}] \sim 10^{20}/(T\,[\text{K}])^2$, so that $t^{\text{rec}} \sim 10^{20}/(3000)^2 \sim 10^{13}\,\text{s}$. In order to estimate the scale of the universe, **then** ($t = t^{\text{rec}}$) compared to **now** ($t = t_0 \equiv 10^{10}\,\text{yrs} \approx 3 \cdot 10^{17}\,\text{s}$) (use matter dominated universe model $a \sim t^{\frac{2}{3}}$)

$$\frac{a_0}{a_{\text{rec}}} = \left(\frac{t_0}{t_{\text{rec}}}\right)^{\frac{2}{3}} \approx \left(\frac{3 \cdot 10^{17}}{10^{13}}\right)^{\frac{2}{3}} \approx 10^3 \tag{2.1}$$

For relativistic particles $T \sim a^{-1}$ holds. So we get

$$T_{\gamma,0} \sim \frac{a_{\text{rec}}}{a_0} \cdot 3000\,\text{K} \approx 3\,\text{K} \tag{2.2}$$

So, the **Hot Big Band Model** predicts a **relic** photon background at 3 K.

Anmerkung: Nach einem Festplattendefekt ist der Rest der Vorlesung leider unwiederholbar verloren gegangen ;-(

Literature

- Mukhanov: Physical foundation of cosmology, CUP 2005
- Weinberg: Gravitation and Cosmology, Wiley 1972
- Cheng: Relativity, Gravitation and Cosmology, OUP 2005