Mitschrift zur Vorlesung: Lattice Gauge Theory

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Mitschrift der Vorlesung Lattice Gauge Theory (englisch)
von Herrn Prof. Dr. Klinkhamer im Wintersemester 2005/2006
von Marco Schreck.

Diese Mitschrift erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

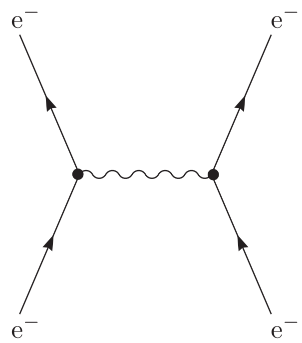
Contents

Chapter 1

Introduction

1.1 Standard model

The standard model contains quarks and leptons. Interactions between particles are described by (non-)Abelian gauge theories. In the perturbation theory cross sections of particle reactions are calculated with the help of Feynman diagrams.



- QCD: $g^2/4\pi$
- Standard model: SU(3) × SU(2) × U(1) (Yang-Mills-Higgs theory)

A non-perturbative part in the standard model is the QCD confinement. There exists a high temperature phase transition (indeed Higgs mechanism). The theoretical frontier of the standard model is non-perturbative physics.

Chapter 2

Euclidian Path Integral and Non-Relativistic Quantum Mechanics

A good article about this topic is [1]. The quantum mechanics of nonrelativistic particles of mass m in a potentia $V(x)$ (imaginary time $t_{\text{Minkowski}} = -\mathrm{i}t_{\text{Euclidian}}$) is described by the Lagrangian

$$\mathcal{L}(x, \dot{x}) = K(\dot{x}) - V(x) \text{ with } K(\dot{x}) = \frac{m}{2}\dot{x}^2,\ V \geq 0 \tag{2.1}$$

and the action

$$S = \int \mathrm{d}t\, \mathcal{L}(\dot{x}(t), x(t)) \tag{2.2}$$

The integral of "all" trajectories $x(t)$ is given by (with $\hbar = c = 1$):

$$Z = \int \text{“}\mathrm{d}x(t)\text{”} \exp(-S) \tag{2.3}$$

In the Minkowski space we get:

$$Z = \int \text{“}\mathrm{d}x(t)\text{”} \exp(\mathrm{i}S) \text{ with } \mathrm{i}S = \mathrm{i}\int \mathrm{d}t \left[\frac{m}{2}\left(\frac{\partial x}{\partial t}\right)^2 - V\right] \tag{2.4}$$

With $t = -\mathrm{i}\tau$ we obtain:

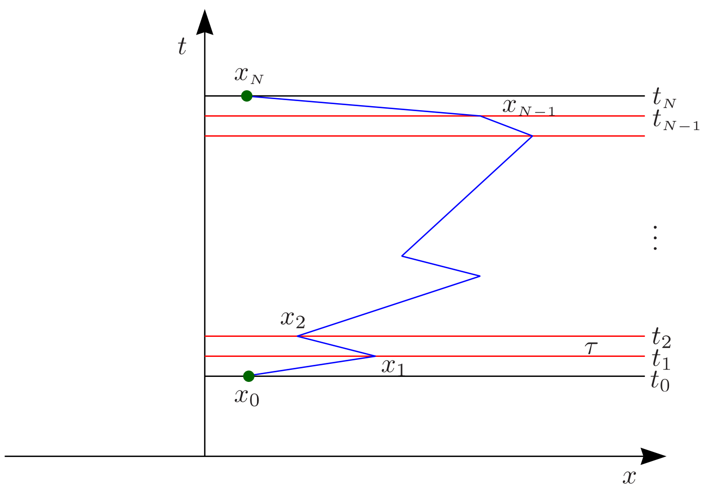
$$\mathrm{i}S \mapsto -\mathrm{i}^2 \int \mathrm{d}\tau \left[-\frac{m}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 - V\right] = -\int \mathrm{d}\tau \left[\frac{m}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + V\right] \tag{2.5}$$

We now want to discretize time and use finite intervals τ with periodical boundary conditions

$$t = ja \text{ with } j = 0, 1, \ldots, N \geq \frac{\tau}{a} \tag{2.6}$$

and write $x(ja) \equiv x_j$, $x_0 = x_N$, so that we get:

$$Z = \int\limits_{[x_0 = x_N]} \prod_{j=0}^{N} \mathrm{d}x_j \exp(-S) \text{ with } S = a\sum_j \left[\frac{m}{2}\left(\frac{x_{j+1} - x_j}{a}\right)^2 + V(x_j)\right] \tag{2.7}$$



Does one recover the canonical Hamiltonian in the continuum limit? Because of the locality of action Z factorizes. So we can write it in the following form:

$$Z = \int\limits_{[x_0=x_N]} \prod_{j=0}^{N} \left(\mathrm{d}x_j \, T_{x_j+1,x_j}\right) \tag{2.8}$$

with the transfer matrix elements

$$T_{x',x} = \exp\left(-\frac{m}{2a}(x'-x)^2 - \frac{a}{2}(V(x') + V(x))\right) \tag{2.9}$$

The $\hat{T}$ operator acts in Hilbert space of square-integrable functions with the basis states $|x\rangle$, which are orthonormal ($\langle x'|x\rangle = \delta(x'-x)$), the inner product

$$\langle \psi'|\psi\rangle = \int \mathrm{d}x \, \psi'^{*}\psi, \; \mathbb{1} = \int \mathrm{d}x \, |x\rangle\langle x| \text{ and } |\psi\rangle = \int \psi(x)|x\rangle \tag{2.10}$$

Furthermore it is $\hat{x}|x\rangle = x|x\rangle$, $[\hat{p},\hat{x}] = -\mathrm{i}$ and $\exp(-\mathrm{i}\hat{p}\Delta)|x\rangle = |x+\Delta\rangle$, so $\hat{p}$ is the generator of translations. The operator $\hat{T}$ is defined by the matrix elements, $\langle x'|\hat{T}|x\rangle \equiv T_{x',x}$.

$$Z = \mathrm{Tr}(\hat{T}^N) = \sum_k \lambda_k^N \text{ for eigenvalues } \lambda_0 > \lambda_1 > \lambda_2 > \ldots = \lambda_0^N \left[1 + O\left(\exp(-N\ln(N\lambda_1))\right)\right] \tag{2.11}$$

$$Z = \int\limits_{[x_0=x_N]} \prod_{j=0}^{N} \mathrm{d}x_j \, \exp(-S) \text{ with } S = a\sum_j \left[\frac{1}{2}m\left(\frac{x_{j+1}-x_j}{a}\right)^2 + V(x_j)\right] \tag{2.12}$$

$$Z = \int\limits_{[x_0=x_N]} \prod_{j=0}^{N} \left(\mathrm{d}x_i T_{x_{i+1},x_i}\right) \text{ with } T_{x,x'} = \exp\left(-\frac{m}{2a}(x'-x)^2 - a\frac{V(x')+V(x)}{2}\right) \tag{2.13}$$

Now we are searching for an operator $\hat{T}$, which works in the Hilbert space with $\langle x'|\hat{T}|x\rangle = T_{x',x}$. We are trying the following ansatz in a sort of integral form:

$$\hat{T} = \int \mathrm{d}\Delta \, \exp\left(-\frac{a}{2}V(\hat{x})\right) \exp\left(-\frac{m}{2q}\Delta^2 - \mathrm{i}\hat{p}\Delta\right) \exp\left(-\frac{a}{2}V(\hat{x})\right) \tag{2.14}$$

Let us check, if this is correct. So we compute the matrix element:

$$\begin{aligned}\langle x'|\hat{T}|x\rangle &= \exp\left(-\frac{a}{2}(V(x')+V(x))\right) \int \mathrm{d}\Delta \, \left\langle x' \middle| \exp\left(-\frac{m}{2a}\Delta^2 - \mathrm{i}\hat{p}\Delta\right) \middle| x \right\rangle = \\ &= \exp\left(-\frac{a}{2}(V(x')+V(x))\right) \int \mathrm{d}\Delta \, \exp\left(-\frac{m}{2a}\Delta^2\right) \langle x'|x+\Delta\rangle =\end{aligned} \tag{2.15}$$

The states $|x\rangle$ are normalized, so we can introduce the Dirac-distribution by $\langle x'|x+\Delta\rangle = \delta(x'-x-\Delta)$ and compute the integral. We can now compute the Gaussian integral in $\hat{T}$:

$$\boxed{\hat{T} = \sqrt{\frac{2\pi a}{m}} \exp\left(-\frac{a}{2}V(\hat{x})\right) \exp\left(-\frac{a}{2}\frac{\hat{p}^2}{m}\right) \exp\left(-\frac{a}{2}V(\hat{x})\right)} \tag{2.16}$$

What we are interested in is the continuum limit, where $a \mapsto 0$. With the Campbell-Baker-Hausdorff-theorem in leading order we get:

$$\boxed{\hat{T} = \sqrt{\frac{2\pi a}{m}} \exp\left(-q\hat{H} + \mathcal{O}(a^2)\right) \text{ with } \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})} \tag{2.17}$$

$\hat{H}$ ist the canonical Hamiltonian, that we plug in in the Schrödinger-equation. The Hamiltonian is the generator of time translations in the same way as $\hat{p}$ is the generator of space translations. What we now want to calculate, are correlation functions.

2.1 Correlation functions

We are now considering two-point-functions with $i - j > 0$. The correlation function is defined a follows:

$$\langle x_i x_j \rangle = \frac{1}{Z} \int \prod_k \mathrm{d}x_k \, x_i x_j \exp(-S) = \frac{1}{Z} \mathrm{Tr} \left(\hat{T}^{N-i+j} \hat{x} \hat{T}^{i-j} \hat{x} \right) \tag{2.18}$$

If we now go to the limit of large N, this has the following structure

$$\left\langle 0 \middle| \hat{x} \left(\frac{\hat{T}}{\lambda_0} \right)^{i-j} \hat{x} \middle| 0 \right\rangle \tag{2.19}$$

where the state $|0\rangle$ is the ground state ($E = 0$) of the system. All the perturbations have died out. λ_0 is the largest eigenvalue of transfer operator. If we now go to the continuum limit and keep $t \equiv (i - j)a$ fixed as $a \mapsto 0$, we now introduce Heisenberg operators in the following way:

$$\hat{x}(t) = \exp(\hat{H}t)\hat{x} \exp(-\hat{H}t) \tag{2.20}$$

For us the time is imaginary, so we have no explicit i in the exponential function.

$$\langle 0| \exp(-\hat{H}t) \exp(\hat{H}t)\hat{x} \exp(-\hat{H}t)\hat{x}|0\rangle = \langle 0|T(\hat{x}(t)\hat{x}(0))|0\rangle = \langle x_i x_j \rangle \tag{2.21}$$

This is the expectation value of the time ordered product.

Chapter 3

Scalar fields

We are now considering a selfconjugate fix scalar field $\varphi(x)$, where x_μ is $\in \mathbb{R}^4$ (with $\mu = 1, 2, 3, 4$; Euclidian space). The Lagrange density is now given by:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{1}{2} m^2 \varphi^2 \tag{3.1}$$

The action S and the path integral Z is defined by:

$$S = \int \mathrm{d}^4 x \, \mathcal{L} \text{ and } Z = \int (\mathcal{D}\varphi) \exp(-S) \tag{3.2}$$

We have an infinitely large space an so an infinitely large number of variables. The UV- and IR-regulation is given by a^{-1} and $(Na)^{-1}$. We want to use a hypercubelattice (in four dimensions) with periodic boundary conditions. The lattice space will be a and the lattice size N. The lattice sites x_μ are given by $x_\mu = an_\mu$, where n_μ is $\in \mathbb{Z}$ and $-N/2 < n_\mu < N/2$. The number of sites is than given by N^4. Furthermore we use the notation $\varphi(x_u) \equiv \varphi_{n_u} \equiv \varphi_{n_1,n_2,n_3,n_4}$.

$$\partial_\mu \varphi(x_u) \mapsto (\varphi_{n_u} + \delta_{u\mu} - \varphi_{n_u})/a \text{ with } \delta_{u\mu} = \begin{cases} 1 & \text{for} \quad \mu = 0 \\ 0 & \text{for} \quad \mu \neq u \end{cases} \tag{3.3}$$

We now want to write the lattice action:

$$S = a^4 \left[\sum_{[m,n]} \frac{1}{2} \left(\frac{\varphi_m - \varphi_n}{a} \right)^2 + \sum_n \frac{m^2}{2} \varphi_n^2 \right] \tag{3.4}$$

where $[m, n]$ are the nearest neighbours. We can now write what we meant by the path integral above:

$$Z = \int \left(\prod_n \mathrm{d}\varphi_n \right) \exp(-S) \tag{3.5}$$

This is a multipole integral, which is a Gaussian integral, since S is quadratic:

$$S = \frac{1}{2} \phi_m M_{mn} \phi_n \tag{3.6}$$

M_{mn} is a M^4-dimensional matrix, which corresponds to the values we have. M_{mn} is pretty much diagonal. With cylindrical coordinates one can find:

$$I = \int_{-\infty}^{+\infty} \mathrm{d}x \, \exp\left(-\frac{1}{2} \mu x^2 \right) = \left(\frac{\mu}{2\pi} \right)^{-\frac{1}{2}} \tag{3.7}$$

We then know that the Gaussian integral is given by:

$$Z = \left(\det \left(\frac{M}{2\pi} \right) \right)^{-\frac{1}{2}} \tag{3.8}$$

In order to find the eigenvalues of the matrix M_{mn} the fourier transformation is very useful (denoted F_{kn}, where n is the site and k the momentum). f_n is a complex-valued function at lattice site n:

$$\widetilde{f}_n = F_{kn} f_n \equiv \sum_n f_n \exp\left(\frac{2\pi \mathrm{i} k_\mu n_\mu}{N} \right) \text{ where } -\frac{N}{2} < k_\mu \leq \frac{N}{2} \tag{3.9}$$

$$f_n = N^{-4} \sum_k \widetilde{f}_k \exp\left(-\frac{2\pi \mathrm{i} k_\mu n_\mu}{N}\right) \tag{3.10}$$

Useful relations are the Parseval-relation

$$\sum_n f_n^* g_n = N^{-4} \sum_k \widetilde{f}_k^* \widetilde{g}_k \tag{3.11}$$

and the relation

$$\sum_{n_\mu} f_{n_\mu + \delta_{\mu u}}^* g_n = N^{-4} \sum_k \widetilde{f}_k^* \widetilde{g}_k \exp\left(\frac{2\pi \mathrm{i} k_u}{N}\right) \tag{3.12}$$

So we can find:

$$S = a^4 N^{-4} \sum_k \frac{1}{2} \widetilde{M}_k |\widetilde{\varphi}_k|^2 \tag{3.13}$$

The matrix M_{mn} is now diagonalized with the eigenvalues $\widetilde{M}_k$:

$$\widetilde{M}_k = m^2 + \frac{2}{a} \sum_u \left(1 - \cos\left(\frac{2\pi k_u}{N}\right)\right) \tag{3.14}$$

With the previous equations (Parseval) we namely obtain:

$$\begin{aligned} S \geq \sum_n \left[\frac{1}{2}\left(\frac{\varphi_{n+u} - \varphi_n}{a}\right)^2 + \frac{1}{2}\left(\frac{\varphi_n - \varphi_{n-u}}{a}\right)^2\right] &= \sum_n \left[\frac{\varphi_n^2}{a^2} - \frac{\varphi_n(\varphi_{n+\hat{u}} + \varphi_{n-\hat{u}})}{a^2}\right] = \\ = N^{-4} \sum_k \frac{|\widetilde{\varphi}_k|^2}{a^2} - N^{-4} \sum_k |\widetilde{\varphi}_k|^2 \frac{1}{a^2} \left[\exp\left(\frac{2\pi \mathrm{i} k_u}{N}\right) + \exp\left(-\frac{2\pi \mathrm{i} k_u}{N}\right)\right] &= \\ = N^{-4} \sum_k \frac{|\widetilde{\varphi}_k|^2}{a^2} - N^{-4} \sum_k |\widetilde{\varphi}_k|^2 \frac{2}{a^2} \cos\left(\frac{2\pi k_u}{N}\right) & \end{aligned} \tag{3.15}$$

$$Z = \prod_k \left(\frac{a^4 \widetilde{M}_k}{2\pi}\right)^{-\frac{1}{2}} \equiv Z(0) \tag{3.16}$$

The first way now is to introduce external sources J_n:

$$S(J) = \frac{1}{2} \varphi_m M_{mn} \varphi_n - J_n \varphi_n \tag{3.17}$$

$$Z(J) = \prod_n \mathrm{d}\varphi_n \exp(-S(J)) \tag{3.18}$$

The Green's functions are:

$$\langle \varphi_{n_1}, \varphi_{n_2}, \dots \rangle = \frac{1}{Z} \int \left(\prod \mathrm{d}\varphi\right) \varphi_{n_1} \varphi_{n_2} \dots \exp(-S) = Z^{-1} \left[\frac{\mathrm{d}}{\mathrm{d}J_{n_1}} \frac{\mathrm{d}}{\mathrm{d}J_{n_2}} \dots\right] Z(J)\Big|_{J=0} \tag{3.19}$$

$$Z(J) = Z(0) \exp\left(\frac{1}{2} J_m (M^{-1})_{mn} J_n\right) \tag{3.20}$$

$$\langle \varphi_m \varphi_n \rangle = (M^{-1})_{mn} = \frac{1}{a^4 N^4} \sum_k \widetilde{M}_k^{-1} \exp\left(\frac{2\pi \mathrm{i} k_0 (m-n)}{N}\right) \tag{3.21}$$

Let's introduce a variable q with the dimension of momentum:

$$q_\mu \equiv \frac{2\pi k_\mu}{Na} \tag{3.22}$$

If we take the limit $N \mapsto \infty$, we find, that the two-point-function is:

$$\langle \varphi_m \varphi_n \rangle = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\exp(-\mathrm{i} q \cdot x)}{m^2 + \frac{2}{a^2} \sum_\nu (1 - \cos(a q_\nu))} \tag{3.23}$$

We will now also define $x_\mu = a(n_\mu - m_\mu)$. For $a \mapsto 0$ we obtain the Euclidian Feynman propagator:

$$\int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\exp(-\mathrm{i}qx)}{m^2 - q^2} \tag{3.24}$$

So our original ansatz was not bad. We now want to add interactions:

$$S(J) = \frac{1}{2}\varphi_m M_{mn} \varphi_n - J_n \varphi_n + \sum_n V_I(\varphi_n) \text{ with } V_I(\varphi) = \lambda_0 \varphi^4 \tag{3.25}$$

λ_0 is the bare coupling constant. So

$$Z(J) = \int \left(\prod \mathrm{d}\varphi_n\right) \exp(-S(J)) \tag{3.26}$$

is perfectly well defined. This integral can be written as a formal perturbation expansion, which goes back to Feynman and Dyson:

$$Z(J) = \exp\left(\sum_n V_I\left(\frac{\mathrm{d}}{\mathrm{d}J_n}\right)\right) Z_{\text{free}}(J) \text{ with } Z_{\text{free}} = Z(0)\exp\left(-\frac{1}{2}J_m (M^{-1})_{mn} J_n\right) \tag{3.27}$$

Z_{free} is the generating function without interactions. This is a power series in λ_0. It is good, when λ_0 is small, so it is a **weak coupling expansion**.

3.1 Strong coupling expansion

We are going to rescale φ_n, namely $\varphi_n \mapsto \lambda_0^{-\frac{1}{4}} \varphi_n$.

$$Z(\lambda^{\frac{1}{4}} J) = \lambda^{-\frac{N^4}{4}} \int \prod \mathrm{d}\varphi_n \exp\left(-\frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\varphi M \varphi - \sum_n \left(\varphi_n^4 - J_n \varphi_n\right)\right) = \tag{3.28}$$

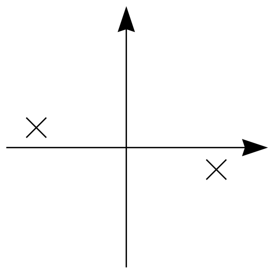
$$= \lambda_0^{-\frac{N^4}{4}} \exp\left(-\frac{1}{2}\frac{1}{\sqrt{\lambda_0}}\frac{\mathrm{d}}{\mathrm{d}J} M \frac{\mathrm{d}}{\mathrm{d}J}\right) \prod_n I(J_n) \text{ where } I(J) \equiv \int_{-\infty}^{+\infty} \mathrm{d}\varphi \exp(-(\varphi^4 - J\varphi)) \approx$$

$$\approx \exp\left(\frac{3}{4} J \left(\frac{J}{4}\right)^{\frac{1}{3}}\right) \tag{3.29}$$

This is a power series in $1/\sqrt{\lambda_0}$, but really this is a power series in $\|M\|/\sqrt{\lambda_0}$, which increases, if $N \mapsto \infty$. In standard Minkowski space we have $q^2 \equiv q \cdot q = {q^0}^2 - |\vec{q}|^2$. So the Feynman propagator is given by:

$$\Delta_F(x^1, x^2, x^3, x^4) = \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{\exp(-\mathrm{i}q \cdot x)}{q^2 - m^2 + \mathrm{i}\epsilon} = \int \frac{\mathrm{d}^3 q \, \mathrm{d}q^0}{(2\pi)^4} \frac{\exp(-\mathrm{i}q \cdot x)}{q_0^2 - (\sqrt{|\vec{q}|^2 + m^2})^2 + \mathrm{i}\epsilon} =$$

$$= \int \frac{\mathrm{d}^3 q \, \mathrm{d}q^0}{(2\pi)^4} \frac{\exp(-\mathrm{i}q \cdot x)}{\left(q_0 + \sqrt{|\vec{q}|^2 + m^2} + \frac{\mathrm{i}\epsilon}{2(q_0 - \sqrt{|\vec{q}^2 + m^2})}\right)\left(q_0 - \sqrt{|\vec{q}|^2 + m^2} + \frac{\mathrm{i}\epsilon}{2(q_0 + \sqrt{|\vec{q}|^2 + m^2})}\right)} \tag{3.30}$$

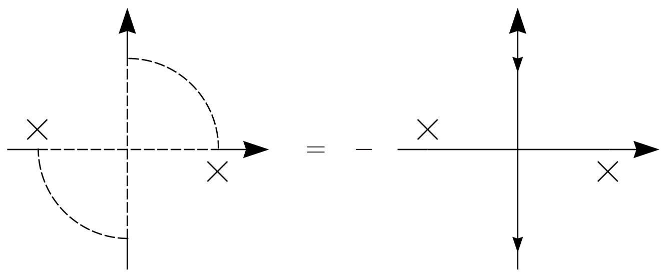


Physically: Positive frequency propagating forwards in time and negative frequency propagating backwards in time (particle and antiparticle)

$$\Delta_F = \int \frac{\mathrm{d}q^0}{(2\pi)^4} \frac{1}{2\sqrt{|\vec{q}|^2 + m^2}} \left(\frac{-1}{q_0 + \sqrt{|\vec{q}|^2 + m^2} + \mathrm{i}\epsilon'} + \frac{1}{q_0 - \sqrt{|\vec{q}|^2 + m^2} + \mathrm{i}\epsilon'}\right) \exp(-\mathrm{i}q_0 x_0) \tag{3.31}$$

$$x_0 \mapsto \infty : \frac{1}{2\sqrt{}} \exp(-\mathrm{i}\sqrt{} x_0),\; x_0 \mapsto -\infty : -\frac{1}{2\sqrt{}} \exp(\mathrm{i}\sqrt{} x_0) \tag{3.32}$$

One can now rotate the integral:



With $q_0 = \mathrm{i}q_4$ and $x_0 = \mathrm{i}x_4$ with $q_4,\, x_4 \in \mathbb{R}$ and the Euclidian metric $q^E \cdot x^E \equiv q^1x^1 + q^2x^2 + q^3x^3 + q^4x^4$ we obtain:

$$\Delta_F \mapsto -\int \mathrm{d}q_4 \, \frac{\exp(\mathrm{i}q \cdot x)}{-q_4^2 - |\vec{q}|^2 - m^2 + \mathrm{i}\epsilon} = \mathrm{i} \int \mathrm{d}q_4 \, \frac{\exp(\mathrm{i}q \cdot x)}{q \cdot q + m^2} \equiv \mathrm{i}\Delta_F^E(x^1, x^2, x^3, x^4) \tag{3.33}$$

So we obtain:

$$\Delta_F^{(M)}(\vec{x}, x^0) = \mathrm{i}\Delta_F^{(E)}(\vec{x}, -\mathrm{i}x_0) \tag{3.34}$$

This procedure is called Wick rotation (Euclidian postulate). $\tau(\ldots, x_k, \ldots)$ are the time-ordered Green's functions of QFT and $S(\ldots, x_k, \ldots)$ are the Euclidian Green's functions.

$$\tau(x_1, \ldots, x_k, \ldots) = \lim_{\varphi \mapsto \pi/2} S(\ldots, (\vec{x}_k, \exp(-\mathrm{i}\varphi)x_k^0), \ldots) \tag{3.35}$$

3.2 Reflection positivity

We are looking for a condition, so that Euclidian Green's functions can be continued to Minkowski space.

$$\phi(x) \sim \exp(Hx^4)\phi(x_0)\exp(-Hx^4) \tag{3.36}$$

This is all right in vector expectation values. The Euclidian time reflection is defined by $\theta(\vec{x}, x^4) = (\vec{x}, -x^4)$ and $\Theta\phi(x) = \overline{\phi(\theta x)}$. Θ is antilinear:

$$\Theta(\lambda F) = \overline{\lambda}\Theta(F) \text{ and } \Theta(FG) = (\Theta F)G + F(\Theta G) \tag{3.37}$$

Θ is the equivalent of hermitian conjugation in Minkowski space. With local support at **positive** Euclidian time ($x^4 > 0$):

$$F = \sum_{j=1}^{N} \int \mathrm{d}x_i \ldots \mathrm{d}x_j \, f_j(x, \ldots, x)\phi(x_i)\ldots\phi(x_j) \tag{3.38}$$

Then the reflection positivity $\langle(\Theta F)F\rangle > 0$ says:

$$\boxed{\sum_{j,k} \int \mathrm{d}x_1 \ldots \mathrm{d}x_j \, \mathrm{d}y_1 \ldots \mathrm{d}y_k \, \overline{f_j(x_1 \ldots x_j)} f_k(y_1 \ldots y_k) \langle \overline{\phi(\theta x_1)} \cdot \ldots \cdot \overline{\phi(\theta x_j)} \phi(y_1) \ldots \phi(y_k) \rangle > 0} \tag{3.39}$$

Reference: Osterwalder & Schrader, CMP <u>31</u> (1973), 83; <u>43</u> (1975), 281

Chapter 4

Gauge fields

We first want to consider electromagnetism as an example. Here we have a vector potential $A_\mu(x)$ and an antisymmetric field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Our Lagrangian density is given by:

$$\mathcal{L} = \frac{1}{4} F_{\mu\nu} F_{\mu\nu} + e j_\mu A_\mu \text{ with } j_\mu = \overline{\psi}\gamma^\mu \psi \tag{4.1}$$

$\delta/\delta A_\mu$ gives us the Maxwell equation $\partial_\mu F_\mu = e j_\nu$. The homogenious equations are given by $\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$. (This is an identity, not a field equation.) The equation $\partial_\mu F_\mu = e j_\nu$ only makes sense, if the current j_ν is conserved: $\partial_\nu j_\nu = 0$.

4.1 Abelian gauge fields

We have a gauge symmetry with arbitrary scalar function $\Lambda(x)$: $A_\mu(x) \mapsto A_\mu(x) + 1/e\partial_\mu \Lambda(x)$. The tensor $F_{\mu\nu}(x)$ is invariant under this transformation. If I demand gauge invariance, so the current $j_\nu(x)$ has to be conserved. $\psi(x)$ transforms in the way $\psi(x) \mapsto \exp(-\mathrm{i}\Lambda(x))\psi(x)$. This is an Abelian transformation group:

$$A_\mu \xrightarrow{\Lambda_1} A_\mu + \frac{1}{e}\partial_\mu \Lambda_1 \xrightarrow{\Lambda_2} A_\mu + \frac{1}{e}\partial_\mu \Lambda_1 + \frac{1}{e}\partial_\mu \Lambda_2 \tag{4.2}$$

$$A_\mu \xrightarrow{\Lambda_2} A_\mu + \frac{1}{e}\partial_\mu \Lambda_2 \xrightarrow{\Lambda_1} A_\mu + \frac{1}{e}\partial_\mu \Lambda_2 + \frac{1}{e}\partial_\mu \Lambda_1 \tag{4.3}$$

Both are the same. We now want to combine it, to show, that this is a group. We then get namely $A_\mu + 1/e\partial_\mu(\Lambda_1 + \Lambda_2)$. So we have $\Lambda_3 = \Lambda_1 + \Lambda_2$.

4.2 Non-Abelian gauge fields

The basic reference is [2]. The idea is to introduce an "isospin" index α, that we put on our field: A^α_μ. So we get:

$$F^\alpha_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial - \nu A^\alpha_\mu + g_0 f^{\alpha\beta\gamma} A^\beta_\mu A^\gamma_\nu \tag{4.4}$$

$f^{\alpha\beta\gamma}$ are the structure constants of a Lie algebra and g_0 is a bare coupling constant. Here we will only consider unitary groups $\mathrm{SU}(n)$ (but also other Lie groups are perfectly possible) and take g in the defining representation of the group G. With parameters ω^α and Hermitian generators λ^α we can write elements $g \in G$ as $g = \exp(\mathrm{i}\omega^\alpha \lambda^\alpha)$. The Lie algebra is given by:

$$\lambda^\alpha, \lambda^\beta = \lambda^\alpha\lambda^\beta - \lambda^\beta\lambda^\alpha = \mathrm{i} f^{\alpha\beta\gamma}\lambda^\gamma \tag{4.5}$$

By definition this is antisymmetric in α and β. Normalization is such, theta $\mathrm{Tr}(\lambda^\alpha\lambda^\beta) = 1/2\delta^{\alpha\beta}$. An example is the group SU(2); it topologically corresponds to the S^3-sphere. SU(2) is the group of 2×2 unitary matrices with determinant 1. The generators of this group are given by $\lambda^\alpha = 1/2\sigma^\alpha$ with $\alpha = 1, 2, 3$. The σ^α are the Pauli matrices:

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma^2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \text{ und } \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

The structure constants are here given by $f^{\alpha\beta\gamma} = \varepsilon^{\alpha\beta\gamma}$, where $\varepsilon^{\alpha\beta\gamma}$ is to totally antisymmetric Levi-Civita-Tensor. The Yang-Mills Lagrangian is now the following:

$$\mathcal{L} = \frac{1}{4} F^\alpha_{\mu\nu} F^\alpha_{\mu\nu} + j^\alpha_\mu A^\alpha_\mu \tag{4.7}$$

It has the same structure as the QED-Lagrangian. One can now calculate the field equations:

$$(D_\mu F_{\mu\nu})^\alpha \equiv \partial_\mu F^\alpha_{\mu\nu} + g_0 f^{\alpha\beta\gamma} A^\beta F^\gamma_{\mu\nu} = j^\alpha_\nu \tag{4.8}$$

So the equation has become nonlinear and the potential A_μ appears because of our non-Abelian gauge theory. Important is also the Bianchi identity:

$$(D_\lambda F_{\mu\nu})^\alpha + (D_\nu F_{\lambda\mu})^\alpha + (D_\mu F_{\nu\lambda})^\alpha = 0 \tag{4.9}$$

4.2.1 Lie-Algebra valued fields

We group the fields $A^\alpha_\mu(x)$ together with the generators λ^α, so that we get $A^\alpha_\mu(x)\lambda^\alpha \equiv A_\mu(x) \in \mathrm{Lie}(G)$. (So the index α vanishes.) So the Yang-Mills field strength also becomes Lie-Algebra valued, namely $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i} g_0 [A_\mu, A_\nu]$. The Lagrangian is now of the following form:

$$\boxed{\mathcal{L} = \frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F_{\mu\nu}) + 2\mathrm{Tr}(j_\mu A_\mu)} \tag{4.10}$$

In this way we have a non-Abelian gauge transformation, namely

$$A_\mu \mapsto g^{-1} \left(A_\mu + \frac{\mathrm{i}}{g_0} \partial_\mu \right) g = g^{-1} A_\mu g + \frac{\mathrm{i}}{g_0} g^{-1} \partial_\mu g \text{ with } g(x) \in G \tag{4.11}$$

Both of the terms are $\in G$. $F_{\mu\nu}$ transforms in a homogenious way: $F_{\mu\nu} \mapsto g^{-1} F_{\mu\nu} g$.

4.2.2 Non-Abelian gauge transformation

$$F_{\mu\nu} \xrightarrow{g_1} g_1^{-1} F_{\mu\nu} g_1 \xrightarrow{g_2} g_2^{-1} g_1^{-1} F_{\mu\nu} g_1 g_2 = (g_1 g_2)^{-1} F_{\mu\nu} (g_1 g_2) \tag{4.12}$$

$$F_{\mu\nu} \xrightarrow{g_2} g_2^{-1} F_{\mu\nu} g_2 \xrightarrow{g_1} g_1^{-1} g_2^{-1} F_{\mu\nu} g_2 g_1 = (g_2 g_1)^{-1} F_{\mu\nu} (g_2 g_1) \tag{4.13}$$

For a general Lie group one has $g_1 g_2 \neq g_2 g_1$.

4.3 Coupling to matter fields

We have a field ϕ_i, that transforms in the way $\phi_i \mapsto R_{ij}[g]\phi_j$ with the representation R_{ij}: $R_{ij}[g_2] R_{jk}[g_1] = R_{ik}[g_2 g_1]$. If the generating matrices are denoted v^α_{ij}, we get:

$$R_{ij}[\exp(\mathrm{i}\omega^\alpha \lambda^\alpha)] = (\exp(\mathrm{i}\omega^\alpha v^\alpha))_{ij} \text{ with } [v^\alpha, v^\beta] = \mathrm{i} f^{\alpha\beta\gamma} v^\gamma \tag{4.14}$$

The covariant derivative is defined by $(D_\mu \phi(x))_i = \partial_\mu \phi_i(x) + \mathrm{i} g_0 A^\alpha_\mu(x) v^\alpha_{ij} \phi_j(x)$. Under a gauge transformation with group elements g this transforms the following way: $(D_\mu \phi)_i \mapsto R_{ij}[g](D_\mu \phi)_j$.

$$(D_\mu \phi(x))_i = \partial_\mu \phi_i + \mathrm{i} g_0 R[A_\mu]_{ij} \phi_j \tag{4.15}$$

One can make a simple check, that this covariant derivative is, what we wanted. We therefore need to know, what $\partial_\mu g^{-1}$ is. This can be computed with $gg^{-1} = 1$ as follows:

$$gg^{-1} = 1 \Rightarrow \partial_\mu (gg^{-1}) = 0 \Rightarrow (\partial_\mu g) g^{-1} + g(\partial_\mu g^{-1}) = 0 \Rightarrow \partial_\mu g^{-1} = -g^{-1} (\partial_\mu g) g^{-1} \tag{4.16}$$

So we get:

$$\begin{aligned} D_\mu (g\phi_i) &= \partial_\mu (g\phi_i) + \mathrm{i} g_0 \cdot \left(g[A_\mu]_{ij} g^{-1} g \phi_j + \frac{\mathrm{i}}{g_0} g(\partial_\mu g^{-1}) g \phi_i \right) = \\ &= (\partial_\mu g)\phi_i + g(\partial_\mu \phi_i) + \mathrm{i} g_0 \cdot g[A_\mu]_{ij} \phi_j - g(\partial_\mu g^{-1}) g \phi_i = g(\partial_\mu \phi_i + \mathrm{i} g_0 [A_\mu]_{ij} \phi_j) = g(D\phi_i) \end{aligned} \tag{4.17}$$

4.3.1 Parallel Transporter

Let's first do the Abelian case. We consider a wave function of particle along a closed curve (in electromagnetic background field), we have to pick up following phase factor:

$$\psi \mapsto \exp\left(\mathrm{i}g_0 \int\limits_{\mathcal{C}} A_\mu \, \mathrm{d}^\mu\right) \psi \equiv U[\mathcal{C}]\psi \tag{4.18}$$

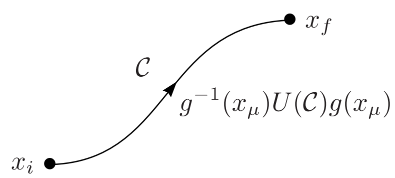
If the particle is a rest (total time T), we have $U = \exp(\mathrm{i}g_0 A_0 T)$. In a non-Abelian case we have to do this more carefully. Be $x_\mu(s)$ with $s \in [0, 1]$ the path. We define a **group** element $U(s)$ for portion of path from $x(0)$ to $x(s)$ through following differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}s} U(s) = \frac{\mathrm{d}x_\mu}{\mathrm{d}s} \mathrm{i}g_0 A_\mu U(s) \text{ with the boundary condition } U(0) = 1 \tag{4.19}$$

Formally we have:

$$U(s) = P \exp\left(\mathrm{i}g_0 \int\limits_0^s \mathrm{d}s' \, \frac{\mathrm{d}x_\mu(s')}{\mathrm{d}s'} A_\mu\right) \tag{4.20}$$

P is the so-called "path-ordering operator". In a power series P orders, so that $A(s'')A(s')$ for $s'' > s'$. U is the "parallel transporter". Under a gauge transformation ($A_\mu \mapsto g^{-1}(A_\mu + d_\mu)g$) we have $U(s) \mapsto g^{-1}(x_\mu(s))U(s)g(x_\mu(0))$.



For a closed contour and taking time we have:

$$W(\mathcal{C})|_{\mathcal{C} \text{ is closed}} \equiv \mathrm{Tr}(U(\mathcal{C})) \mapsto W(\mathcal{C}) \tag{4.21}$$

So it's invariant under the transformation. $W(\mathcal{C})$ is the so-called "Wilson loop operator".

4.3.2 Covariant derivative of a matter field

The naive definition is:

$$\lim_{\Delta \mapsto 0} \frac{\phi(x + \Delta) - \phi(x)}{\Delta} \tag{4.22}$$

But now we want to use:

$$\lim_{\Delta \mapsto 0} \frac{U(x, x + \Delta)^{-1}\phi(x + \Delta) - \phi(x)}{\Delta} \tag{4.23}$$

Under gauge transformations only fails $g(x)$:

$$g^{-1}(x)U^{-1}g(x + \Delta)g^{-1}(x + \Delta)\phi(x + \Delta) - g^{-1}(x)\phi(x) = g^{-1}(x)[U^{-1}\phi - \phi] \tag{4.24}$$

So we obtain:

$$\partial_\mu \phi - \frac{\mathrm{i}g_0 \Delta A \phi(x)}{\Delta} = (\partial_\mu - \mathrm{i}g_0 A)\phi(x) \equiv D\phi \tag{4.25}$$

4.4 Lattice gauge theory

An outstanding reference to this topic is [3]. So we now want to use the parallel transporter:

$$U(x, y) = P \exp\left(\mathrm{i}g_0 \int \mathrm{d}x_\mu \, A_\mu\right) \text{ with } U(x, y) \mapsto g^{-1}(x)U(x, y)g(y) \tag{4.26}$$

There are three ingredients now.

1.) **Hypercubic lattice**: That means, that we have nearest neighbours i, j and lattice spacing a.

2.) Variables: For every nearest neighbour there exists $U_{ij} \in G$ for the link between (ij). It is $U_{ji} \equiv U_{ij}^{-1}$.

$$U_{ij} \approx \exp\left(\mathrm{i}g_0 a A_\mu\left(\frac{x^i + x^j}{2}\right)\right) \tag{4.27}$$

The variables are group elements, which are associated with each link.

3.) The action can be written in a very simple form (the one, that Wilson chose):
The smallest Teil of the lattice is a so-called " plaquette", for which we want to use the symbol "□" from now on.

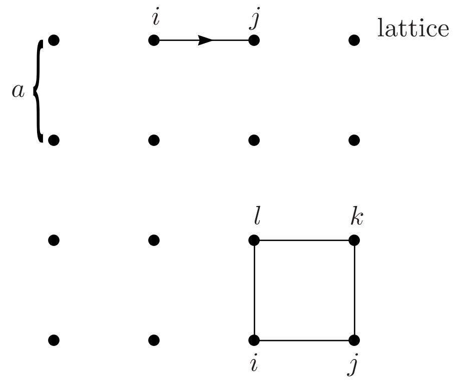
$$S_W = \sum_{\text{all } \square} S_\square \tag{4.28}$$

For one single plaquette we have

$$S_{ijkl} = \beta\left(1 - \frac{1}{n}\mathrm{Re}[\mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li})]\right) \tag{4.29}$$

where n is the dimension of the group matrices and β ist an inverse coupling constant. Because of $\mathrm{Tr}(U_{ij}U_{jk}U_{kl}U_{li})$ the action is also gauge invariant. An alternative notation ist given by:

$$S_W = \sum_{i=1}^{N}\sum_{1\le\mu<\nu\le4} S_{i\mu\nu} \text{ with } S_{i\mu\nu} = \beta\left(1 - \frac{1}{n}\mathrm{Re}[\mathrm{Tr}(U_{i,i+\hat\mu}U_{i+\hat\mu,i+\hat\mu+\hat\nu}U^{-1}_{i+\hat\nu,i+\hat\mu+\hat\nu}U^{-1})]\right) \tag{4.30}$$



This now has the following properties.

1.) quasi-local: We only have interactions with nearest neighbours.

2.) gauge invariance: $U_{ij} \mapsto g(x_i)^{-1}U_{ij}g(x_j)$
This transformation leaves the action invariant.

3.) For a lattice spacing $a \mapsto 0$ the theory reproduces the Yang-Mills action S_{YM}.

4.) The transfer matrix is positive definite and we have reflection positivity.

We now want to prove property 3. Therefore we consider a (12)-plaquette centered at x^μ.

$$\begin{aligned} S_\square \sim \beta\Bigg(1 - \frac{1}{n}\mathrm{Re}\Big\{\mathrm{Tr}\Big[&\exp\left(\mathrm{i}g_0 a A_1\left(x_\mu - \frac{a}{2}\delta_{\mu 2}\right)\right)\cdot\exp\left(\mathrm{i}g_0 A_2\left(x_\mu + \frac{a}{2}\delta_{\mu 1}\right)\right) \times \\ &\times\exp\left(-\mathrm{i}g_0 a A_1\left(x_\mu + \frac{a}{2}\delta_{\mu 2}\right)\right)\cdot\exp\left(-\mathrm{i}g_0 a A_2\left(x_\mu - \frac{a}{2}\delta_{\mu 1}\right)\right)\Big]\Big\}\Bigg) \end{aligned} \tag{4.31}$$

$$S = \frac{\beta g_0^2}{2n}\int \mathrm{d}^4x\,\frac{1}{2}\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu}) + \dots \tag{4.32}$$

Let's first discuss the Abelian part of these exponentials, where we make a Taylor expansion in the argument:

$$\exp\left[\mathrm{i}g_0 a\left(-\frac{a}{2}\partial_2 A_1 + \frac{a}{2}\partial_1 A_2 - \frac{a}{2}\partial_2 A_1 + \frac{a}{2}\partial_1 A_2\right)\right] = \exp\left(\mathrm{i}g_0 a^2(\partial_1 A_2 - \partial_2 A_1)\right) = \exp\left(\mathrm{i}g_0 a^2 F_{12}^{\mathrm{lin}}\right) \tag{4.33}$$

Now we are coming to the non-Abelian part:

a.) Appeal to gauge covariance: $F^{\text{lin}} \mapsto F^{\text{YM}}$

b.) We want to look at field at the same point, which are not commuting. So we have to use the Baker-Campbell-Hausdorff formula. Be A and B group elements:

$$\exp(A)\exp(B) = \exp(C) \text{ with } C = A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \dots \tag{4.34}$$

With this we obtain:

$$\exp(\mathrm{i}g_0 a A_1(x))\exp(\mathrm{i}g_0 a A_2(x))\exp(-\mathrm{i}g_0 a A_1(x))\exp(-\mathrm{i}g_0 a A_2(x)) =$$

$$= \exp\left(\mathrm{i}g_0 a(A_1 + A_2) - \frac{1}{2}g_0^2 a^2 [A_1, A_2] + \mathcal{O}(a^3)\right) \cdot \exp\left(-\mathrm{i}g_0 a(A_1 + A_2) - \frac{1}{2}g_0^2 a^2 [A_1, A_2] + \mathcal{O}(a^3)\right) = \tag{4.35}$$

$$= \exp(-g_0^2 a^2 [A_1, A_2]) = \exp(-g_0^2 a^2 F_{12}^{\text{nonlin}}) \tag{4.36}$$

So we get:

$$S_\Box = \beta\left[1 - \frac{1}{n}\text{Re}\Big[\text{Tr}\big(\exp(\mathrm{i}g_0 a^2 F_{12}^{\text{YM}} + \mathcal{O}(a^3))\big)\Big]\right] = \tag{4.37}$$

$$= \beta\left[1 - \frac{1}{n}\text{Re}\Big[\text{Tr}\big(\mathbb{1}_n + \mathrm{i}g_0 a^2 F_{12}^{\text{YM}} - \frac{g_0^2 a^4}{2}(F_{12}^{\text{YM}})^2 + \mathcal{O}(a^6)\big)\Big]\right] =$$

$$= \beta\left[1 - \frac{1}{n}\cdot\text{Tr}(\mathbb{1}_n) - \frac{1}{n}\text{Re}[\mathrm{i}g_0 a^2 \text{Tr}(F_{12}^{\text{YM}})] + \frac{g_0^2 a^4}{2n}\text{Tr}(F_{12}^{\text{YM}})^2 + \mathcal{O}(a^6)\right] = \frac{\beta g_0^2 \cdot a^4}{2n}\text{Tr}(F_{12}^2) + \mathcal{O}(a^6) \tag{4.38}$$

The expression $\sim a^2$ always cancels out, because the real part of an imaginary number is equal to zero. So one does not need $\text{Tr}(\lambda^a) = 0$.

$$\boxed{S_W = \frac{\beta g_0^2}{2n}\int \mathrm{d}^4x\, \frac{1}{2}\text{Tr}(F_{\mu\nu}F_{\mu\nu}) \text{ with } \beta \equiv \frac{2n}{g_0^2}} \tag{4.39}$$

β has to be identified with $2n/g_0^2$!

4.5 Path integral

So we have do define:

$$Z = \int \prod_l [\mathrm{d}U_l]\exp(-S_W) \tag{4.40}$$

where dU is the invariant group measure (see later). We now want to write up the measure concretely for SU(2). The group elements g can be written with the Pauli matrices:

$$g = g_0\mathbb{1} + \vec{a}\cdot\mathrm{i}\vec{\sigma} \in \text{SU}(2) \tag{4.41}$$

So the elements of the group are parameterized by four coordinates a_0, a_1, a_2 and a_3 with $a_0^2 + |\vec{a}|^2 = 1$. So the group is isomorphic to the S^3 (four dimensional sphere).

$$\int_{\text{SU}(2)} \mathrm{d}U = \frac{1}{\pi^2}\int \mathrm{d}^4a\, \delta(a^2 - 1) \tag{4.42}$$

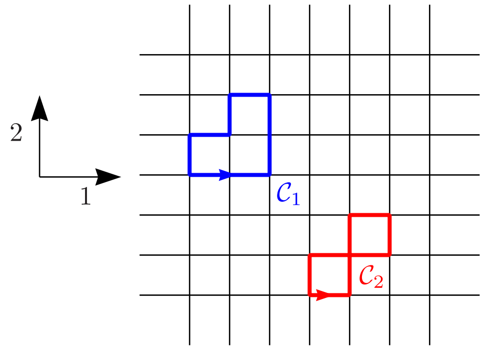
$$Z_{\text{SU}(2)} = \int \prod_l \left(\frac{1}{\pi^2}\int \mathrm{d}a_0^l\, \mathrm{d}a_1^l\, \mathrm{d}a_2^l\, \mathrm{d}a_3^l\, \delta((a^l)^2 - 1)\right) = \exp\left(-S\left[U_l \equiv a_\mu^l \sigma_\mu\right]\right) \tag{4.43}$$

This is the partition function for the SU(2) with l links. The expectation value of a function $H(u)$ ist now defined in the following way:

$$\langle H\rangle = \frac{1}{Z}\int \prod_l \mathrm{d}U_l\, H(U)\exp(-S_W) \text{ with } \langle\mathbb{1}\rangle = 1 \tag{4.44}$$

An example is the so-called Wilson loop:

$$W(\mathcal{C}) \equiv \text{Tr}\left(\prod_{l\in\mathcal{C}} U_l\right) \tag{4.45}$$



The whole theory is defined by the numbers $\{\langle W(\mathcal{C}_1)\rangle, \langle W(\mathcal{C}_2)\rangle, \langle W(\mathcal{C}_3)\rangle, \dots\}$ like for example the mass spectrum or confinement.
We are now interested in a group integration over a compact Lie group G. There exists a **unique** invariant measure $\mathrm{d}g$, which is called the Haar measure. What are the properties?

1.) Linearity:
For arbitrary functions f and h over G and constants a, $b \in \mathbb{C}$ the property of this measure is the following:

$$\int_G \mathrm{d}g\,(af(g) + bh(g)) = a\int_G \mathrm{d}g\, f(g) + b\int_G \mathrm{d}g\, h(g) \tag{4.46}$$

2.) Positivity:
If $\forall\ g\ f(g) > 0$, then $\int_G \mathrm{d}g\, f(g) > 0$.

3.) Left-invariance ("shift of variables"):

$$\int_G \mathrm{d}g\, f(g) = \int_G \mathrm{d}g\, f(\overline{g}g)\,\forall\,\overline{g} \in G \text{ fixed} \tag{4.47}$$

4.) Normalization:

$$\int_G \mathrm{d}g\, \mathbb{1} = 1 \tag{4.48}$$

on the S^3 there is now special point.

Now we are coming to the explicit formula; but we first have to introduce the metric tensor on the group G:

$$M_{ij} = \mathrm{Tr}(g^{-1}(\partial_i g)g^{-1}\partial_j g) \tag{4.49}$$

where G is parameterized by $\{\alpha_i\}$, so that $G = \{g(\alpha)|\alpha \in D \subseteq \mathbb{R}^n\}$ and $\partial_i g \equiv \partial/\partial\alpha_i g(\alpha)$. Then the measure is the following:

$$\int \mathrm{d}g\, f(g) = K\int_D \mathrm{d}^n\alpha\,|\det(M)|^{\frac{1}{2}} f(g(\alpha)) \tag{4.50}$$

This is analogue to general relativity. One can also prove "right-invariance". (But this is not a necessary condition.) Furthermore because of the uniqueness of the measure it is:

$$\int \mathrm{d}g\, f(g^{-1}) = \int \mathrm{d}g\, f(g) \tag{4.51}$$

4.6 Lattice gauge theory order parameters

In the Wilson lattice gauge theory we have the U_l (they are like the "spins" of the system) and β (which is a four-spin coupling constant). So the question is, if there exists spontaneous magnetization? The answer is no. We want to prove this:
For a given link (ij) we want to look at the following expectation value:

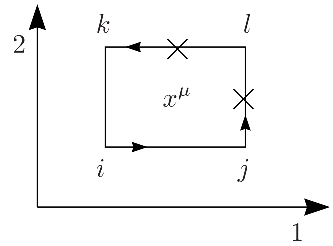
$$\langle U_{ij}\rangle = \frac{1}{Z}\int\left(\prod_l \mathrm{d}U_l\right) U_{ij}\exp(-S_W) \text{ with } S_W = \sum_{\text{all }\Box} S_\Box \tag{4.52}$$

In this theory we have gauge invariance for **any** link (kl). That means $U_{kl} \mapsto g_k U_{kl} g_l^{-1}$, where g_k, g_l are $\in G$. The g_k and g_l are completely arbitrary. The trick is now, to change all other links starting at site i in the following way: $U_{ik} \mapsto U_{ij}U_{ik}\mathbb{1}\ \forall\ k \neq j$. When we do that, we will find, that the action S_W becomes independent of the variable U_{ij}. Also the measure does not change. That means:

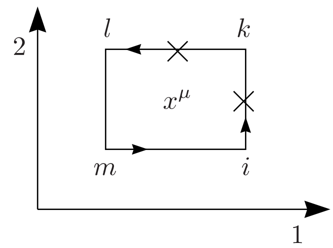
$$\langle U_{ij}\rangle = \frac{Z}{Z}\int \mathrm{d}U_{ij}\, U_{ij} = 0 \tag{4.53}$$

The U_{ij} has only none trivial representation of the group. This means, that the integral vanishes. Especially for the SU(2) we have:

$$\int \mathrm{d}^4 a_{ij}\,\delta()(a_{ij}\cdot\sigma) \tag{4.54}$$



$$\mathrm{Tr}(U_{ij}\mathrm{XX}U_{ki}) \mapsto \mathrm{Tr}(U_{ij}\mathrm{XX}U_{ki}U_{ji}) = \mathrm{Tr}(\mathrm{XX}U_{ki}) \tag{4.55}$$



$$\mathrm{Tr}(U_{mi}U_{ik}\mathrm{XX}) \mapsto \mathrm{Tr}(U_{mi}(U_{ij})^{-1}(U_{ij})U_{ik}\mathrm{XX}) = \mathrm{Tr}(U_{mi}U_{ik}\mathrm{XX}) \tag{4.56}$$

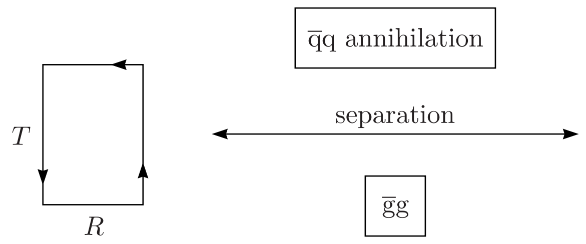
The Wilson action is completely independent on the value of the special link. This is a local transformation; it does not hold for the Ising model. Generally, the vacuum expectation value of any gauge invariant variable vanishes [4]. One of the conclusions of this is, that there does not exist a **local** order parameter like magnetization. One could even calculate the average plaquette:

$$P = \left\langle\left(1 - \frac{1}{n}\mathrm{Tr}(U_\Box)\right)\right\rangle \tag{4.57}$$

This is **always** non-zero. So it is not an order parameter. A **nonlocal** order parameter (for pure gauge theories) is the **Wilson loop**:

$$W(\mathcal{C}) = \left\langle \mathrm{Tr}\left(\prod_{l\in\mathcal{C}} U_l\right)\right\rangle \tag{4.58}$$

These are complex numbers, which only depend on the shape of the loops $\mathcal{C}$. The interpretation is, that $W(\mathcal{C})$ measures the response of gauge fields to an external quark source passing along $\mathcal{C}$. We want to consider a "timelike" loop:



It is action = energy × time.

$$W(R\times T)\sim \exp\left(-E(R)T\right) \tag{4.59}$$

$E(R)$ is the gauge field energy for static $q\overline{q}$ pair at separation R. For **linear** confinement we find $E(R)\sim KR$. That is, why the quarks don't exist separately, because one needs an infinitely large energy to separate them. So we find:

$$W(R\times T)\sim \exp(-KTR)=\exp(-K\cdot A_{\mathcal{C}}) \tag{4.60}$$

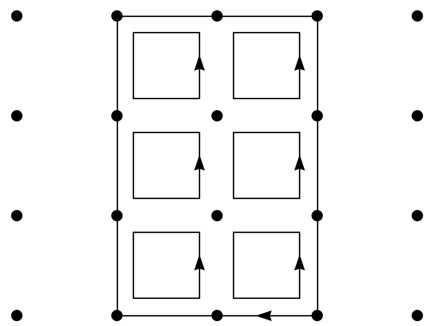
$A_{\mathcal{C}}$ ist the minimal area for a given curve $\mathcal{C}$. Generally, if $W(\mathcal{C})$ is $\propto \exp(-K\cdot A_{\mathcal{C}})$, quarks are linearly confined. If $W(\mathcal{C})$ is $\propto \exp(-kP_{\mathcal{C}})$, so have have no linear confinement. $P_{\mathcal{C}}$ is the perimeter of a given curve $\mathcal{C}$. The string tension K is the order parameter for confinement. It has also been "measured" from Regge trajectories (where α' is the Regge constant):

$$K\equiv \frac{\alpha'}{2\pi}\sim\frac{1\,\mathrm{GeV}^{-2}}{2\pi}\sim(400\,\mathrm{MeV})^2 \tag{4.61}$$

The area law can be established **analytically** in the strong confinement. For SU(N) theories in the **weak coupling** limit, the area law has only been found **numerically**.

4.6.1 Strong coupling expansion

One can find a very nice discussion in [5]. The idea is in our Boltzmann-like factor. We want to make a Taylor expansion in β (high temperature expansion in statistical mechanics). We have here a pure SU(N) Wilson lattice gauge theory. Let's now use a rectangular loop $\mathcal{C}=I\times J$.



$W(2,3)$

$$W(I,J)=Z^{-1}\int_{\mathcal{C}=\text{rectangle}(I,J)}\left(\prod_l \mathrm{d}U_l\right)\exp(-S_W)\frac{1}{N}\mathrm{Tr}\left(\prod_{l'\in\mathcal{C}}U_{l'}\right) \tag{4.62}$$

The shifted action can be written in the following form:

$$S=-\sum_{\text{all }\Box}\frac{\beta}{2N}\left(\mathrm{Tr}(U_\Box)+(\mathrm{Tr}(U_\Box))^*\right) \tag{4.63}$$

We first make an interesting observation, namely that $W\mapsto 0$ as $\beta\mapsto 0$, since $\int \mathrm{d}U\,U=0$ for every link of the curve $\mathcal{C}$. All loop variables need to be paired, in order to have a non-vanishing result. Let's look at the particular loop (2 × 3).

We need to pull down these plaquettes of the action. (For SU(2) the orientations are irrelevant.) We need following group integrals:

$$\int \mathrm{d}g=1 \text{ and } \int \mathrm{d}g\, g_{ab}g^{-1}_{cd}=\frac{1}{N}\delta_{ad}\delta_{bc} \tag{4.64}$$

We take the terms $\sim \beta^6$, because we have six tiles. For the leading term one obtains:

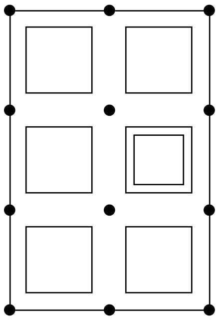
$$W(I,J) = \begin{cases} \left(\frac{\beta}{2N^2}\right)^{IJ} & \text{for} \quad N \geq 3 \\ \left(\frac{\beta}{N^2}\right)^{IJ} & \text{for} \quad N = 2 \end{cases} \tag{4.65}$$

Write $W(I,J) = \exp(-KA)$, wehre $A \equiv IJa^2$ and then one finds:

$$K = -\frac{1}{a^2} \log\left(\frac{\beta}{2N^2}\right) \text{ for } N \geq 3 \tag{4.66}$$

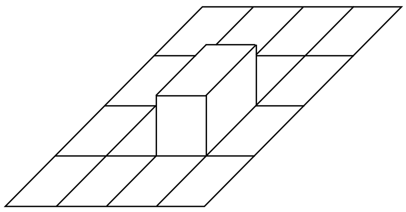
So we have found area law behaviour for $\beta \mapsto 0$, so in this part of the theory we have confinement. The same holds for loops of all shapes.
One gets higher order corrections by mainly two effects.

1.) One can have multipole tilings.



Here one gets an extra factor of β.

2.) The surface may not be minimal.

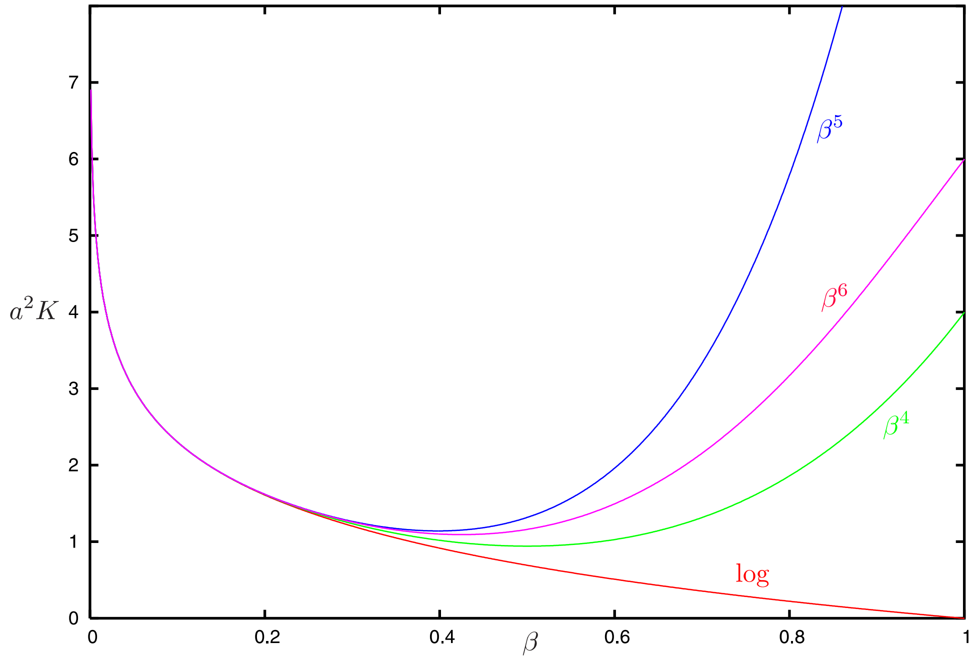


Here one obtains an extra factor of β^4.

$$\left(\frac{\beta}{2N^2}\right)^{IJ} \left(1 + \text{const.}\beta^4 IJ\right) = \left(\frac{\beta}{2N^2}\right)^{IJ} \exp(IJ\text{const.}\beta^4) = \exp\left[-IJ\left(-\log\left(\frac{\beta}{2N^2}\right) - \text{const.}\beta^4\right)\right] \tag{4.67}$$

In the SU(3) one gets the result:

$$a^2 K = -\log\left(\frac{\beta}{18}\right) + 4\left(\frac{\beta}{18}\right)^4 + 12\left(\frac{\beta}{18}\right)^5 - 10\left(\frac{\beta}{18}\right)^6 + \mathcal{O}(\beta^7) \tag{4.68}$$



This is a series with a finite radius of convergence, namely $\beta^{(K)}_{\mathrm{SU}(3)} \sim 5$. Around $\beta = 0$ the expansion is analytic. (This can be proven.) The weak coupling expansion (Feynman) is, at least, **asymptotic**. Let's consider the definition of G.H. Hardy: "A series $a_0 + a_1 x + a_2 x^2 + \ldots$ is said to be an asymptotic series for $f(x)$ near $x = 0$, if $f(x) = a_0 + a_1 x + \ldots + a_n x^n + \mathcal{O}(x^{n+1})$ for each little n and small x." Typically we are interested in $x \geq 0$, but this also hold for $x = r\exp(\mathrm{i}\zeta)$ with $-\pi + \delta \leq \zeta \leq \pi - \delta$. An asymptotic series can either be convergent or divergent. An infinity of functions can be represented by the same series, so for example:

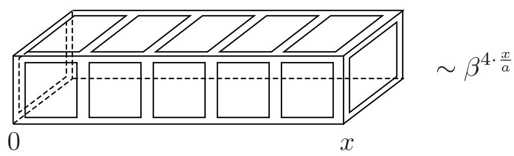
$$F(x) = f(x) + C\exp\left(-\frac{A}{x}\right) \tag{4.69}$$

In the standard-model is a term of the form $C\exp(-A/x)$. (It was found by t'Hooft.) Some other observable, that is of interest, would be the so-called mass gap. That can be obtained from the correlation of two plaquettes, which are at large distant:

$$\langle (\mathrm{Tr}(U_{\square\ \mathrm{at}\ 0}))\,(\mathrm{Tr}(U_{\square\ \mathrm{at}\ x}))\rangle \tag{4.70}$$

One contribution would precisely be the following:

This does not depend on the distance, so it is not the correlation.



This would be the lowest order contribution, where both plaquettes are connected.

$$\langle = \exp(-mx) = \exp\left(-ma\cdot\frac{x}{a}\right) \propto \beta^{4\cdot\frac{x}{a}} = \exp\left(4\log(\beta)\cdot\frac{x}{a}\right) \Rightarrow ma = -4\log(\beta) + \mathcal{O}(\beta^2) \tag{4.71}$$

Again, we have found a finite radius of convergence. For a strong-coupling lattice gauge theory, we have found linear confinement and a mass gap. (There are no such analytic results at weak coupling.) The leading terms give for SU(3):

$$\frac{ma}{\sqrt{ka^2}} \approx 4 \Rightarrow m \approx 4\sqrt{k} \sim 4\cdot 400\,\mathrm{MeV} \approx 1,6\,\mathrm{GeV} \tag{4.72}$$

So the lattice spacing cancels out. That would be the scale of the " glue ball". Why not say, that we have solves QCD? Because we have established analytically linear confinement and a mass gap. But it has not been proven, that this is QCD.

4.6.2 Intermezzo: The Wilson loop and the static quark potential

For a moment let us return to the Minkowski space time and continuum fields. (We'll take the signature (+,-,-,-).) Furthermore, we want to use the Hamiltonian formalism. The Yang-Mills action was in the continuum:

$$S = -\frac{1}{4}\int \mathrm{d}^4x\, F^a_{\mu\nu}F^{a\mu\nu} \text{ with } a = 1, \ldots, n_g \tag{4.73}$$

n_g is the dimension of the Lie algebra. For that case let us go to temporal gauge, namely $A_0 = 0$. The momenta conjugate to A^a_i (with $i = 1, 2, 3$) are given by:

$$\Pi^a_i = \frac{\delta S}{\delta(\partial_0 A^{ai}(x))} = -F^a_{0i} = F^{a0i} \equiv -E^a_i \tag{4.74}$$

This is a non-Abelian electric field. Then we can write down the Hamiltonian

$$H = \frac{1}{2}\int \mathrm{d}^3x\; (E^a_i E^a_i + B^a_i B^a_i) \tag{4.75}$$

with the non-Abelian magnetic field $B_i \equiv -1/2\varepsilon_{ijk}F^{jk}$. This generates the equations of motion, but **not Gauß**' law $D^iE_i = 0$, which has to be imposed as **initial condition**. (From equations of motion I have, that $\partial_t D^iE_i = 0$. So the condition is valid at all times.) So we do the canonical quantization in "coordinate representation", with a state vector represented by a wave functional $\psi[A^a(\vec{x})]$. The electric field becomes a differential operator:

$$E^a_i(\vec{x}) = \mathrm{i}\frac{\delta}{\delta A^{ai}(\vec{x})} \tag{4.76}$$

The coordinates are the gauge fields. A time independent gauge transformation with a parameter $\Lambda(x) \in G$ is represented by operator $R(\Lambda)$, so that $(R(\Lambda)\psi)[A] = \psi[\Lambda^{-1}(A + d)\Lambda]$. So the argument is shifted. This is a kind of displacement in group space. Also interesting are the infinitesimal transformations. For such a transformation we have with T^a as the generators of the Lie algebra: $\Lambda \equiv 1 + \mathrm{i}\omega^a T_a$. So one finds:

$$R(\Lambda) = 1 - \mathrm{i}\omega^a \frac{1}{g} \underbrace{D^iE^a_i}_{\text{generators}} + \mathcal{O}(\omega^2) \tag{4.77}$$

D^i is a covariant derivative in the coordinate space and E^a_i a derivative in the function space. Hence Gauß' law (i.e. the **absence** of external charges) is equivalent to **gauge invariance** of the wave functional. It's a facto, in order to **introduce external** charges, we are forced to consider **non-invariant** wave functionals. Suppose, that $\psi_\alpha[A]$ transforms according to the fundamental representation of Λ at some fixed space point $\vec{x}$. That means:

$$R[\Lambda]\psi_\alpha = \Lambda_{\alpha\beta}(\vec{x})\psi_\beta \tag{4.78}$$

Then $\operatorname{div}\vec{E}$ acts as follows: $D_iE^{ai} = gT^a\delta^{(3)}(\vec{x})$. This implies the presence of an external charge in the fundamental representation located at $\vec{x}$. (That is precisely, what the phenomenologists call a quark!)
Let's now return to the Euclidian lattice gauge theory. There we have a wave function $\psi[U]$, which depends on links b at $t = 0$. We use the temporal gauge, namely $U(b) = 1$ with $b \in Bt + 1, t$. We are looking at a time-independent gauge transformation $U'(\vec{y}, \vec{x}) = \Lambda^{-1}(\vec{y})U(\vec{y}, \vec{x})\Lambda(\vec{x})$. Then a state with a static quarks at point $\vec{x}$ and a static antiquark at point $\vec{y}$ transforms as follows:

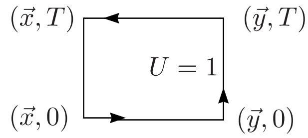
$$\psi_{\alpha\beta}[U'] = \Lambda_{\alpha\gamma}(\vec{x})\Lambda^{-1}_{\delta\beta}(\vec{y})\psi_{\gamma\delta}[U] \tag{4.79}$$

The states form a Hilbert space, which we want to call $\mathcal{H}_{\vec{x}\vec{y}}$. In this space we have a complete set of eigenvectors $\psi^{(n)}$, so that $H\psi^{(n)} = E_n\psi^{(n)}$. $\psi^{(0)}$ is the ground state. The lowest eigenvalue E_0 depends on the distance R between $\vec{x}$ and $\vec{y}$. Let's assume, that $\vec{x}$ and $\vec{y}$ are one of the axes of our hypercubic lattice. The static quark potential is then $V(R) \equiv E_0 = \min_{\mathcal{H}_{\vec{x}\vec{y}}} E(H)$. For a generic state $\Psi \in \mathcal{H}_{\vec{x}\vec{y}}$ (meaning, that there is some overlap with the ground state) we would actually do the following:

$$\langle\Psi\,|\exp(-TH)|\,\Psi\rangle = \sum_n |\langle\psi^{(n)}|\Psi\rangle|^2\exp(-TE_n) \xrightarrow[T\mapsto\infty]{} |\langle\psi^{(0)}|\Psi\rangle|^2\exp(-TV(R)) \tag{4.80}$$

Take the test function $\widetilde{\psi}_{\alpha\beta}[U] = U_{\alpha\beta}(\vec{x}, \vec{y})\Omega[U]$, where $\Omega[U]$ is the gauge invariant vacuum wave functional and $U_{\alpha\beta}(\vec{x}, \vec{y})$ is the parallel transporter between $\vec{x}$ and $\vec{y}$. With this test function we write down the path integral for the matrix element:

$$\langle\widetilde{\psi}|\exp(-TH)|\widetilde{\psi}\rangle = \frac{1}{Z}\int\prod_l(\mathrm{d}U_l)\,\exp(-S(U))U^\dagger_{\alpha\beta}(\vec{x}+T\hat{e}_4,\vec{y}+T\hat{e}_4)\mathbb{1}U_{\alpha\beta}(\vec{x},\vec{y})\mathbb{1} =$$
$$= \frac{1}{Z}\int\prod_l(\mathrm{d}U_l)\,\exp(-S(U))\mathrm{Tr}[U(\mathcal{C}_{R,T})] \equiv \langle W(\mathcal{C}_{R,T})\rangle \tag{4.81}$$



So we find:

$$\boxed{V(R) = -\lim_{T\mapsto\infty}\frac{1}{T}\log\langle W(\mathcal{C}_{R,T})\rangle} \tag{4.82}$$

This is the static quark potential.

4.6.3 Weak coupling expansion

We have our partition function:

$$Z = \int\left(\prod_l \mathrm{d}U_l\right)\exp\left[-\beta\sum_{\text{all }\square}\left(1-\frac{1}{n}\mathrm{Re}(\mathrm{Tr}(U_\square))\right)\right] \tag{4.83}$$

We are now interested in this quantity for $\beta \equiv 2n/g_0^2 \mapsto \infty$. (This means $g_0 \mapsto 0$. So we call this the weak coupling expansion.) Typically it is $U_\square \mapsto 1$. What we then have, is a settle point approximation around $U_l = 1$. Let's parametrize this a follows:

$$U_\square = \exp(\mathrm{i}\lambda^\alpha\omega^\alpha_\square) \text{ where } [\lambda^\alpha,\lambda^\beta] = \mathrm{i}f^{\alpha\beta\gamma}\lambda^\gamma \text{ and } \mathrm{Tr}(\lambda^\alpha\lambda^\beta) = \frac{1}{2}\delta^{\alpha\beta} \tag{4.84}$$

$\omega^\alpha_\square$ is considered to be small now:

$$1-\frac{1}{n}\mathrm{R}(\mathrm{Tr}(U_\square)) = \frac{1}{4n}\omega^\alpha_\square\omega^\alpha_\square + \mathcal{O}(\omega^3_\square) \tag{4.85}$$

So we get:

$$Z = \int\left(\prod_l \mathrm{d}U_l\right)\exp\left[-\frac{\beta}{4n}\sum_{\text{all }\square}\omega^\alpha_\square\omega^\alpha_\square + \mathcal{O}(\beta\omega^3_\square)\right] \tag{4.86}$$

The sum runs over the dimension of the Lie group, which is eight (and corresponds to eight gluons). For $\beta \mapsto \infty$, dominant contribution has "$\omega_\square$" $= \mathcal{O}(1/\sqrt{\beta}) = \mathcal{O}(g_0)$, so that in the integrand one obtains: $\exp[\mathcal{O}(1)+\mathcal{O}(g_0)]$. One way to the gauge fixing is $U_{\vec{x},\vec{x}+\hat{e}_4} = 1$ (like the temporal gauge).

4.6.4 Details on gauge fixing in lattice gauge theory

Let us consider $P(U)$, which is gauge invariant. We also define a δ-function over the group G by the following properties:

$$\int \mathrm{d}g\,f(g)\delta(g',g) = \int \mathrm{d}q\,f(g)\delta(g,g') = f(g') \text{ and } \delta(g,g') = \delta(g_0 g g_1^{-1}, g_0 g' g_1^{-1}) \tag{4.87}$$

Consider **one** link (i,j) and do **not** integrate over the corresponding link variable, but fix it to a particular arbitrary value g.

$$I(P,q) = \frac{1}{Z}\int\prod_l(\mathrm{d}U_l)\,\delta(U_{ij},g)P(U)\exp(-S(U)) \tag{4.88}$$

One gets the Green's function by integrating over g:

$$G(P) = \int \mathrm{d}g\,I(P,g) \tag{4.89}$$

This theory is still gauge invariant, so we make a gauge transformation, namely $U_{ij} \mapsto g_i U_{ij} g_j^{-1}$. Then we find, because P and S are invariant: $I(P,g) = I(P, g_i^{-1} g g_j)$. Since g_i and g_j are arbitrary, we conclude, that $I(P,g) = I(P)$! So for all g we have $I(P,g) \sim G(P)$. This can be generalized for all links of a tree T(no loops). So we have:

$$G(P) = \frac{1}{Z} \int \left(\prod_l U_l \right) \prod_{(ij)\in T} \delta(U_{ij}, g_{ij}) P(U) \exp(-S(U)) \tag{4.90}$$

g_{ij} are arbitrary – but fixed – elements of G. Now we return to the weak coupling expansion. Then approximately 1/4 of links are fixed an on the remaining links we expand it in the Lie algebra: $U_{ij} = 1 + \mathrm{i}\lambda^\alpha \omega_{ij}^\alpha + \mathcal{O}(\omega^2)$.

$$\omega_\Box^\alpha = \sum_{(i,j)\in\Box} \omega_{ij}^\alpha + \mathcal{O}(\omega^2) \text{ and } \mathrm{d}U_{ij} = \left[J + \mathcal{O}(\omega_{ij}^2)\right] \mathrm{d}^{n_g}\omega_{ij} \tag{4.91}$$

J is a constant and n_g is the dimension of Lie(G). Then we find, that our partition function looks a follows:

$$Z = K \int \prod_{(ij)\notin T} \mathrm{d}\omega_{ij} \, \exp\left[-\frac{1}{2}\beta\omega D^{-1}\omega + \mathcal{O}(\beta\omega^3) \right] \tag{4.92}$$

$\omega D^{-1}\omega$ is the propagator and $\mathcal{O}(\beta\omega^3)$ gives us the vertices. D^{-1} is a square matrix with $3N^4 n_g$ rows and columns.

$$Z = K' \left| \frac{D}{\beta} \right|^{\frac{1}{2}} (1 + \mathcal{O}(\beta^{-1})) = K' |D|^{\frac{1}{2}} \beta^{-3n_g \frac{N^4}{2}} \tag{4.93}$$

$K'|D||^{\frac{1}{2}}$ is a constant. Average plaquette:

$$P \equiv \left\langle 1 - \frac{1}{n}\mathrm{Tr}(U_\Box) \right\rangle = \frac{1}{6N^4} \frac{\partial}{\partial\beta} \log(Z) = \frac{n_g}{4\beta} + \mathcal{O}(\beta^{-2}) \tag{4.94}$$

It is

$$\frac{1}{6} = \frac{\text{Number of sites}}{\text{Number of plaquettes}} \text{ for } d = 4 \tag{4.95}$$

$$\frac{3n_g N^4 \cdot \frac{1}{2\beta}}{6N^4} = \frac{n_g}{4\beta} \tag{4.96}$$

4.7 Continuum limit

Removing the regulator (a), observables should approach the " physical values". On the lattice convenient to measure dimension**ful** quantities in lattice units. The correlation length is $\xi = (ma)^{-1}$ with m as the mass gap. Hence we are interested in the limit $\xi \mapsto \infty$. (In statistical mechanics this corresponds to a second order phase transition.) So we need to find points in coupling constants parameter space with **critical** behavior. This does **not** hold for strong coupling lattice gauge theory, because $\xi = -1/(4\log(\beta/18)) \mapsto 0$ for $\beta \mapsto 0$ in the strong coupling limit. Continuum theory will turn out to be in the opposite limit, namely $\beta \mapsto \infty$ (weak coupling expansion).

4.7.1 Renormalization concept

The bare couplings acquire a cutoff dependence, such that physical quantities have a finite limit, when the cutoff is removed. For a renormalizable theory, this procedure gives a **unique finite** limit for **all** observables. There are many possible renormalization schemes. Let us look for example at the QED. If m_e and the charge e are fixed, then one finds $m_0(\Lambda)$ and $e_0(\Lambda)$. However, this is more difficult in a confining theory (for example pure Yang-Mills). $K \sim M_{\text{mass gap fixed}} \mapsto g_0(1/a)$

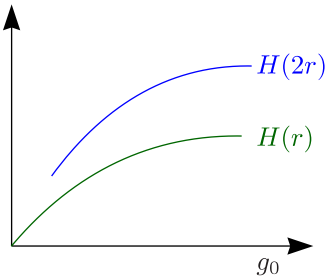
We want to simplify this discussion by using only one bare coupling constant g_0. We have dimensionless oberservables H defined over a physical scale r in the theory with cutoff a, so $H = H(r, a, g_0(a))$. In the continuum limit $a \mapsto 0$ H should retain a nontrivial r-dependence. This can only happen at the critical point, where g_0 has a special value, namely g_F and which involves physics even over a vast range of scales.

So we reduce a by the factor 2. If a is small enough, then the observables will not change much. So we get:

$$H\left(r, \frac{a}{2}, g_0\left(\frac{a}{2}\right)\right) = H(r, a, g_0(a)) + \mathcal{O}(a^2) \tag{4.97}$$

Because H is dimensionless, it can only depend on the ratio of r and $a/2$, so we can also write $H = H(2r, a, g_0(a/2))$. We are now going to make some assumptions.

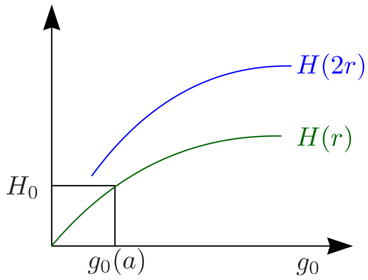
1.) For fixed values of r and a, there are two known functions with respect to g_0, namely $H(r, a, g_0)$ and $H(2r, a, g_0)$.



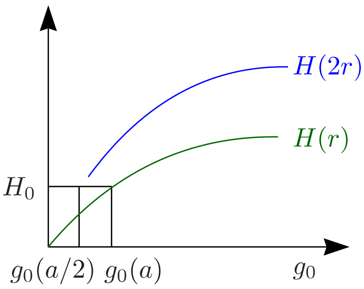
2.) In the continuum limit we have:

$$\lim_{a \mapsto 0} H(r, a, g_0(a)) = H_0(r) = H_0 \tag{4.98}$$

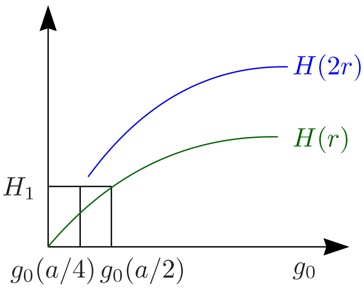
3.) We neglect all terms $\mathcal{O}(a^2)$.



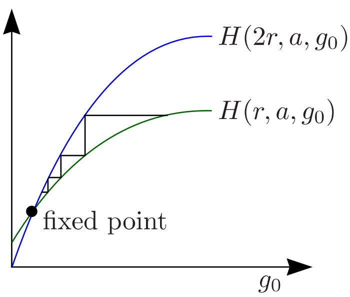
Then $H = (2r, a, g_0(a/2)) = H_0$ gives:



Next define $H_1 = H(r, a, g_0(a/2))$ and then $H(2r, a, g_0(a/4)) = H_1$.



When we make the lattice finer, so we get an UV fixed point.



At the renormalization group fixed point it is $H(r,a,g_F) = H(2r,a,g_F)$ and the physics is scale invariant ($\xi \mapsto \infty$). The renormalization prescription is to set the observable H at scale r to H_0 for all values of cutoff a.

$$0 = a\frac{\mathrm{d}}{\mathrm{d}a}H(r,a,g_0(a)) = a\frac{\partial H}{\partial a} + \gamma(g_0)\frac{\partial H}{\partial g_0} \text{ with } \gamma(g_0) \equiv a\frac{\mathrm{d}}{\mathrm{d}a}g_0(a) \tag{4.99}$$

For a fixed point g_F we have:

$$\lim_{a\mapsto 0} g_0(a) = g_F \Rightarrow \gamma(g_F) = 0 \tag{4.100}$$

γ is renormalization scheme dependent (here: choice of H and r), but its zeros must be universal, if the continuum limit is to be unique.
Consider strong-coupling lattice gauge theory again: So let us take the renormalization group description with K fixed.

$$K = \frac{1}{a^2}\log(3g_0^2(a)) + \mathcal{O}(g_0^{-2}) \tag{4.101}$$

$$0 = a\frac{\mathrm{d}K}{\mathrm{d}a} = -2K + \frac{2}{a^2 3g_0^2}\gamma^K(g_0) \Rightarrow \boxed{\gamma^K(g_0) = 3g_0\log(3g_0^2)} \tag{4.102}$$

For $g_0 \mapsto \infty$ we find, that $\gamma^K \neq 0$, so $g_0 = \infty$ ist not the fixed point! Other renormalization group prescription is the mass gap fixed.

$$m = \frac{4}{a}\log(3g_0^2) + \mathcal{O}(g_0^{-2}) \tag{4.103}$$

$$0 = a\frac{\mathrm{d}m}{\mathrm{d}a} = -m + \frac{8}{3ag_0}\gamma^m(g_0) \Rightarrow \gamma^m(g_0) = \boxed{\frac{3}{2}g_0\log(3g_0^2)} \tag{4.104}$$

Because of the additional factor 1/2 here, the functions γ are non-univsersal. Asymptotic freedom: weak coupling Yang-Mills theory. For one loop calculations one obtains $\gamma(g_0) = \mathcal{O}(g_0^3)$, so $g_0 = 0$ is a zero of the renormalization group function. It is UV attractive if and only if $\gamma = \gamma_0 g_0^3 + g_1 g_0^5 + \ldots$ with $\gamma_0 > 0$. For SU(N) gauge theory with n_f fermions in the fundamental representation we find:

$$\gamma_0 = \frac{1}{16\pi^2}\left(\frac{11}{3}n - \frac{2}{3}n_f\right) \text{ and } \gamma_1 = \left(\frac{1}{16\pi^2}\right)^2\left(\frac{34}{3}n^2 - \frac{10}{3}nn_f - n_f\frac{n^2-1}{n}\right) \tag{4.105}$$

We require, that $n_f < 11/2n$, because only then γ_0 is > 0. $\gamma(g_0)$ is scheme dependent, **but** not the coefficients γ_0 and γ_1. Consider two schemes $g_0(a)$ and $g_0'(a)$. So let us write the relation between them: $g_0' = g_0 + cg_0^3 + \mathcal{O}(g_0^5)$ and $g_0 = g_0' - cg_0'^3 + \mathcal{O}(g_0'^5)$.

$$\begin{aligned}\gamma'(g_0') = a\frac{\mathrm{d}}{\mathrm{d}a}g_0'(a) = \frac{\partial g_0'}{\partial g_0}\gamma(g_0) = (1+3cg_0^2)(\gamma_0 g_0^3 + \gamma_1 g_0^5) = (1+3cg_0'^2+\ldots)(\gamma_0 g_0'^3(1-cg_0'^2)^3 + \gamma_1 g_0'^5) = \\ = \gamma_0 g_0'^3 + \gamma_1\gamma_0'^5 + \mathcal{O}(g_0'^7)\end{aligned} \tag{4.106}$$

What is the renormalization group equation?

$$\frac{\mathrm{d}g_0}{\gamma_0 g_0^3 + \gamma_1 g_0^5} = \mathrm{d}\log(a) \Rightarrow g_0^{-2} = \gamma_0\log(a^{-2}\Lambda_0^{-2}) + \frac{\gamma_1}{\gamma_0}\log[\log(a^{-2}\Lambda_0^{-2})] + \mathcal{O}(g_0^2) \tag{4.107}$$

Check for $\gamma_1 = 0$:

$$\frac{-2\,\mathrm{d}g_0}{g_0^3} = \gamma_0 a^2\Lambda_0^2\frac{-2}{a^3\Lambda_0^2}\,\mathrm{d}a \Rightarrow \frac{\mathrm{d}g_0}{\gamma_0 g_0^3} = \mathrm{d}\log(a) \tag{4.108}$$

Asymptotic freedom means, that $g_0 \mapsto 0$ for $a \mapsto 0$ ($\pi/a \mapsto \infty$). Λ_0 is scheme dependent:

$$\log\left(\frac{\lambda_0^2}{\Lambda_0}\right) = \frac{2c}{\gamma_0} \tag{4.109}$$

We want to check this.

$$g_0^{-2} - g_0'^{-2} = \gamma_0\log\left(\frac{a^2\Lambda_0'^2}{a^2\Lambda_0^2}\right) \tag{4.110}$$

$$g_0^{-2} - g_0'^{-2} = g_0'^{-2}(1 - cg_0'^2)^{-2} - g_0'^{-2} = g_0'^{-2}(1 + 2cg_0'^2) - g_0'^{-2} = 2c \Rightarrow \log\left(\frac{\Lambda_0'^2}{\Lambda_0^2}\right) = \frac{2c}{\gamma_0} \tag{4.111}$$

In order to fix the number c we need the one loop calculation. For renormalized coupling from the three gluon vertex in the Feynman gauge with all legs carrying the same momentum $\mu^2 = r^{-2}$, the result of the calculation was:

$$\frac{\Lambda_R}{\Lambda_0^{\text{Wilson lattice}}} = \begin{cases} 57,\ldots & \text{for} \quad SU(2) \\ 83,\ldots & \text{for} \quad SU(3) \end{cases} \tag{4.112}$$

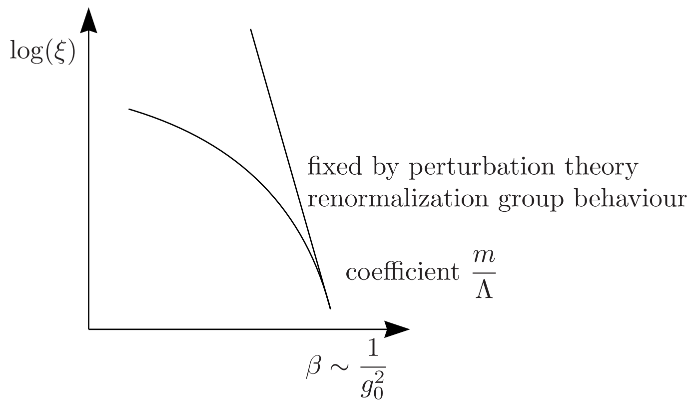
$$g_0^{-2} = \gamma_0 \log(a^{-2}\Lambda_0^{-2}) + \ldots \Rightarrow \exp(-g_0^2) = (a^2\Lambda_0^2)^{-\gamma_0} \tag{4.113a}$$

$$\Rightarrow a = \Lambda_0^{-1}(g_0^2\gamma_0)^{-\frac{\gamma_1}{2\gamma_0^2}} \exp\left(-\frac{1}{2\gamma_0 g_0^2}\right)\left[1 + \mathcal{O}(g_0^2)\right] \tag{4.113b}$$

If we multiply this with the mass gap, we obtain the inverse correlation length:

$$\xi^{-1} = \left(\frac{m}{\Lambda_0}\right)(g_0^2\gamma_0)^{-\frac{\gamma_1}{2\gamma_0^2}} \exp\left(-\frac{1}{2\gamma_0 g_0^2}\right) \tag{4.114}$$

$\exp(-1/(2\gamma_0 g_0^2))$ ist an essential singularity. That means, that all Taylor coefficients vanish. So this term we will never get in perturbation theory.



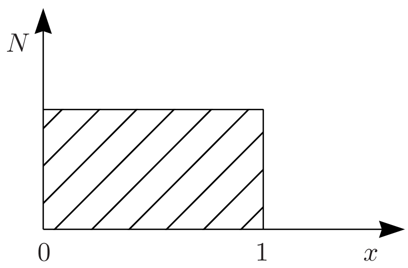
Λ_0 is universal and different correlation functions ξ_1 and ξ_2 allow us to calculate the ratio m_1/m_2, which is uniquely determined by the theory (no free parameters).

4.8 Monte Carlo

The problem we have on a finite lattice is a multipole integral (or sum if one performs the calculation in a discrete way). Let us take a lattice with the number of sites $= 10^4$. That means, that we have 40000 links, because we always work in four dimensions. The minimal "gauge" group G ist $\mathbb{Z}_2 = \{\mathbb{1}, -\mathbb{1}\}$. Then, the partition function is a sum over 2^{40000} configurations. We need, what is called **importance sampling**. The goal of the Monte Carlo approach is, to provide a small number of configurations, which are **typical**. Outgoing from some initial configuration, one gets with a pseudorandom change the probability density W of the configurations C, which approaches the distribution $W_{\text{eq}}(C) \propto \exp(-\beta S(C))$.

4.8.1 Intermezzo on "random numbers"

Given is a function $0 \leq f(x) \leq 1$ over $x \in [0, 1]$ and also a random number generator with uniform distribution W.



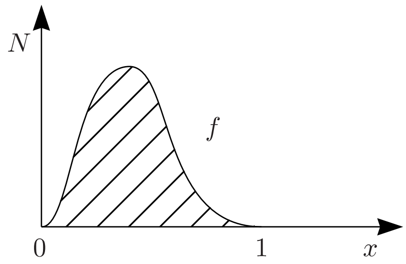
The **goal** is to obtain random numbers with the distribution function $W \propto f(x)$. The algorithm is as follows:

1.) Get random x_1, y_1.

2.) Calculate $f(x_1)$.

3.) Accept x_1, if $y_1 < f(x_1)$.

So one can really get a distribution of random numbers according to the given function f:



4.8.2 Heat bath algorithm

Successively touch a heat bath to all links of the lattice. We do a lattice gauge theory with the group $G = Z_2 = \{-\mathbb{1}, \mathbb{1}\}$ and consider one link l_1. The probability for it to be left in state ① is $P(1) = \exp(-\beta S(1))/[\exp(-\beta S(1)) + \exp(-\beta S(-1))]$. $\exp(-\beta S(1))$ is the action with all other links fixed. The algorithm now is:

1.) Get random number $r \in [0, 1]$ from uniform distribution.

2.) $P(1) > r$: $U'_{l_1} = 1$ or $P(1) < r$ $[P(-1) > -2]$: $U'_{l_1} = -1$

3.) Then consider another link l_2 etc. for the whole lattice. This constitutes one Monte Carlo iteration.

This is one step in a so-called **Markov chain**.
Now we have an ensemble consisting of an infinite number of field configurations with density $W[\varphi]$. We also have defined a measure $[\mathrm{d}\varphi]$. The canonical ensemble is then $W_{\mathrm{can}}[\varphi] \propto \exp(-\beta S[\varphi])$ (Boltzmann factor).

i.) Markov process: The update is a **stochastic** process $[\varphi] \mapsto [\varphi']$ with transition probability $P([\varphi'] \leftarrow [\varphi])$ going from $[\varphi]$ to $[\varphi']$. Then one has a change of the ensemble density:

$$W'[\varphi'] = \sum_{[\varphi]} P([\varphi'] \leftarrow [\varphi]) W[\varphi] \tag{4.115}$$

(We want to write this in a shorter form $W' = PW$ in future.) There are following conditions on P:

1.) unitarity: $\sum_{[\varphi']} P([\varphi'] \leftarrow [\varphi]) = 1$
The sum of all propabilities to reach each $[\varphi']$ is one.

2.) strong ergodicity: For any combination $[\varphi]$, $[\varphi']$: $P([\varphi'] \leftarrow [\varphi]) > 0$

3.) $\sum_{[\varphi]} W[\varphi] = 1$

ii.) Equilibrium approach: $PW_{\mathrm{can}} = W_{\mathrm{can}}$ $(*)$ (The canonical ensemble is a fix point of this transition probability or in other words an eigenvector of the transition probability with eigenvalue one.)

Provided, (2) holds, $(*)$ is a necessary and sufficient condition for any ensemble to approach the Boltzmann distribution.

Proof:

We consider two ensembles E with E' with their own densities $W[C]$ and $W'[C]$. We define the distance $\|E - E'\|$ in the following way:

$$\|E - E'\| = \sum_C |W[C] - W'[C]| \tag{4.116}$$

Suppose, that E' came from E by the above Monte Carlo process with property $(*)$:

$$W_{\text{can}}[C] = \sum_{C'} P([C] \leftarrow [C'])W_{\text{can}}[C'] \tag{4.117}$$

Then we have

$$W'[C] = \sum_{C''} P([C] \leftarrow [C''])W[C''] \tag{4.118}$$

and now we obtain:

$$\|E' - E_{\text{can}}\| \equiv \sum_C |W'[C] - W_{\text{can}}[C]| = \sum_C \Big|\sum_{C'} P([C] \leftarrow [C'])(W[C'] - W_{\text{can}}[W'])\Big| \leq \tag{4.119}$$

$$\leq \sum_{C,C'} P([C] \leftarrow [C'])|W[C'] - W_{\text{can}}[W']| = \|E - E_{\text{can}}\| \tag{4.120}$$

Remark:

1.) Practical algorithms use the sufficient condition of **detailed balance**:

$$P([C'] \leftarrow [C]) \exp(-\beta S(C)) = P([C] \leftarrow [C']) \exp(-\beta S(C')) \tag{4.121}$$

We can check this by doing the sum over C':

$$\sum_{C'} P([C'] \leftarrow [C]) \exp(-\beta S(C)) = \exp(-\beta S(C)) = \sum_{C'} P([C] \leftarrow [C']) \exp(-\beta S(C')) =$$

$$= \exp(-\beta S(C)) \tag{4.122}$$

2.) Heat bath $P([C'] \leftarrow [C]) \propto \exp(-\beta S(C'))$ (C independent)

4.8.3 Metropolis algorithm

The article in which this algorithm is developed is [6].

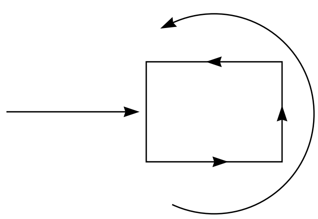
- Select trial link variable U' instead of U with probability distribution $\overline{P}(U',U)$, which is arbitrary but symmetric, namely $\overline{P}(U'U) = \overline{P}(UU')$.
- Calculate the tentative new action $S(U')$.
- If $S(U') < S(U)$, then I accept. If $S(U') > S(U)$, I conditionally accept with probability $\exp(S(U') - S(U))$.

Remark:

1.) Accept/Reject with random number r.

2.) $U' = R \cdot U$, where R is a random group element from a table.

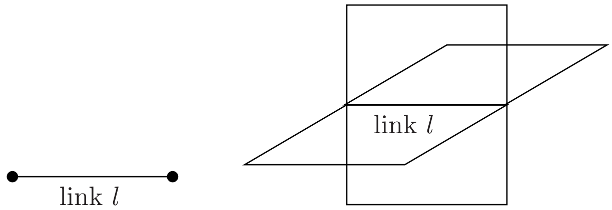
3.) Test more than once for U'



If the number of hits goes to ∞, we have the heat bath.

4.8.4 Example: Heat bath for SU(2)

We consider a link l. For such a link there are six plaquettes involved with products $\overline{U}_p$ with $p = 1, \ldots, 6$.



Then the probability is:

$$\mathrm{d}W[U] \propto \mathrm{d}U \exp\left(\frac{1}{2}\beta \mathrm{Tr}\left[U \sum_p \overline{U}_p\right]\right) \tag{4.123}$$

Recall, that SU(2) is homeomorph to the hyper sphere $S^3 = \{a_0 + \mathrm{i}\vec{\sigma}\cdot\vec{a} | a_0^2 + |\vec{a}|^2 = 1\}$. So we have $\mathrm{d}U \sim \mathrm{d}^4 a\, \delta(a^2 - 1)$ and also:

$$\sum_p \overline{U}_p = k\overline{U} \text{ with } \overline{U} \in \mathrm{SU}(2) \text{ and } k^2 = \det\left(\sum_p \overline{U}_p\right) \tag{4.124}$$

We use the invariance of the measure to absorb $\overline{U}$ (Haar measure):

$$\mathrm{d}W[U\overline{U}^{-1}] \sim \mathrm{d}U \exp\left(\frac{1}{2}\beta k \mathrm{Tr}(U)\right) \sim \mathrm{d}^4 a\, \delta(a^2 - 1)\exp(\beta k a_0) \tag{4.125}$$

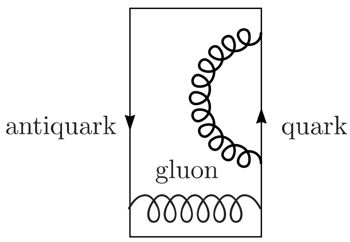
We find such a U and replace the link on the lattice by a product $U'_f = U\overline{U}^{-1}$.

4.9 Monte Carlo results

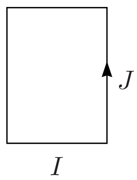
"The window of opportunity" is small, because the number of sites is finite and $a \mapsto 0$. On the other hand, the asymptotic freedom experimentally holds already at rather low momentum transfer (2 GeV). If the string tension persists in the weak coupling limit, we should verify the behaviour

$$a^2 K = \frac{K}{\Lambda_0^2}(\gamma_0 g_0^2)^{-\frac{\gamma_1}{\gamma_0}} \exp\left(-\frac{1}{\gamma_0 g_0^2}\right)(1 + \mathcal{O}(g^2)) \tag{4.126}$$

($\exp(-1/(\gamma_0 g_0^2))$ has an essential singularity.) The technical problem is, that the Wilson loop $\langle W[C]\rangle$ has also perimeter contributions (for example from divergent self energies and gluon exchange).



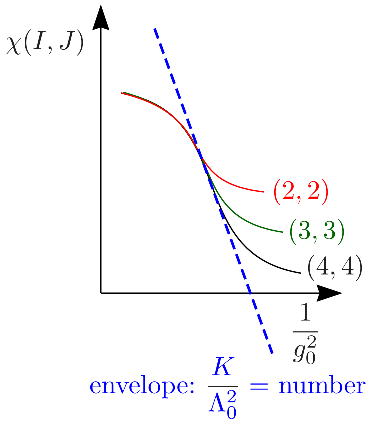
We consider rectangle loops $W(I, J)$.



Greutz has now found the following ratio:

$$\chi(I, J) = -\ln\left(\frac{W(I,J)W(I-1,J-1)}{W(I,J-1)W(I-1,J)}\right) \tag{4.127}$$

As I and $J \mapsto \infty$ for $W \propto \exp[-Ka^2 IJ - E_0 a 2(I - J)]$, where the first term is the area and the second the perimeter, one obtains $\chi = Ka^2$. If on the other hand I and J are fixed and a, $g_0(a) \mapsto 0$, then I find $K = \mathcal{O}(g_0^2)$, so this is a **regular** behaviour.



The results, that were obtained, are the following. For SU(3) we found $\Lambda_0 \sim 6 \cdot 10^{-3}\sqrt{K}$ and $\Lambda_R = 84\Lambda_0$, so $\Lambda_R \sim 0,5\sqrt{K}$. Regge slope α' [GeV^2]:

$$\alpha' = \frac{1}{2\pi K} \Rightarrow \sqrt{K} \sim 400\,\text{MeV} \Rightarrow \Lambda_R \sim 200\,\text{MeV} \tag{4.128}$$

For the mass gap (pure SU(3) lattice gauge theory) one obtains $m/\Lambda_0 \sim 300$. Also one finds the deconfinement temperature $T_C/\Lambda_0 \sim g_0$, hence $m \sim 2\sqrt{K} \sim$ GeV and $T_C \sim 1/2\sqrt{K} \sim 200\,\text{MeV}$. In a four-dimensional SU(3) Yang-Mills theory there is no other way, to calculate the mass gap (which is formed by a massive glue ball). For a SU(∞) Yang-Mills theory one can find $T_C \approx 1/2\sqrt{K}$.

Chapter 5

Large-N-reduced Models

5.1 Prelimaries

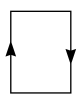
We start from pure SU(3) Yang-Mills theory. The clever thing, that war realized by t'Hooft in 1974, is, that there is a free parameter, namely SU(N). So we expand around $N = \infty$. Specifically, take $N \mapsto \infty$ with g^2N fixed. Perhaps "quarkless QCD" is defined as the SU(∞) theory plus $\mathcal{O}(1/N^2)$ corrections. So for $N = \infty$ there is a vast simplification of Feynman diagrams, only the planar diagrams remain. Assuming, there is confinement at $N = \infty$, then it really seems that QCD is the theory that comes out. Then many phenomenological properties can be explained, as long as there is confinement.

- All dynamics for the mesons are understood (for example Zweig's rule).
- Many properties of the glue balls are explained.
- The baryons arise as solitons. (These are the famous Skyrmions.)
- It also explains the η' problem.

Large N Yang-Mills theory has **factorization**. For e.g. Wilson loops we have

$$\frac{\langle W_1 W_2 \dots W_n \rangle}{\langle W_1 \rangle \langle W_2 \rangle \dots \langle W_n \rangle} = O\left(\frac{1}{N^{2n-2}}\right) \tag{5.1}$$

Because of factorization, Schwinger-Dyson equations are loop equations for Wilson loops (i.e. **closed** system).

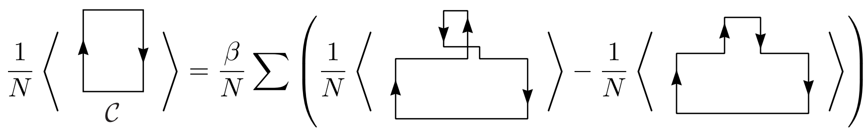


$\mathcal{C}$

stands for

$$\mathrm{Tr}\left(\prod_{l \in \mathcal{C}} U_l\right) \tag{5.2}$$

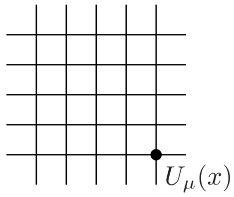
The Schwinger-Dyson equations on the lattice are:



(5.3)

5.2 The Eguchi-Kawai model

Eguchi and Kawai (EK) realized, that because of factorization the SU(∞) lattice, gauge theory is equivalent to a single-point model [7]. That means, that the Schwinger Dyson equations are the same. We make a reduction R to go from lattice gauge theory to the Eguchi-Kawai model.



We have an infinite lattice and on this lattice the link $U_\mu(x)$. The reduction is, to replace $U_\mu(x)$ by a matrix $V_\mu \in \mathrm{SU}(N)$ für $N \mapsto \infty$.

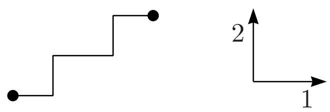
$$W(\mathcal{C}) = \mathrm{Tr}\left(U_\mu(x)U_\nu(x+\hat{\mu})\ldots\right) \mapsto W_R(\mathcal{C}) \equiv \mathrm{Tr}\left(V_\mu V_\nu \ldots\right) \tag{5.4}$$

In the Eguchi-Kawai model calculate:

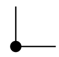
$$\langle W_R(\mathcal{C})\rangle_{\mathrm{EK}} = \frac{1}{Z_{\mathrm{EK}}} \int \prod_{\mu=1}^{4} \mathrm{d}V_\mu \, W_R(\mathcal{C}) \exp\left(-S_{\mathrm{EK}}\right) \text{ with } S_{\mathrm{EK}} = \beta \sum_{\mu,\nu} \mathrm{Tr}(\mathbb{1} - V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger) \tag{5.5}$$

Then the numbers $\langle W_R(\mathcal{C})\rangle_{\mathrm{EK}}$ are in the large-N-limit equal to $\langle W(\mathcal{C})\rangle_{\mathrm{LGT}}$. For lattice gauge theory in the Schwinger Dyson equations open strings vanish.

$$\left\langle \prod_{\mathrm{open}} U_\mu(x) \right\rangle = 0 \text{ because of gauge invariance} \tag{5.6}$$



On the Eguchi-Kawai model the same holds for $\langle \prod_{\mathrm{open}} U_\mu \rangle$. This vanishes by the $\mathrm{U}(1)^4$ invariance $V_\mu \mapsto \exp(\mathrm{i}\vartheta_\mu)V_\mu$, provided there is no spontaneous symmetry braking.



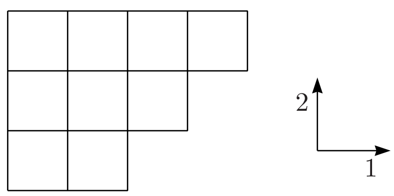
But in the original Eguchi-Kawai model there **is** spontaneous symmetry braking as $\beta \mapsto \infty$.

5.3 The twisted Eguchi-Kawai (TEK) model

One only needs six complex phase factors to save the $\mathrm{U}(1)^4$.

$$z_{\mu\nu} \equiv \exp\left(2\pi\mathrm{i}\frac{n_{\mu\nu}}{N}\right) \in \mathbb{Z}_n \subset \mathrm{SU}(N) \text{ with } n_{\mu\nu} = -n_{\nu\mu} \in \mathbb{Z} \tag{5.7}$$

$$W_{\mathrm{TEK}}(\mathcal{C}) = \prod_{\mu,\nu} (z_{\mu\nu})^{\text{Number of plaquettes } \mathcal{C}} \mathrm{Tr}(V_\mu V_\nu \ldots) \tag{5.8}$$



$$\langle W_{\mathrm{TEK}}(\mathcal{C})\rangle = \frac{1}{Z_{\mathrm{TEK}}} \int \prod_{\mu} \mathrm{d}V_\mu \, W_{\mathrm{TEK}}(\mathcal{C}) \exp(-\beta S_{\mathrm{TEK}}) \text{ with } S_{\mathrm{TEK}} = \beta \sum_{\mu,\nu} \mathrm{Tr}\left(\mathbb{1} - z_{\mu\nu} V_\mu V_\nu V_\mu^\dagger V_\nu^\dagger\right) \tag{5.9}$$

One can prove, that for large N it holds: $\langle W_{\mathrm{TEK}}(\mathcal{C})\rangle = \langle W(\mathcal{C})\rangle_{\mathrm{LGT}}$.

5.3.1 Symmetric twist

The corresponding reference is [8]. This works for $N = L^2$ with $L \in \mathbb{N}$ and $L \mapsto \infty$. The twist is now the following, namely

$$n_{\mu\nu} = \begin{cases} L & \text{for} \quad \nu > \mu \\ -L & \text{for} \quad \nu < \mu \end{cases} \tag{5.10}$$

The effective space-time volume is $V = L^4 = N^2$ (hypercube with periodic boundary conditions).

5.3.2 Hot twist

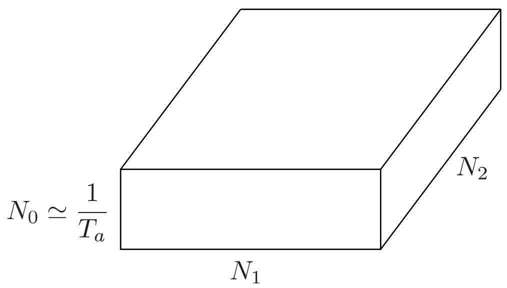
First, we note the reference: [9].

$$N = N_0^2 \cdot 2K(4K^2 - 1) \text{ with } N_0,\, K \in \mathbb{N} \tag{5.11}$$

The idea is, that N_0 is an arbitrary number and K goes to infinity in steps. With $P = 2K+1$ and $M = 2K-1$ the full twist matrix is:

$$(n_{\mu\nu}) = \begin{pmatrix} 0 & -2K^2PM & 2KPM & 2K^2PM \\ 2K^2PM & 0 & 2KP & PM \\ -2KPM & -2KP & 0 & 2KM \\ -2K^2PM & -PM & -2KM & 0 \end{pmatrix} \tag{5.12}$$

The effective space-time volume is equal to $V = N_0N_1N_2N_3$ with $N_1 = N_02K(2K-1)$, $N_2 = N_0(4K^2-1)$ and $N_3 = N_02K(2K+1)$.



With a Monte Carlo simulation it is able to get $\sqrt{K}/\Lambda_0$ and T_c/Λ_0.

Chapter 6

Chiral fermions (Weyl fermions)

6.1 Euclidian chiral gauge theory in continuum

Be γ_μ the Dirac matrices with the properties $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $(\gamma_\mu)^\dagger = \gamma_\mu$, and $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. We want to consider a theory with left-handed fermions:

$$\boxed{P_-\psi(x) = \psi(x),\ \overline{\psi}(x)P_+ = \overline{\psi}(x) \text{ with } P_\pm = \frac{1}{2}(\mathbb{1}_4 + \gamma_5)} \tag{6.1}$$

$\psi(x)$ and $\overline{\psi}(x)$ are independent four-component spinor fields (Dirac spinors). Hence, in the path integral one has to separately integrate over them. The gauge transformations work as follows:

$$\psi'(x) = R[\Lambda(x)]\psi(x) \text{ and } \overline{\psi}'(x) = \overline{\psi}(x)R[\Lambda(x)]^{-1} \tag{6.2}$$

We want to consider two chiral gauge theories, namely:

1.) non-Abelian group: $G = \mathrm{SU}(2)$, $R_{\text{left}} = \underline{2} + \underline{2}$ with $f = 1, 2$

2.) Abelian group: $G = \mathrm{U}(1)$, $R_{\text{left}} = (1, 1, 1, 1, 1, 1, 1, 1, -2)$ with $f = 1, \ldots, q$

So we have $R[\Lambda] = \Lambda^q$. The action depends on the gauge and the Weyl fields:

$$S[A, \overline{\psi}, \psi] = \int \mathrm{d}^4x \left(\frac{1}{4g_0^2} F^a_{\mu\nu}F^a_{\mu\nu} + \overline{\psi}_f\gamma_\mu D_\mu\psi_f\right) \tag{6.3}$$

with the covariant derivative $D_\mu \equiv \partial_\mu + A^a_\mu(x)R[T^a]$. The gauge fields are real. The theory is well defined; there are no anomalies. The fermion propagator is given by:

$$\langle\psi_f(x)\overline{\psi}_f(y)\rangle_{g=0} = -\mathrm{i}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\,\frac{\gamma_\mu p_\mu}{p^2} P_+ \exp(\mathrm{i}p(x-y)) \tag{6.4}$$

In particular we would like to calculate an effective action:

$$\exp(-S_{\text{eff}}[A]) = \int [D\psi]_{\text{left}}[D\overline{\psi}]_{\text{left}} \exp(A, \overline{\psi}, \psi) \tag{6.5}$$

If the action changes because of a gauge transformation, there is a gauge anomaly. As a result of this, the action as to be regularized somehow.

6.2 Lattice gauge theory with Wilson fermions

We consider Dirac fields at sites of the lattice: $\psi(x)$ with $x = a(n_0, n_1, n_2, n_3)$ whereas $n_\mu \in \mathbb{Z}$. $\psi(x)$ can be written as a Fourier transform:

$$\psi(x) = \int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4p}{(2\pi)^4}\,\widetilde{\psi}(p)\exp(\mathrm{i}px) \tag{6.6}$$

The UV-cutoff is given by $p = \pi/a$ (one edge of the Brillouin zone). That is why we work on the lattice. We have link variables $U(x,\mu)$, which transform in the way $U(x,\mu) \mapsto \Lambda(x)U(x,\mu)\Lambda(x+a\hat{\mu})^{-1}$. Furthermore we define a forward- and backward difference operator:

$$\nabla_\mu \psi(x) = \frac{R[U(x,\mu)]\psi(x+a\hat{\mu}) - \psi(x)}{a}, \quad \nabla^*_\mu \psi(x) = \frac{\psi(x) - R[U(x-a\hat{\mu},\mu)]^{-1}\psi(x-a\hat{\mu})}{a} \tag{6.7}$$

These operators transform covariantly under gauge transformations. Now we are able to write up the lattice action:

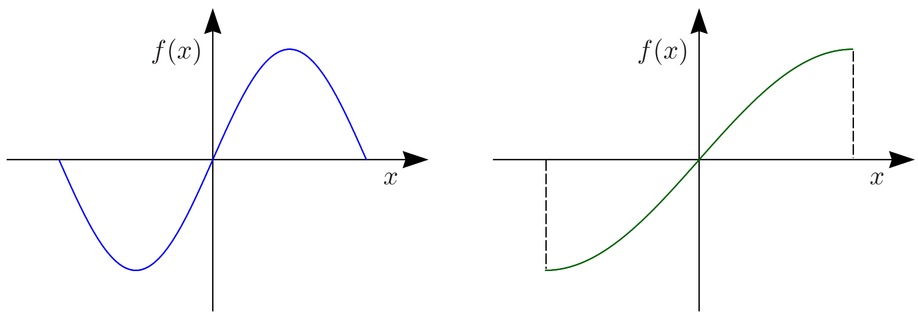
$$S[U,\overline{\psi},\psi] = a^4 \sum_x \overline{\psi}(x) D_W \psi(x) \tag{6.8}$$

D_W is the Wilson operator (something like a Dirac operator). It is a second-derivative operator, which is on the lattice given by:

$$D_W = \frac{1}{2}\sum_\mu (\gamma_\mu(\nabla_\mu + \nabla^*_\mu) - a\nabla^*_\mu\nabla_\mu) \tag{6.9}$$

The correction $a\nabla^*_\mu\nabla_\mu$ is small on a fine lattice. We can now calculate the propagator.

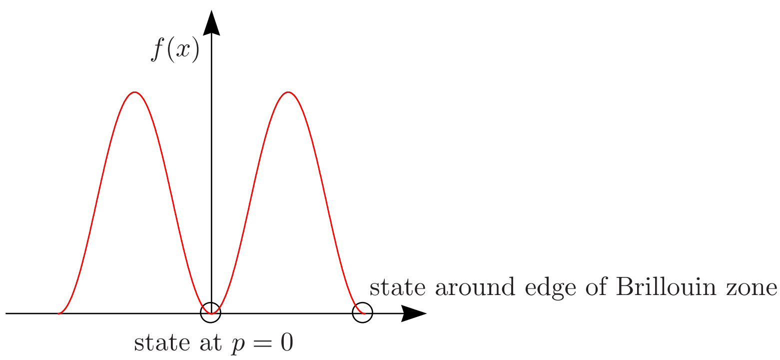
$$\langle\psi(x)\overline{\psi}(y)\rangle = -\mathrm{i}\int_{-\pi/a}^{\pi/a} \frac{\mathrm{d}^4 p}{(2\pi)^4}\, \frac{\gamma_\mu \mathring{p}_\mu + \mathrm{i}\frac{a}{2}\widehat{p}^2}{(\mathring{p})^2 + \frac{a^2}{4}(\widehat{p}^2)^2}, \quad \mathring{p}_\mu \equiv \frac{\sin(ap_\mu)}{a}, \quad \widehat{p}_\mu \equiv \frac{\sin\left(\frac{a}{2}p_\mu\right)}{\frac{a}{2}} \tag{6.10}$$



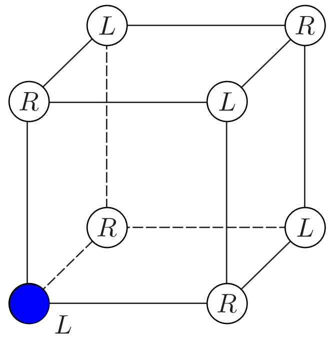
For $a = 0$ we obtain the propagator of the continuum theory. The additional terms $\sim \widehat{p}$ come from the Wilson term. The denominator only vanishes for $p = 0$. Without the Wilson term we have additional singularities. The theory is alright for vector-like theories (like QCD).
For chirally constrained fields

$$P_+ D_W P_- = P_+ \gamma_\mu \frac{1}{2}(\nabla_\mu + \nabla^*_\mu) \tag{6.11}$$

appears in the action. So the Wilson term cancels out and in the denominator of the propagator we only have $(\mathring{p})^2$.



The state at the edge of the Brillouin zone is a so-called uninvited guest (doubler). This problem is called the "fermion doubling problem" (2^4 fermions in $d = 4$).



The real problem is that the uninvited guests are not exact copies. We have eight left-handed and eight right-handed fermions, so the theory is vector-like. This leads us to the Nielsen-Ninomiya no-go theorem (1981): "There are always unphysical poles in the chiral fermion propagator with $P_\pm$ as long as the lattice Dirac operator is local." (So it is major problem, not a problem because of the Wilson operator.) An unexpected solution came years later.

6.3 Chiral lattice fermions

Instead of $P_+ D_W P_-$ we consider a modified Dirac operator D, that is **postulated** to satisfy the so-called Ginsparg-Wilson relation (1982): $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$. So the relation becomes nonlinear. We also have the hermiticity $D^\dagger = \gamma_5 D \gamma_5$. The standard continuum theory does not satisfy the Ginsparg-Wilson relation! We can now write up the action:

$$S[U, \overline{\psi}, \psi] = \int \overline{\psi} D \psi \tag{6.12}$$

We want to consider infi"chiral" transformations: $\psi \mapsto \psi + \epsilon\widehat{\gamma}_5\psi$ and $\overline{\psi} \mapsto \overline{\psi} + \epsilon\overline{\psi}\gamma_5$ with $\widehat{\gamma}_5 \equiv \gamma_5(1 - aD)$. It should be checked that the action is invariant under this transformations:

$$\delta S = \epsilon \int [\overline{\psi}\gamma_5 D\psi + \overline{\psi} D\gamma_5(1 - aD)\psi] = \epsilon \int \overline{\psi}[\gamma_5 D + D\gamma_5 - aD\gamma_5 D]\psi = 0 \tag{6.13}$$

With $\widehat{\gamma}_5$ satisfying $(\widehat{\gamma}_5)^\dagger = \widehat{\gamma}_5$, $(\widehat{\gamma}_5)^2 = 1$ and $\gamma_5 D = -D\widehat{\gamma}_5$ we define projectors:

$$\widehat{P}_\pm \equiv \frac{1}{2}(\mathbb{1}_4 \pm \widehat{\gamma}_5) \text{ and } P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \gamma_5) \tag{6.14}$$

The "chirally" constrained left-handed field has:

$$\boxed{\widehat{P}_- \psi = \psi \text{ and } \overline{\psi} P_+ = \overline{\psi}} \tag{6.15}$$

That is the solution of the fermion doubling problem, provided there exists an operator D! The explicit realization of this operator was found by Neuberger in 1998:

$$D[U] = a^{-1}(1 - V[U]), \quad V^\dagger = \gamma_5 V \gamma_5, \quad \widehat{\gamma}_5 = \gamma_5 V, \tag{6.16a}$$

$$V \equiv X(X^\dagger X)^{-\frac{1}{2}} \equiv \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{\pi} X(t^2 - X^\dagger X)^{-1}, \quad X \equiv 1 - aD_W \tag{6.16b}$$

The operator D depends on the gauge fields and also the projectors. D^{-1} has no unphysical poles. Furthermore it is local and unitary.

6.4 Chiral lattice gauge theory

The goal is to compute correlation functions of gauge invariant operators $\varphi_n(x)$ (this can for example be Wilson loops). We want to calculate this as a path integral:

$$\langle \varphi_1(x_1) \dots \varphi_n(x_n) \rangle = \frac{1}{Z} \int [DU][D\psi]_{\text{left}}[D\overline{\psi}]_{\text{left}} \varphi_1 \dots \varphi_n \exp(-S) \tag{6.17}$$

Introduce an orthonormal basis $v_j(x)$, we have:

$$\psi(x) = \sum_j v_j(x) c_j \tag{6.18}$$

with Grassmann coefficients c_j. Then we have:

$$[D\psi]_{\text{left}} = \prod_j \mathrm{d}c_j \tag{6.19}$$

With a similar basis $\overline{v}_k(x)$ holds:

$$[D\overline{\psi}]_{\text{left}} = \prod_k \mathrm{d}c_k \tag{6.20}$$

Since $\widehat{\gamma}_5$ depends on $[U]$, so does $[D\psi]_{\text{left}}$. The related phase ambiguity in the measure needs to be fixed. This has been done (but completely satisfying only for Abelian chiral gauge theories).

Bibliography

[1] R. P. Feynman, "*Space-time approach to nonrelativistic quantum mechanics*," Rev. Mod. Phys. **20**, 367 (1948).

[2] C. N. Yang and R. L. Mills, "*Conservation of isotopic spin and isotopic gauge invariance*," Phys. Rev. **96**, 191 (1954).

[3] K. G. Wilson, "*Confinement of Quarks*," Phys. Rev. D **10**, 2445 (1974).

[4] S. Elitzur, "*Impossibility Of Spontaneously Breaking Local Symmetries*," Phys. Rev. D **12**, 3978 (1975).

[5] J. M. Drouffe and J. B. Zuber, "*Strong Coupling And Mean Field Methods In Lattice Gauge Theories*," Phys. Rept. **102**, 1 (1983).

[6] N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller and E. Teller, "*Equation of state calculations by fast computing machines*," J. Chem. Phys. **21**, 1087 (1953).

[7] T. Eguchi and H. Kawai, "*Reduction Of Dynamical Degrees Of Freedom In The Large N Gauge Theory*," Phys. Rev. Lett. **48**, 1063 (1982).

[8] A. Gonzalez-Arroyo and M. Okawa, "*The Twisted Eguchi-Kawai Model: A Reduced Model For Large N Lattice Gauge Theory*," Phys. Rev. D **27**, 2397 (1983).

[9] F. R. Klinkhamer and P. van Baal, "*Hot Twists For The Single Point Model Of Large N QCD*," Nucl. Phys. B **237**, 274 (1984).