Mitschrieb zur Vorlesung: Riemannsche Geometrie

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Mitschrieb der Vorlesung Riemannsche Geometrie
von Herrn Provatdozent Dr. Baues im Wintersemester 2005/2006
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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Kapitel 1

Introduction

Be $M \subseteq \mathbb{R}^n$, $x \in \mathbb{M}$ and $\{T_xM = \{x'(0)|c : I \mapsto M$, c a smooth curve $c(0) = x\}$ with $I = (-1, 1)$ and $c'(0) = \partial/\partial_t|_{t=0}c(t) \in \mathbb{R}^n$. The tangential space T_xM ist a k-dimensional vector subspace of $\mathbb{R}^n$. The tangent bundle is defined by

$$TM = \bigcup_{x \in T} T_xM = \bigcup_{x \in M} \{x\} \times T_xM$$

and the natural projection is given by

$$\pi : TM \mapsto M,\ v = (x, v_x) \in \{x\} \times T_xM \mapsto x \in M$$

$\pi^{-1}(x) = \{x\} \times T_xM$ is a vector bundle over a manifold, the so-called „bundle projection"(bundle map).

Proposition:

TM is a submanifold of $\mathbb{R}^n \times \mathbb{R}^k$ and π: $TM \mapsto M$ is a smooth map.

Proof:

It is

$$TM = \bigcup_{x \in M} \{x\} \times T_xM \subseteq \mathbb{R}^n \times \mathbb{R}^n$$

by definition. For $x \in M$ and $W \subseteq \mathbb{R}^n$ f: $W \mapsto \mathbb{R}^{n-k}$, where $0 \in \mathbb{R}^{n-k}$ is a regular value for f, we receive $M \cap W = f^{-1}(0)$, because M is a manifold. The result of $T_xM = \mathrm{d}f|_x^{-1}(0)$ with $x \in W$ is:

$$TM \cap (W \times \mathbb{R}^n) = (f\,\mathrm{d}f)^{-1}(0) = F^{-1}(0)$$

where F: $W \times \mathbb{R}^n \mapsto \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$ and $(x, v) \mapsto (f(x), \mathrm{d}f|_x(n))$. It remains to show, theta $0 \in \mathbb{R}^{n-k} \times \mathbb{R}^{n-k}$ is a regular value.

1.) $\mathrm{d}f|_x$: $\mathbb{R}^{n-k} \mapsto \mathbb{R}^{n-k}$ is surjective $\forall\ x \in M \cap W$

2.) $\partial_{(0,v)}|_{(x,v)}\mathrm{d}f = \mathrm{d}f|_x(v)$

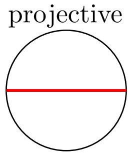
(1)+(2) show, that $\mathrm{d}F|_{(x,b')}$ is surjective $\forall\ x \in M \cap W$. Then 0 is a regular value for F.

Exercise:

Use the „local chart definition" of submanifolds, to prove, that $TM \subseteq \mathbb{R}^n \times \mathbb{R}^n$ is a submanifold.
Be $M \subseteq \mathbb{R}^n$ and $N \in \mathbb{R}^{n'}$ submanifolds. Then $M \times N \subseteq \mathbb{R}^n \times \mathbb{R}^{n'}$ is a submanifold with $TM \times TN = T(M \times N)$ and $T_xM \times T_{x'}N = T_{(x,x')}M \times N$.

1.1 Quotients

Be $\mathbb{P}^n\mathbb{R} := \{$all 1-dimensional vector spaces in $\mathbb{R}^{n+1}\} = S^n/\{+1,-1\}$ the so-called real projective space, where S^n is the unitary sphere in n dimensions. This is a topological space, but is it also a submanifold? For $n = 1$ the projective space is given by $\mathbb{P}^1\mathbb{R} = S^1/\{+1,-1\} \simeq S'$.



(homoemorphic projection)

$\mathbb{P}^n\mathbb{C}$ is the complex projective space with $\mathbb{P}^n\mathbb{C} = S^{2n+1}/S'$, where $S' = \{z \in \mathbb{C}||z| = 1\} \subseteq \mathbb{C}^\star$ subgroup. $S^{2n+1} = \{(z_1, z_2, \ldots, z_{2n+1})| \sum |z_i|^2 = 1\}$ is $\subseteq \mathbb{C}^{n+1}$.

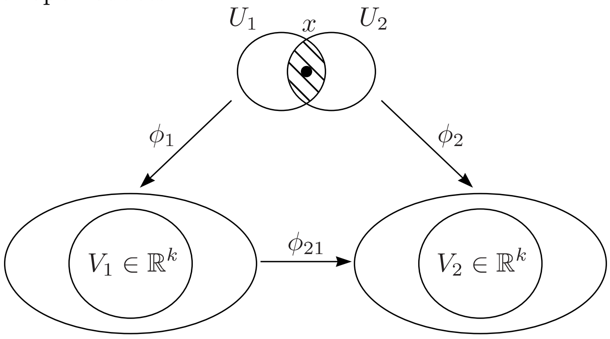
Example:

For $n = 1$, $\mathbb{P}^1\mathbb{C} = S^3/S' \xrightarrow{\simeq} S^2$.

1.1.1 Final remark on submanifolds

Lemma:

Let $M \subseteq \mathbb{R}^n$ be a k-dimensional submanifold. Let (U_1, ϕ_1) and (U_2, ϕ_2) be local charts for M. Define the coordinate change $\phi_{21} = \phi_2 \circ \phi_1^{-1}$: $\phi_1(U_1 \cap U_2) \mapsto \phi_2(U_1 \cap U_2)$, where $U_1 \cap U_2 \subseteq \mathbb{R}^k$. Then ϕ_{21} is a diffeomorphism of open subets in $\mathbb{R}^k$.



Proof:

$\phi_{21} = \phi_2 \circ \phi_1^{-1}$ is a composition of smooth maps and so ϕ_{21} is smooth. $\phi_{21} = \phi_1 \circ \phi_2^{-1}$, $\phi_2(U_1 \cap U_2) \mapsto \phi_1(U_1 \cap U_2)$ is a smooth inverse of ϕ_{21} and so ϕ_{21} is a diffeomorphism.

1.2 Abstract manifolds

Let X be a topological space. The space is called HAUSDORFF, when $\forall$ x, y exists $Ux = x$ and $Uy = y$ with $U_x \cap Uy = \varnothing$, where $x \neq y$ are open subsets of X (separable neughbourhood).

Example:

$\mathbb{R}^n$ is a HAUSDORFF space. X is countable at infinity $\Leftrightarrow$ $\exists$ a countable generating set for the topology on X. ($B_{1/2}(q)$, $q \in \mathbb{Q}^n$ generate the topology.)
We are concerned about the definition of the smooth manifolds. Be $M = (X, \overline{A})$, where X is a topological space, which is Hausdorff and has a countable basis. $\overline{A}$ is a smooth structure on X. This means, that $\mathcal{A}$ is a maximal smooth atlas. $\mathcal{A}$ is a smooth atlas with the charts (U, ϕ), where ϕ is a homeomorphism ϕ: $U \xrightarrow{\simeq} V \subseteq^0 \mathbb{R}^K$. If we have two charts (U_1, ϕ_1), $(U_2, \phi_2) \in A$, we receive $\phi_2 \circ \phi_1^{-1}$: $\phi_1(U_1 \cap U_2) \mapsto \phi_2(U_1 \cap U_2) \subseteq^0 \mathbb{R}^K$. This is a **smooth diffeomorphism** and M is called a k-dimensional smooth manifold (differentiable manifold).

Example:

Submanifolds are smooth manifolds.

Proposition:

Let $M \subseteq \mathbb{R}^n$ be a n-dimensional submanifold. Then M is a smooth manifold.

Proof:

M is a subset of $\mathbb{R}^n$ and has a subspace topology inherited from $\mathbb{R}^n$. So the open sets U in M are of the form $U = M \cap \widetilde{U}$, where $\widetilde{U}$ is open in $\mathbb{R}^n$. Then M is a Hausdorff topology space and also has a countable basis for the topology. By definition of the submanifold M has a smooth atlas with charts taking values in $\mathbb{R}^K$ (compare with Lemma in last lecture). □

1.2.1 Real projective spaces

$\mathbb{P}^n\mathbb{R} = \{$all 1-dimensional subspaces in $\mathbb{R}^{n+1}\}$. So $x \in \mathbb{P}^n\mathbb{R}$ can be described by a $n+1$-tupel of numbers, namely $x = (x_1 : \ldots : x_{n+1}) = (\lambda x_1 : \ldots : \lambda x_{n+1})$ with $\lambda \in \mathbb{R} \setminus \{0\}$ and $(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$. The x_i are homogeneous coordinates for the line $x = \{\lambda(x_1, \ldots, x_{n+1}) | \lambda \in \mathbb{R}\}$.

Remark:

For the projective space we have $\mathbb{P}^n\mathbb{R} = S^n/\{+1, -1\}(= \mathbb{R}^{n+1} \setminus \{0\}/n)$, $(x_1, \ldots, x_{n+1}) \mapsto (x_1 : x_2 : \ldots : x_{n+1})$. $\mathbb{P}^n\mathbb{R}$ with the quotient topology with respect to π is Hausdorff and also has a countable basis (exercise for definition of quotient topology).
Now let's look for affine charts for projective space:

$U_i = \{(x_1 : \ldots : x_i : \ldots : x_{n+1}) | x_i \neq 0\}$ with $i = 1, \ldots, n+1$

U_i is an open set in $\mathbb{P}^n\mathbb{R}$. So we define a chart

$$\phi_i : U_i \mapsto \mathbb{R}^n,\ (x_1 : \ldots : x_i : \ldots, x_{n+1}) \mapsto \left(\frac{x_1}{x_i}, \ldots, \frac{\hat{x}_i}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_{n+1}}{x_i}\right)$$

The inverse chart is given by $\psi_i = \phi_i^{-1}$: $\mathbb{R}^n \mapsto \mathbb{P}^n\mathbb{R}$, $(y_1, \ldots, y_n) \mapsto (y_1 : \ldots : y_{i-1} : 1 : y_i : \ldots : y_n)$. Therefore ϕ_i is a homoeomorphism. ϕ_i are called **affine charts**. All of them cover the projective space:

$$\mathbb{P}^n\mathbb{R} = \bigcup_{i=1}^{n+1} U_i$$

We now want to compute the coordinate changes for this atlas:

$U_1 \cap U_j = \{(x_1, \ldots, x_{n+1}) | x_i \neq 0,\ x_j \neq 0\}$ with $j < i$

Then ist follows, that $\phi_i(U_1 \cap U_j) = \{(y_1, \ldots, y_j, \ldots, y_n) | y_j \neq 0\}$. We now have to compute:

$$\phi_j \circ \phi_i^{-1}(y_1, \ldots, y_n) = \phi_j(y_1 : \ldots : y_{i-1} : 1 : y_i : \ldots : y_n) = \left(\frac{y_1}{y_j}, \ldots, \frac{\hat{y}_j}{y_j}, \ldots, \frac{y_{i-1}}{y_j}, \frac{1}{y_j}, \ldots, \frac{y_n}{y_j}\right)$$

This is smooth on that set, because $x_j \neq 0$. So $\phi_j \circ \phi_i^{-1}$ is a diffeomorphism.

1.3 Smooth functions and maps in manifolds

Definition:

Be M a smooth manifold and $U \subseteq M$ an open subset. Furthermore be f: $U \mapsto \mathbb{R}$ a continuous function. f is **called smooth** at $x \in U$, if there exists a chart ϕ: $U \mapsto V \subseteq \mathbb{R}^K$ with parametrization $\psi = \phi^{-1}$: $V \mapsto U$ such that $f \circ \psi$: $V \mapsto \mathbb{R}$ is smooth.

Remark:

This definition does not depend on the choice of chart in M. In fact, if $f \circ \phi_1^{-1}$: $V_1 \mapsto \mathbb{R}$ is smooth, than this implies, that $f \circ \phi_2^{-1}$: $V_2 \mapsto \mathbb{R}$ near $\phi_2(x)$, where (U_1, ϕ_1) and (U_2, ϕ_2) are charts near $x \in U_1 \cap U_2$. The reason for this is, that the coordinate change is smooth.

$f \circ \psi_2 = (f \circ \psi_1) \circ (\phi_1 \circ \psi_2)$

$f \circ \psi_1$ is smooth and $\phi_1 \circ \psi_2$ is a smooth coordinate change. So $f \circ \psi_2$ is smooth.

Definition:

A continuous function f ($\in C^0(U)$) is called smooth, if it is smooth at all $x \in U$. $C^\infty(U)$ is a set (ring) of all smooth functions.

Lemma:

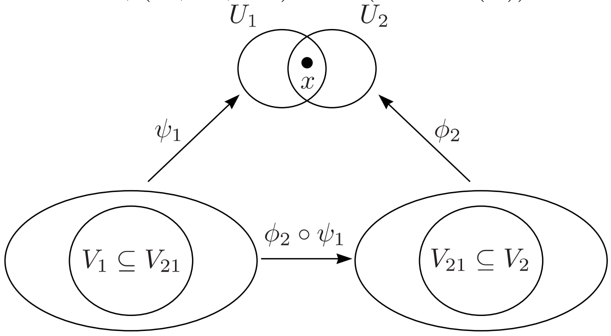
(Smooth maps in Euclidian spaces)
Be $\Phi\colon U \mapsto V$, where $U \subseteq^0 \mathbb{R}^K$, $V \subseteq^0 \mathbb{R}^{K'}$ is smooth, if and only if for all functions $f \in \varphi^\infty(V)$ it is $f \circ \phi \in C^\infty(U)$.

Proof „⇒":

If Φ and f are smooth, than this implies, that $f \circ \Phi$ are smooth (chain rule).

Proof „⇐":

$\Phi\colon U \mapsto V \subseteq \mathbb{R}^{K'}$ is smooth, if all component functions Φ_i (where $\Phi = (\Phi_1, \ldots, \Phi_{K'})$) are smooth. $\Phi_i = x_i \cdot \Phi$, $x_i\colon V \mapsto \mathbb{R}$, $(x_1, \ldots, x_{K'}) \mapsto x_i$ ($x_i \in C^\infty(V)$)



Definition:

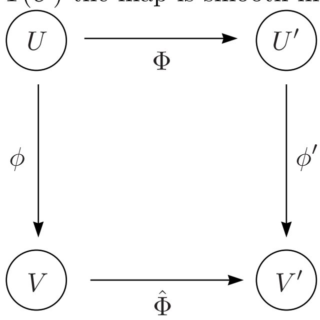
Let $\Phi\colon M \mapsto N$ be a continuous map. M, N are **smooth** manifolds. Than Φ is called smooth if $\forall\, f \in C^\infty(U)$ **and all** U, where $U \subseteq^0 N$ open, $f \circ \Phi\colon \Phi^{-1}(U) \mapsto \mathbb{R}$ is in $C^\infty(\Phi^{-1}(U))$.

Remark:

With this definition the coordinate charts $\Phi\colon U \mapsto V \subseteq^0 \mathbb{R}^K$ and parametrization charts $\psi = \phi^{-1}\colon V \mapsto U$ are smooth maps.

Proposition:

A map $\Phi\colon M \mapsto N$ is smooth if and only if for all charts (U, ϕ) for M and charts (U', ϕ') containing the image $\Phi(U)$ the map is smooth in the charts. That means:



Lemma:

Let $\Phi\colon M \mapsto N$, $\Psi\colon N \mapsto L$ be smooth maps, than $\Psi \circ \Phi\colon M \mapsto L$ is smooth. The consequence is, theta X has only a countable number of connected components.
Be $\tau \subseteq P(x)$, where τ is the set of all open sets of X. $M \subseteq \tau$ is an open covering, $\cup N = X$ and $N \in M$.

a.) $U \in \tau$, ϕ: $\mathrm{U} \xrightarrow{\simeq} V \subseteq \mathbb{R}^k$ (open), where ϕ is a homoemorphism. Then (U, ϕ) is called a coordinate chart of X.

b.) Let $\mathcal{A} = \{(U, \phi) | U \in M\}$, where M is a covering of X, be a system of coordinate charts. $\mathcal{A}$ is called an atlas.

Definition:

A topological, HAUSDORFF, (countable basis of topology) space X, which admits an atlas $\mathcal{A}$, is called a **topological manifold**.

Example:

$\mathbb{R}$, $\mathbb{R}^N$, intervals in $\mathbb{R}$ are submanifolds of $\mathbb{R}^n$.

Example:

- $X = \mathbb{R} \cup \{\overline{0}\}$, open sets in x
- all open sets of $\mathbb{R}$
- $\{\overline{I} = (I/\{0\} \cup \overline{0} | I$ is open in $\mathbb{R}, 0 \in I\}$

X admits an atlas, but X is not HAUSDORFF.
Let X be a topological space (manifold) and $\mathcal{A}$ be an atlas for X. Then A is called smooth, if the coordinate changes ϕ_{21}: $\phi_1(U_1 \cap U_2) \mapsto \phi_2(U_1 \cap U_2)$ $(\phi_{21} = \phi_2 \circ \phi_1^{-1})$ are diffeomorphisms of open subsets in $\mathbb{R}^n$ for all pairs of charts (U_1, ϕ_1), (U_2, ϕ_2) contained in A.
Be M a smooth manifold. We are looking at π_E: $E \mapsto M$, where E is a topological space and π_E is continuous. (E, π_E) has the structure of a smooth vector bundle (of rang n), if the following conditions are all satisfied:

- There exists an open covering of M.
- For $U \in \mathcal{A}$ there exist bundle charts (also called local trivialisations) h_U: $\pi^{-1}(n) \mapsto U \times \mathbb{R}^m$

  $h_U(\overline{v}) = (\pi_E(\overline{v}), h_U^2(\overline{v}))$, $h_U^2 : \pi^{-1}(u) \mapsto \mathbb{R}^m$

  so that the change of trivialisation

  $h_{U_j} \circ h_{U_i}^{-1} : U_i \cap U_j \times \mathbb{R}^m \mapsto U_i \cap U_j \times \mathbb{R}^m$

  of the form $h_{U_j} \circ h_{U_i}^{-1}(x, v) = (x, g_{ij}(x)(v))$, where g_{ij}: $U_{ij} \mapsto \mathrm{GL}_m(\mathbb{R})$ is a smooth map. (Remark: The g_{ij} are called transition functions of the bundle and h_U are homeomorphisms.)

Definition:

(E, π_E) with an atlas of bundle charts is called a smooth vector bundle. It is $\pi_E^{-1}(x) =: E_X$ and the fiber over $x \in M$ is a vector space over the real numbers (such that if $x \in U$ and h_U is a local trivialisation, $h_U^2 | E_X$: $E_X \mapsto \mathbb{R}^m$ is an isomorphism of vector spaces.

Proposition:

Let π_E: $E \mapsto M$ be a vector bundle, then E is a smooth manifold in a unique way, such that the projection π_E and all local trivialisations h_U are smooth.

Proof:

1.) E is a Hausdorff space. So use, that the h_U are homeomorphisms and also that M is Hausdorff. (easy exercise)

2.) Construct a smooth structure on E, which satisfies the requirements. Let $\mathcal{A}$ be a covering for M with associated local trivialisations h_U for $U \in \mathcal{A}$. By enlarging $\mathcal{A}$ if necessary, we can assume, that for alle $U \in \mathcal{A}$ there exists a smooth chart ϕ_U: $U \mapsto \mathbb{R}^K$ (for the manifold M). Define charts $\overline{\phi}_U$: $\pi^{-1}(u) \mapsto \mathbb{R}^k \times \mathbb{R}^m$, $\phi/v \mapsto (\phi_U(\pi_E(\overline{v})), h_U^2(\overline{v})$. So the $(\pi^{-1}(u), \overline{\phi}_U)$ define a smooth atlas for E.

$\overline{\phi}_{U_j} \circ \overline{\phi}_{U_i}^{-1} : \phi_{U_i}(U_i \cap U_j)(\subseteq^0 \mathbb{R}^k) \times \mathbb{R}^m \mapsto \phi(U_j \cap U_i)(\subseteq^0 \mathbb{R}^k) \times \mathbb{R}^m$, $(y, v) \mapsto (\phi_{U_j} \circ \phi_{U_i}(y), g_{ji}(\phi_{U_i}^{-1}(y)))$

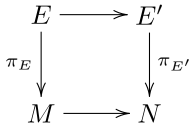
This is smooth, because the transition functions of the bundle are smooth. This implies also, that the maps h_U are smooth, in fact $h_U(\overline{v}) = \phi_U^{-1} \times \mathrm{id}_{\mathbb{R}^m} \circ \overline{\phi}_U$. Also π_E is smooth. □

Example:

The so-called **trivial bundle** is given by $E := M \times \mathbb{R}^m \mapsto M$, $\pi_E(x,v) \mapsto v$.

Definition:

Let $E \mapsto M$ and $E' \mapsto N$ be smooth vector bundles and f: $M \mapsto N$ be a smooth map. A smooth map F: $E \mapsto E'$ is called a vector bundle morphism (or just bundle map) over F, if the diagram



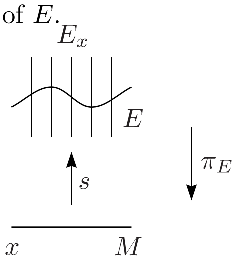
is commutative and the maps $F_x = F|_{E_x}$: $E_x \mapsto E'_{f(x)}$ are linear for all $x \in M$.

Definition:

Be E and E' bundles over M. Then a bundle map F: $E \mapsto E'$ over $f = \mathrm{id}_M$ is called a **homomorphism** of bundles. $E \mapsto M$ and $E' \mapsto M$ are called **equivalent**, if there exist homomorphisms F: $E \mapsto E'$ and F': $E' \mapsto E$, so that $F \circ F' = \mathrm{id}_E$ and $F' \circ F = \mathrm{id}_E$. The bundle is called trivial, if it is equivalent to a bundle of the form $M \times \mathbb{R}^m$.

Definition:

Let $E \xrightarrow{\pi} M$ be a vector bundle. Then a smooth map S: $M \mapsto E$, which satisfies $\pi \circ s = \mathrm{id}_M$ is called a section of E.



Example: Zero-section: $S(x) = O_x \in E_x$ (O_s in the vector space E_x)

1.4 The tangent bundle

$TM = \dot{\bigcup}_{x \in M} T_xM$, where T_xM is the tangent space of $x \in M$.

$\pi : TM \mapsto M$, $T_xM \ni \overline{v} \mapsto \pi(\overline{x}) = x$

Proposition:

The tangent bundle has a natural structure of a smooth vector bundle, such that, if f: $M \mapsto N$ is a smooth map, the differential $\mathrm{d}f$: $TM \mapsto TN$ ($\mathrm{d}f|_x$: $T_xM \mapsto T_{f(x)}N$) is then a morphism of vector bundles.

Corollary:

TM has the structure of a smooth manifold and the differentials of smooth maps are smooth.

Proof:

Let $\mathcal{A}$ be a smooth atlas for M. Define h_U: $\pi^{-1}(u) \mapsto U \times \mathbb{R}^k$, where $k = \dim(M)$ and $(U, \phi) \in \mathcal{A}$, $\pi^{-1}(u) \subseteq TM$.

$h_U : \overline{v} \mapsto (\pi(\overline{v}), \mathrm{d}\phi|_{\pi(\overline{v})}(\overline{v}))$

The h_U are bijections of $\pi^{-1}(u)$ onto $U \times \mathbb{R}^k$. Topology on TM: $W \subseteq TM$ is open, if and only of $h_U(w)$ are open in $U \times \mathbb{R}^k$ for alle U belonging to the atlas $\mathcal{A}$. This implies, that π is continuous. So we compute the transition functions with respect to the $\{h_U\}$:

$h_{U_j} \circ h_{U_i}^{-1} : (U_i \cap U_j) \times \mathbb{R}^k \mapsto (U_i \cap U_j) \times \mathbb{R}^k$, $(x, v) \mapsto (x, \mathrm{d}\phi_{ji}|_x(v))$

where $\phi_{ji} = \phi_{U_j} \circ \phi_{U_i}^{-1}$ is a coordinate change for the atlas $\mathcal{A}$. This implies, that $g_{ji}(x)(v) = \mathrm{d}\phi_{ji}|_x(v)$ are the transition functions, which are linear (x fixed) and depend smoothly on x. So TM is a smooth vector bundle. If $f\colon M \mapsto N$, then we have following diagram:

$$\begin{array}{ccc} TM & \xrightarrow{\mathrm{d}f} & TN \\ {\scriptstyle \pi_M}\downarrow & & \downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow[f]{} & N \end{array}$$

which commutes and $\mathrm{d}f|_x\colon T_xM \mapsto T_xN$ is linear. Also $\mathrm{d}f$ is smooth with resprect to the bundle structures on T_M, T_N!

1.5 Möbius-Band

We are looking a the Möbius-Band Möb $= [0,1] \times \mathbb{R}/\sim$, $(0,v) \sim (1,-v)$. The infinite cyclic group of diffeomorphisms generated by $\gamma \in \mathrm{Diff}(\mathbb{R}\times\mathbb{R})$ with $\gamma(t,v) = (t+1,-v)$ ist $\Gamma = \langle\gamma\rangle$. The quotient space $\mathbb{R}\times\mathbb{R}/\Gamma$ is $\{[(t,v)] | (t,v) \sim (t',v')$ if and only if there exists $\widetilde{\gamma} \in \Gamma$ with $\widetilde{\gamma}(t,v) = (t',v')\}$. The quotient space is the same thing as the Möbius band.

Lemma:

The space Möb is a smooth vector bundle over S^1.

Proof:

For the proof we recall, that $S^1 = \mathbb{R}/\mathbb{Z} = \{[t] | t \sim t'$ if and only if there exists $n \in \mathbb{Z}$, so that $t' = n + t\}$. (This is another way, to describe a circle.) There is a projection $\pi_{M\ddot{o}b}$: Möb $\mapsto S^1$, $[(t,v)] \mapsto [t]$. This is a well defined continuous map.
To proof now, that Möb is a smooth vector bundle, we want to define smooth charts. $U_1 = \{[t] | -1/2 < t < 1/2\}$ and $U_2 = \{[t] | 0 < t < 1\}$ are open sets of the circle S^1. U_1 looks like a circle with two points removed and U_2 looks like a circle with the upper point removed. $U_1 \cap U_2 \times \mathbb{R} \mapsto U_1 \cap U_2 \times \mathbb{R}$ defines bundle charts ϕ_1: $\pi_{M\ddot{o}b}^{-1}(U_1) \mapsto U_1 \times \mathbb{R}$, $[t,v] \mapsto (t,v)$ and ϕ_2: $\phi_{M\ddot{o}b}^{-1}(U_2) \mapsto U_2 \times \mathbb{R}$, $[t,v] \mapsto (t,v)$. We now calculate the coordinate change. Bundle charts for Möb: $\phi_2 \circ \phi_1^{-1}$: $U_1 \cap U_2 \times \mathbb{R} \mapsto U_1 \cap U_2 \times \mathbb{R}$, $(t,v) \mapsto (t,-v)$. The coordinate change in the second component is linear. □

Proposition:

The bundle $\pi_{M\ddot{o}b}$: Möb $\mapsto S^1$ does not admit a non-vanishing section.

Proof:

$$\begin{array}{ccc} \mathbb{R} & \xleftarrow[\pi_1]{} & \mathbb{R}\times\mathbb{R} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{M\ddot{o}b}} \\ S^1 & \xleftarrow{\pi_{M\ddot{o}b}} & \text{Möb} \end{array}$$

This is a commutative diagram. $\pi_1\colon \mathbb{R}\times\mathbb{R}$, $(t,v) \mapsto t$ (trivial bundle over $\mathbb{R}$). The quotient map $f_{M\ddot{o}b}$ is a map of bundles over f, which is an isomorphism on the fiber vector spaces. Given any section $s\colon S^1 \mapsto$ Möb, there exists a section $\widetilde{s}\colon \mathbb{R} \mapsto \mathbb{R}\times\mathbb{R}$ with $\widetilde{s}(t) = (t, X(t))$, such that $f_{\text{Möb}}\widetilde{s}(t) = s(t)$.

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{\widetilde{s}} & \mathbb{R}\times\mathbb{R} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{M\ddot{o}b}} \\ S^1 & \xrightarrow[s]{} & \text{Möb} \end{array}$$

is a commutative diagram. The map X is smooth, because s is smooth and it satisfies $X(t+1) = -X(t)$. In particular, $X\colon \mathbb{R} \mapsto \mathbb{R}$ has a zero. And therefore s has a zero. For example, there exists $x \in S^1$ with $s(x) = 0 \in \text{Möb}_x$. □

Proposition:

Let $E \mapsto S^1$ be a vector bundle of rank 1. Then E is isomorphic to the trivial bundle or to the bundle Möb.

Definition:

A manifold is called orientable, if it admits a smooth atlas, such that all coordinate changes have a positive determinant for their differentials. A maximal atlas for M with this property is called an orientation of M. (Recall: $\Phi\colon U \mapsto V$ is a local diffeomorphism of a (connected) open subset of $\mathbb{R}^n$ to $V \subseteq \mathbb{R}^n$. Then $f(x) = \det(\mathrm{d}\Phi|_x)$ is a (positive) or negative function, called the determinant of the differential.)

Exercise:

Show, that if M is orientable, then there exist precisely two orientations.

Lemma:

The Möbius band is **not** orientable. (Exercise series 6)

1.6 Operations on vector bundles

Given a vector bundle E over an manifold M, we can define natural bundles associated with it, for example the dual bundle $E^* \mapsto M$. We can also look at $\otimes^p E \mapsto M$ (p-th tensor power of E) or $\Lambda^p E \mapsto M$ (p-th exterior power of E). Given another bundle $E' \mapsto M$, we can form $E \oplus E' \mapsto M$ (direct sum), $E \otimes E' \mapsto M$ (tensor product) and $E \wedge E' \mapsto M$ (alternating or wedge product). The way to define these things is obvious.

1.6.1 Dual bundle of a vector bundle

If V is a real vector space, then $V^* = \{\lambda : V \mapsto \mathbb{R} | \lambda \text{ linear}\}$ is called the **dual space** of V. If $V \xrightarrow{\varphi} W$ is a linear map, then $\varphi^*\colon W^* \mapsto V^*$ with $\varphi^*(\lambda) = \lambda \circ \varphi$ is the induced map. If $\varphi\colon V \mapsto W$ is an isomorphism, you also get an isomorphism $\varphi_*\colon V^* \mapsto W^*$ with $\varphi_*(\lambda) = \lambda \circ \varphi^{-1}$. If $E \mapsto M$ is a vector bundle, then the dual bundle is a bundle $E^* \mapsto M$, such that all fibers $(E^*)_x$ identify with $(E_x)^*$.
Now we are coming to the construction of the dual bundle E^*. Let $\mathcal{A}$ be an atlas of bundle charts for E. We have a covering $\mathscr{A}$ of M and for all $U_i \in \mathscr{A}$, $\phi_i\colon \pi_E^{-1}(U_i) \mapsto U_i \times \mathbb{R}^m$ with $m = \operatorname{rank}(E)$. We have coordinate changes $\phi_j \circ \phi_1^{-1}\colon U_i \cap U_j \times \mathbb{R}^m \mapsto U_i \cap U_j \times \mathbb{R}^m$, $(x, v) \mapsto (x, g_{ji}(x)v)$, where $g_{ji}\colon U_i \cap U_j \mapsto \mathrm{GL}(m, \mathbb{R})$. We define the bundle $E^* \underset{\pi_E^*}{\longrightarrow} M$ by declaring the coordinate changes for charts $\psi_i\colon {\pi_E^*}^{-1}(U_i) \mapsto U_i \times \mathbb{R}^m$ to be the maps $\psi_j \circ \psi_i^{-1}\colon U_i \cap U_j \times \mathbb{R}^m \mapsto U_i \cap U_j \times \mathbb{R}^m$, $(x, v) \mapsto (x, ((g_{ji}(x))^\intercal)^{-1} v)$. $\square$
Recall, that the local trivializations ϕ_i, $\phi_i\colon \pi_E^{-1}(U_i) \mapsto U_i \times \mathbb{R}^m$ are of the form $\phi_i(v) = (x, L(x, v))$, if $v \in E_x$, and the map $L(x) = L(x, \bullet)\colon E_x \mapsto \mathbb{R}^m$ is a linear isomorphism. Identifying $\mathbb{R}^m$ with its dual space $(x = (x_1, \ldots, x_m) \mapsto \lambda_x \in (\mathbb{R}^m)^*$, where $\lambda_x(v) = \sum_{i=1}^m x_i v_i)$ $\psi_i(\lambda_x) = (x, L(x)_*(\lambda_x))$ with $\lambda_x \in E^*$.

1.7 Differential forms

Be M a manifold and $\Lambda^k T^* M := \mathrm{Alt}^k(TM)$ the bundle of alternating k-dimensional maps on TM with $\mathrm{Alt}^k(TM)_x = \mathrm{Alt}^k(TM_x)$ and $\mathrm{Alt}^k(TM) \mapsto M$. $\Omega^k(M)$ is the space of sections of $\mathrm{Alt}^k(TM)$. Furthermore we are considering $\omega \in \Omega^k(TM)$ with $\omega\colon M \mapsto \mathrm{Alt}^k(TM)$, $x \mapsto \omega_x \in \mathrm{Alt}^k(TM_x)$. $\Omega^k(M)$ is the space of smooth differential k-forms on M.

Example:

An orientation form on M is a nowhere vanishing n-form $\omega \in \Omega^n(M)$, where $n = \dim(M)$. That is, ω_x für $x \in M$ provides an orientation of the vector space $T_x M$.

Example:

Another example are differential forms on $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$ be an open subset. Every $\omega \in \Omega^1(U)$ may be written in the form $\omega = f_1 \,\mathrm{d}x^1 + f_2 \,\mathrm{d}x^2 + \ldots + f_n \,\mathrm{d}x^n$, where $f_i \in C^\infty(n)$ and $\mathrm{d}x^i \in \Omega^1(U)$ are differentials of the coordinate functions $x^i\colon U \mapsto \mathbb{R}$, $x = (x_1, \ldots, x_n) \mapsto x_i$.

Remark:

If $f\colon M \mapsto \mathbb{R}$ is a smooth function, then the assignment $v \in T_xM \mapsto \mathrm{d}f|_x(v) \in T_{f(x)}\mathbb{R} = \mathbb{R}$ shows, that $\mathrm{d}f$ is identified with an element of $\Omega^1(M)$.
Explicitly: $\mathrm{d}x^i|_x(v) = v_i$, $v = (v_1, \ldots, v_i, \ldots, v_n) \in \mathbb{R}^n = T_xU$. Similarly we have for $\mathrm{d}x^i \wedge \ldots \wedge \mathrm{d}x^n \in \Omega^n(U)$:

$$(\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n)(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n) := \det(\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n) \text{ with } \overline{v}_i \in \mathbb{R}^n = T_xU$$

Every $\omega \in \Omega^n(U)$, where $\omega = f\,\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$, $f \in C^\infty(U)$ (because $\mathrm{Alt}^n(\mathbb{R}^n)$ is one-dimensional and $\det \neq 0$).

Remark on the general case:

$\Omega^k(U)$ admits a basis $\mathrm{d}x^{i_1} \wedge \mathrm{d}x^{i_2} \wedge \ldots \wedge \mathrm{d}x^{i_k}$, where $i_1 < i_2 < \ldots < i_k$ over $C^\infty(U)$.

Proposition:

Be M a smooth manifold. M is orientable if and only if there exists a smooth nowhere vanishing n-form $\omega \in \Omega^n(M)$, where $n = \dim M$.

Proof:

We have already proofed that a non-vanishing n-form implies and atlas („$\Leftarrow$“).

1.8 Partition of unity on a smooth manifold

- Let $\mathcal{A}$ be a covering of M and $\mathcal{B}$ another covering. Then $\mathcal{B}$ is called subordinate to $\mathcal{A}$, if for all $U \in \mathcal{B}$ there exists a $V \in \mathcal{A}$ with $U \subseteq V$.
- A covering $\mathcal{B}$ is called locally finite, if for all $x \in M$ there exists a neighborhood $U_x \ni x$, so that $\{U \in \mathcal{B} | U \cap U_x \neq \varnothing\}$ is a finite set.

Definition:

A partition of unity on M is a locally finite covering $\{U_i\}_{i \in I}$, where I is an index set, together with smooth functions $\alpha_i\colon M \mapsto \mathbb{R}$, $\alpha_i \in C^\infty(M)$, $i \in I$ with the property, that

1.) $\mathrm{Supp}(\alpha_i) = \overline{\{x \in M | \alpha_i(x) \neq 0\}} \subseteq U_i$

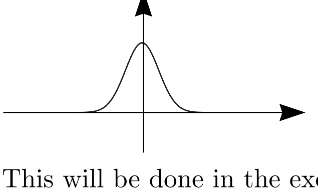
2.) $\sum_{i \in I} \alpha_i(x) = 1 \;\forall\, x \in M$ (Remark: The sum is finite.)

Theorem:

Let $\mathcal{A}$ be a covering of M. Then there exists a partition of unity $(U_i, \alpha_i)_{i \in I}$, such that the covering $(U_i)_{i \in I}$ is subordinate to $\mathcal{A}$ („subordinate partition of unity“).

Proof:

To prove the theorem, you need test functions („bump“functions) on $\mathbb{R}^n$ with support contained in a neighborhood of 0.



This will be done in the exercises.

Important remark:

Given a local „object“ on a set U_i, for example a local section $S\colon U_i \mapsto E$, where $E \xrightarrow{\pi} M$ is a vector bundle, we can define a global section of E, $\hat{S}_i\colon M \mapsto E$, by declaring $\hat{S}_i(x) = \alpha_i(x)S_i(x)$ for $x \in U_i$ and $\hat{S}_i(x) = 0$ for $x \in M \setminus U_i$. (We want to use the abbreviation $\hat{S}_i = \alpha_i S_i$.) And thus, $\alpha_i S_i$ is a smooth section of E. Moreover, having local data S_i on all U_i, we patch them together by putting $S = \sum_{i\in I} \alpha_i S_i$, which is a smooth section of E.
We proof now: Let M be a smooth oriented manifold. Then M admits a nowhere vanishing smooth n-form $\omega \in \Omega^n(M)$, $n = \dim M$.

Proof:

Because M is oriented, we can choose a smooth oriented atlas (U_i, ϕ_i) for M. This means, that coordinate changes ϕ_{ji} satisfy $\det(\mathrm{d}\phi_{ji}) > 0$. So choose a partition of unity, which is subordinate to this atlas. By restricting charts this partition of unity defines also an oriented atlas. Hence, we can assume from the beginning, that there is a partition of unity $(U_i, \alpha_i)_{i\in I}$. Look at the chart $\phi_i\colon U_i \mapsto \mathbb{R}^n$. On $\mathbb{R}^n$ we have $\omega_0 = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n \in \Omega^n(\mathbb{R}^n)$ (canonical non-vanishing form). So we can pull back to define $\phi_i^*\omega_0\colon U_i \mapsto \mathrm{Alt}^n(TM)$, where $\phi_i^*\omega_0$ is a local action on M_i.

$(\phi_i^*\omega_0)|x(v_1, \ldots, v_n) = \omega_0|_{\phi_i(x)}(\mathrm{d}\phi_i|_x(v_1), \ldots, \mathrm{d}\phi_i|_x(v_n))$ with $x \in U_i$ and $v_i \in T_xM$

We can put

$$\omega := \sum_{i\in I} \alpha_i(\phi_i^*\omega_0)$$

We claim, that this ω is an orientation form for M. Look at ω in a local chart (U_k, ϕ_k) and compute $(\phi_k)_*\omega|_{U_k} \in \Omega^n(\phi_k(U_k))$, where $\omega_i = \alpha_i\phi_i^*\omega_0 = \alpha_i(\phi_i)_*^{-1}\omega_0$ and $\alpha_i^*(\phi_k(x)) = \alpha(x)$:

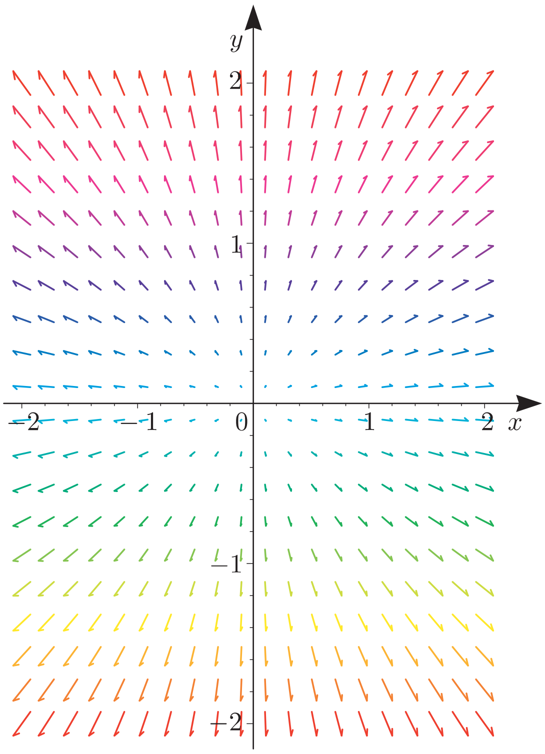
$$\begin{aligned}(\phi_k)_*\omega|_{U_k} &= \sum_{i\in I}(\phi_k)_*\omega_i = \sum_{i\in I}(\phi_k)_*\alpha_i\phi_i^*\omega_0 = \sum_{i\in I}(\phi_k)_*(\alpha_i(\phi_i)_*^{-1}\omega_0) = \sum_{i\in I}\alpha_i^*(\phi_k\circ\phi_i^{-1})_*\omega_0 = \sum_{i\in I}\alpha_i^*(\phi_{ki})_*\omega_0 = \\ &= \sum_{i\in I}\alpha_i^*\det(\mathrm{d}\phi_{ki})\omega_0 = \left(\sum_{i\in I}\alpha_i^*\det(\mathrm{d}\phi_{ki})\right)\omega_0 = \mu\omega_0 \text{ with } \mu \in C^\infty,\ \mu > 0\end{aligned}$$

Because of $\det(\phi_{ki}) > 0$ and $\alpha_i > 0$, μ is a positive function. So the ω doesn't vanish in all local charts and therefore $\omega_x \neq 0$ for all $x \in M$. So ω is an orientation form on M. □

1.9 Vector fields

Let's just look on further examples of vector fields.

1.) Be $M = U = \mathbb{R}^2 \setminus \{0\}$. For $x \in U$ we define $X(x) = x$; this vector field is called the position vector field.



$c_x(t) = \exp(t)x$ for $t \in \mathbb{R}$ is an integral curve for X. X is a complete vector field and the flow is given by $\phi_t(x) = \exp(t)x$.

2.) Every vector field on $\mathbb{R}^n$ is written in the form $X = \sum_{i=1}^{n} x_i \partial/(\partial x_i)$, where $x_i \in C^\infty(\mathbb{R})$ and $\partial/(\partial x_i)$ are the coordinate vector fields on $\mathbb{R}^n$. The integral curves of $\partial/(\partial x_i)$ are given by $c_x(t) = x + te_i$, where e_i is the standard coordinate vector. The flow of $\phi_t(x)$ of $\partial/(\partial x_i)$ is equal to $x + te_i$.

3.) Be $\mathrm{GL}_n(\mathbb{R}) = U \overset{0}{\subseteq} \mathbb{R}^{n'<n}$. For $\varphi \in \mathrm{Mat}(n \times n, \mathbb{R})$ we define a vector field $X_\varphi(g) = g\varphi$ with $g \in \mathrm{GL}_n(\mathbb{R})$ (matrix product). So X_φ ist a vector field of $\mathrm{GL}_n(\mathbb{R})$. Recall: We have the exponential map

$$\exp : \mathrm{Mat}(n \times n, \mathbb{R}) \mapsto \mathrm{GL}_n(\mathbb{R}),\ \varphi \mapsto \exp(\varphi) = \sum_{i=0}^{\infty} \frac{1}{i!}\varphi^i$$

$c_g(t) = g(\exp(t\varphi))$ is a curve $\in \mathrm{GL}_n(\mathbb{R})$ with $c_g(0) = g$. c_g in an integral curve for the vector field X_φ, hence X_φ is complete with the flow $\phi_t(g) = g \cdot (\exp(t\varphi))$.

Definition:

A smooth map $\mathbb{R} \mapsto \mathrm{Diff}(M)$ of the form $t \mapsto \phi_t$, $\phi_t \in \mathrm{Diff}(M)$, where $\mathrm{Diff}(M)$ denotes the group of diffeomorphisms of M, is called a one parameter group of diffeomorphisms, if

i.) $\phi_0 = \mathrm{id}_M$

ii.) $\phi_s \circ \phi_t = \phi_{s+t}\ \forall\ s, t \in \mathbb{R}$

Remark:

We say, that ϕ_t is smooth, if $\mathbb{R} \times M \mapsto M$, $(t, x) \mapsto \phi_t(x)$ is smooth.
Let X be a smooth vector field on M, which is complete. The map Φ: $\mathbb{R} \times M \mapsto M$, $(t, x) \mapsto c_x(T)$ with $c_x(0) = 0$ (which is an integral curve for X) is smooth. It is called the (global) flow of X. So a general vector field admits only local flow maps near x, Φ: $I \times U \mapsto M$, where $I =]a, b[$ with $a < 0 < b$ and U is a neighbourhood of $x \in M$.

Proposition:

Let X be a complete vector field with global flow Φ. Then

i.) For $t \in \mathbb{R}$ fixed, $x \mapsto \phi_t(x) := \Phi(t, x)$ is a diffeomorphism of M.

ii.) The map $t \mapsto \phi_t \in \mathrm{Diff}(M)$ is a one parameter group (which is called the „flow“ of X).

Remark:

Let $(\phi_t)_{t\in\mathbb{R}}$ be a one parameter group of diffeomorphisms. Then we define a vector field on M by declaring

$$X(x) := \frac{\partial}{\partial t}\Big|_{t=0} \phi_t(x)$$

Verify, that $c_x(t) = \phi_t(x)$ is an integral curve for the vector field X with $c_x(0) = x$. (Use, that ϕ_t is a one-parameter group.)

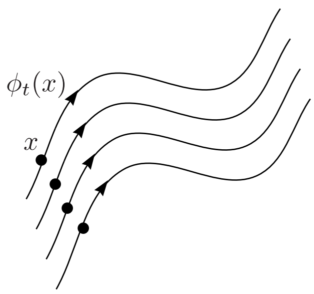
Remark:

Concerning the third example, recall, that $\mathrm{GL}_n(\mathbb{R})$ is a group and that multiplication and taking inverses in $\mathrm{GL}_n(\mathbb{R})$ is smooth. In particular, for $g \in \mathrm{GL}_n(\mathbb{R})$, R_g: $h \mapsto h \cdot g$ (right-multiplication) and L_g: $h \mapsto g \cdot h$ (left-multiplication) are smooth maps. Now, for a smooth homeomorphism $\mathbb{R} \xrightarrow{\gamma} \mathrm{GL}_n(\mathbb{R})$ (γ is called a one parameter subgroup) we can define $t \mapsto L_{\gamma(t)}$, which is a one-parameter group of diffeomorphisms in $\mathrm{Diff}(\mathrm{GL}_n(\mathbb{R}))$.

Proof:

i.) $x \mapsto \phi_t(x)$ is smooth. (That's clear from the definition.) And so is the map $x \mapsto \phi_{-t}(x)$. Assuming namely $\phi_s(\phi_t(x)) = \phi_{s+t}(x)$ we get $\phi_t \circ \phi_{-t} = \phi_0 = \mathrm{id}_M = \phi_{-t} \circ \phi_t$. This implies, that ϕ_t is a diffeomorphism with a smooth inverse. Obviously, it is $\phi_0 = \mathrm{id}_M$, because $\phi_0(x) = \Phi(0, x) = c_x(0) = x$.

Picture of ϕ_t.



We move along the flow up to the time t.

ii.) Look at $c(t) = \phi_t \circ \phi_s(x)$, where s is fixed. By definition of ϕ_t, where $c(t) = c_{\phi_s(x)}(t)$ is an integral curve with $c(0) = \phi_s(x)$. Now look at $\widetilde{c}(t) = \phi_{t+s}(x) = c_x(t+s)$ with $c_x(0) = x$, which is an integral curve for X. Now we compute:

$$\frac{\partial}{\partial t}\widetilde{c}(t) = \frac{\partial}{\partial t}c_x(t+s) = \dot{c}_x(t+s) = X(c_x(t+s)) = X(\widetilde{c}(t))$$

So $\widetilde{c}$ is an integral curve and $\widetilde{c}(0) = c_x(s) = \phi_s(x)$. By uniqueness of the locals, this implies $c(t) = \widetilde{c}(t)$ for all $t \in \mathbb{R}$. This is equivalent to equation $(*)$. □

Remark:

The proof shows, that a local flow for X, $(t, x) \mapsto \phi_t(x)$ (with $(t, x) \in I \times U$) has the property, that ϕ_t: $U \mapsto \phi_t(U)$ is a diffeomorphism and for all s, $t \in I$ it is $\phi_s \circ \phi_t = \phi_{t+s}$ (as defined).

1.9.1 Action of diffeomorphisms on vector fields

Be $\psi \in \mathrm{Diff}(M)$, where X is a vector field. Define $\psi_* X$ (push-forward, image of X by ψ) by $(\psi_* X)|_{\psi(x)} := \mathrm{d}\psi|_{\psi^{-1}(x)}(X(\psi^{-1}(x)))$. (This is again a smooth vector field, because the differential is smooth.) We define $\psi^* X$ (pull-back) by declaring $\psi^* X := (\psi^{-1})_* X$.

Lemma:

Let ψ, ψ_1, ψ_2 be $\in \mathrm{Diff}(M)$.

i.) $(\psi_1 \circ \psi_2)_* = (\psi_1)_* \circ (\psi_2)_*$ and $(\psi_1 \circ \psi_2)^* = (\psi_2)^* \circ (\psi_1)^*$

ii.) Let c_x be an integral curve for X at x. Then $\Phi \circ c_x$ is an integral curve for $\Phi_* X$ at $\Phi(x)$.

iii.) If ϕ_t is the flow for X, then $\psi \circ \phi_t \circ \psi^{-1}$ is the flow for $\psi_* X$.

Definition:

Let $\Phi \in \mathrm{Diff}(M)$. Then a vector field is called invariant by ϕ, if $\Phi_* X = X$.

Example:

In the third example the vector fields X_φ are invariant by $\Phi = L_g$ for all $g \in \mathrm{GL}_n(\mathbb{R})$.

Remark:

If X is invariant by ψ, then $\psi \circ \phi_t \circ \psi^{-1} = \phi_t$, so the flow commutes with ψ.

Proof of the lemma:

i.) Exercise!

ii.) $\widetilde{c}(t) := \Phi \circ c_x(t)$, $\widetilde{c}(0) = \Phi(x)$

$$\frac{\partial}{\partial t}\widetilde{c}(t) = \mathrm{d}\Phi|_{c_x(t)} \circ \dot{c}_x(t) = \mathrm{d}\Phi|_{\Phi^{-1}\circ\widetilde{c}(t)}(X(c_x(t))) = \mathrm{d}\Phi|_{\Phi^{-1}\circ\widetilde{c}(t)}(X(\Phi^{-1}\widetilde{c}(t))) = (\Phi)_*(X)(\widetilde{c}(t))$$

This implies, that $\widetilde{c}(t)$ is an integral curve for $\Phi_* X$.

iii.) Plug in the definition and use ii). Do it as an exercise, to get used to these calculations!

Kapitel 2

Riemannian metrics

Definition:

Be M a differential manifold. If for all $p \in M$ there exists a scalar product $\langle , \rangle_p \equiv g_p(,)$in T_pM, such that „g varies smoothly with respect to p“ ($*$, the dependence on p is „differentiable‘), then (M, g) is called a Riemannian manifold and g is called a Riemannian metric on M. ($*$): Given local coordinates φ: $U \mapsto \mathbb{R}^n$, $q \mapsto (x^1(q), \ldots, x^n(q))$, then the functions g_{ij}: $U \mapsto \mathbb{R}$ given by $g_{ij}(q) = \langle \partial/\partial x^i|_q, \partial/\partial x^j|_q \rangle$ for $1 \leq i,\ j \leq n$ are C^∞. In other words, the matrix $(g_{ij}(q))$, which represents g_q in local coordinates with respect to the basis $\partial/\partial x^1|_q$, …, $\partial/\partial x^n|_q$ of T_qM has C^∞ entries.

Examples:

1.) The first easy example is $\mathbb{R}^n$ with the standard scalar product $\langle , \rangle$. So $(\mathbb{R}^n, \langle , \rangle)$ is a Riemannian manifold.

2.) The second example is the hyperbolic space $\mathbb{H}^n$ with $\mathbb{H}^n = \{x \in \mathbb{R}^n | x^n > 0\}$. So we have a diagonal matrix representing g_{ij}:

$$g_{ij}(x) = \begin{cases} \frac{1}{(x^n)^2} & \text{für} \quad i = j \\ 0 & \text{für} \quad i \neq j \end{cases}$$

3.) The immersion ϕ: $M \mapsto (N, \langle\langle , \rangle\rangle)$ introduces a Riemannian metric on M. This is just done by the pull-back: $\langle , \rangle := \phi^* \langle\langle , \rangle\rangle$. Example for this are surfaces in $\mathbb{R}^3$ oder the standard sphere $S^n \subseteq \mathbb{R}^{n+1}$.

4.) We want to consider Riemannian products. $(M_1, \langle , \rangle^{(1)})$, $(M_2, \langle , \rangle^{(2)}\rangle$ be Riemannian manifolds. So the product manifold is given by:

$$T_{(p,q)}(M_1 \times M_2) = T_pM_1 \oplus T_qM_2, v = (v_1, v_2)$$

We define the projection by π_i: $M_1 \times M_2 \mapsto M_i$, $\mathrm{d}\pi_i|_{(p,q)}(v) = v_i$ for $i = 1$, 2. For u, $v \in T_{(p,q)}(M_1 \times M_2)$ we get:

$$\langle u, v \rangle_{(p,q)} := \langle \mathrm{d}\pi_1|_{(p,q)}(u), \mathrm{d}\pi_1|_{(p,q)}(v) \rangle_p^{(1)} + \langle \mathrm{d}\pi_2|_{(p,q)}(u), \mathrm{d}\pi_2|_{(p,q)}(v) \rangle_q^{(2)}$$

Note, that $T_pM_1 \perp T_qM_2$ for $T_pM_1 = \mathrm{d}\pi_1|_{(p,q)}(T(M_1 \times M_2))$. Let us consider some examples:

a.) $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

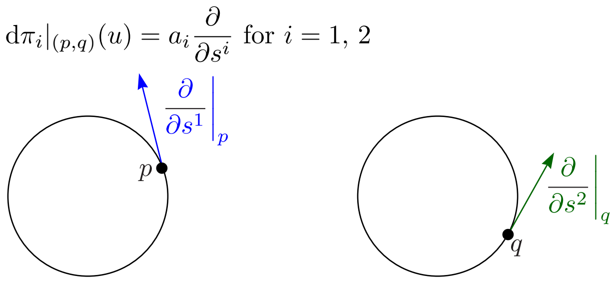
b.) Are more interesting example is the flat torus. We take the product of two such $M_i = S^1 = \{x \in \mathbb{R}^2 | \|x\| = 1\}$. What we then get, is $T^2 = S^1 \times S^1$. We endow every factor with a Riemannian metric (for example the one induced for $\mathbb{R}^2$). We denote $\partial/\partial s^1$, $\partial/\partial s^2$ the unit tangent vector fields on S^1. Then we get:

$$T_{(p,q)}T^2 = T_{(p,q)}(S^1 \times S^1) = \mathbb{R}\, \frac{\partial}{\partial s^1}\bigg|_p \oplus \mathbb{R}\, \frac{\partial}{\partial s^2}\bigg|_q$$

So $u \in T_{(p,q)}T^2$ can be written as

$$u = a_1\, \frac{\partial}{\partial s^1}\bigg|_p + a_2\, \frac{\partial}{\partial s^2}\bigg|_q \quad \text{where } a_1,\ a_2 \in \mathbb{R}$$

$$\mathrm{d}\pi_i|_{(p,q)}(u) = a_i \frac{\partial}{\partial s^i} \text{ for } i = 1,\, 2$$



Now we can compute the coefficients of the matrix, which stands for the matric on the Torus T^2. The basis for $T_{(p,q)}T^2$ is the following:

$$\left\{ \left(\frac{\partial}{\partial s^1}\Big|_p, 0 \right), \left(0, \frac{\partial}{\partial s^2}\Big|_q \right) \right\}$$

$$g_{11}(p,q) = \left\| \mathrm{d}\pi_1 \left(\frac{\partial}{\partial s^1}\Big|_p, 0 \right) \right\|^2 + \underbrace{\left\| \mathrm{d}\pi_2 \left(\frac{\partial}{\partial s^1}\Big|_p, 0 \right) \right\|^2}_{0 \in T_q M_2} = \left\| \frac{\partial}{\partial s^1}\Big|_p \right\|^2 + 0 = 1$$

This is, because we have chosen a unique vector field.

$$g_{22}(p,q) = \left\langle \mathrm{d}\pi_1 \left(\frac{\partial}{\partial s^1}\Big|_p, 0 \right), \mathrm{d}\pi_1 \left(0, \frac{\partial}{\partial s^2}\Big|_q \right) \right\rangle + \left\langle \mathrm{d}\pi_2 \left(\frac{\partial}{\partial s^1}\Big|_p, 0 \right), \mathrm{d}\pi_2 \left(0, \frac{\partial}{\partial s^2}\Big|_q \right) \right\rangle = 0$$

Finally, we get:

$$g_{ij}(p,q) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \delta_{ij}$$

So the two tangent spaces M_1 and M_2 are orthogonal. The metric is flat, so the torus T^2 is locally isometric to $\mathbb{R}^2$. But this is not a global property, because the torus is compact and $\mathbb{R}^2$ not.

Definition:

Be $(M, \langle , \rangle)$ and $(N, \langle\langle , \rangle\rangle)$ two manifolds. A local diffeomorphism $\phi\colon U \mapsto V$ is called a local isometry if $\forall$ $p \in U$ and $\forall$ $u,\, v \in T_pM$ it is $\langle u, v \rangle = \langle\langle \mathrm{d}\phi|_p(u), \mathrm{d}\phi|_p(v) \rangle\rangle_{\phi(p)}$.

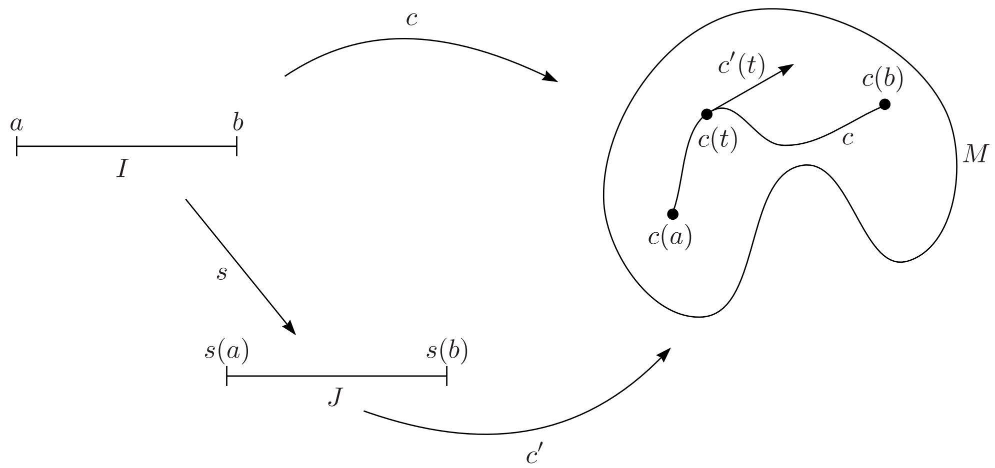
Remark:

If $\phi\colon \mathbb{R}^2 \mapsto S^1 \times S^1 = T^2$, $(s,t) \mapsto (\cos(s), \sin(s)) \times (\cos(t), \sin(t))$, then ϕ is a local isometry from $(\mathbb{R}^2, \langle , \rangle_{eucl}\rangle)$ to $(T^2, \langle , \rangle)$. But T^2 is **not** globally isometric to $\mathbb{R}^2$ (since T^2 is compact, but $\mathbb{R}^2$ is not).

2.1 Lengths of smooth curves

Let $c\colon I \mapsto (M, \langle , \rangle)$ be a smooth curve. Then the tangent vector field of c is:

$$c'(t) \equiv \frac{\mathrm{d}c}{\mathrm{d}t}(t) := \mathrm{d}c|_t \left(\frac{\partial}{\partial t} \right) \in T_{c(t)}M \text{ and } t \in I$$



C' is a differentiable vector field along c in M. The **length** of c: $[a, b] \mapsto M$ is defined as:

$$L(c) := \int_a^b \sqrt{\left\langle \frac{\mathrm{d}c}{\mathrm{d}t}, \frac{\mathrm{d}c}{\mathrm{d}t} \right\rangle_{c(t)}} \, \mathrm{d}t = \int_a^b \|c'(t)\|_{c(t)} \, \mathrm{d}t$$

Remark:

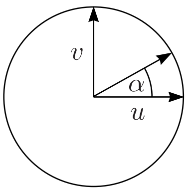
1.) $L(c)$ is independent of the parametrization (for example a property of the image $c([a, b])$ in M. So let's proof this. Be s: $I = [a, b] \mapsto J = [s(a), s(b)]$, $t \mapsto s(t)$ a diffeomorphism. We consider c: $I \mapsto M$, c': $J \mapsto M$, $c' = c \circ s$ and use the chain rule:

$$L(c) = \int_a^b \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\|_{c(t)} \mathrm{d}t = \int_a^b \left\| \frac{\mathrm{d}c'(s(t))}{\mathrm{d}t} \right\| \cdot \left| \frac{\mathrm{d}s}{\mathrm{d}t} \right| \mathrm{d}t = \int_{s(a)}^{s(b)} \left\| \frac{\mathrm{d}c'}{\mathrm{d}s} \right\|_{c'(s)} \mathrm{d}s = L(c')$$

2.) If ϕ: $(M, \langle , \rangle) \mapsto (N, \langle\langle , \rangle\rangle)$ is a Riemannian isometry and c: $I \mapsto M$ a smooth curve of length l. Then $\phi \circ c$ is also a smooth curve in N of the same length l. This can be expressed by the property $\langle u, v \rangle = \langle \mathrm{d}\phi|_p(u), \mathrm{d}\phi|_p(v) \rangle_{\phi(q)}$.

3.) Every smooth curve c: $I \mapsto M$ with $c'(t) \neq 0$ for alle $t \in I$ can be parametrized by arclength. That means, that there exists $\widetilde{c}$: $J \mapsto M$ with $c = \widetilde{c} \circ s$, s: $I \mapsto J$ as above, such that $\|\widetilde{c}'(s)\| = 1 \; \forall \; s \in J$. Let us consider the following example, namely $(S^n, \langle , \rangle_{ind})$ (Riemannian metric induced from $\mathbb{R}^{n-1}$). A great circle on S^n is an intersection of S^n with a two-dimensional subspace of $\mathbb{R}^{n+1}$, say $[u, v]$, where u, $v \in \mathbb{R}^{n+1}$ with $\|u\| = \|v\| = 1$.

$G: \; [0, 2\pi] \mapsto S^n$, $t \mapsto \cos(t) \cdot u + \sin(t) \cdot v$



$G'(t) = -\sin(t) \cdot u + \cos(t) \cdot v$

$\|G'(t)\|^2 = \|-\sin(t) \cdot u + \cos(t) \cdot v\|^2 \overset{u \perp v}{=} \langle -\sin(t) \cdot u, -\sin(t) \cdot u \rangle + \langle \cos(t) \cdot v, \cos(t) \cdot v \rangle = \sin^2(t) + \cos^2(t) = 1$

So we have a parametrization by arclength.

$$L(G|_{[0,\alpha]}) = \int_0^\alpha \|G'(t)\| \, \mathrm{d}t = \alpha$$

2.2 Existence of Riemannian metrics

Theorem 1:

On every n-dimensional differentiable manifold M, there exists a Riemannian metric g on M.

Proof:

We choose a smooth atlas $\mathcal{A} = \{(U, \Phi_U)\}$ and a subordinate partition of unity, (M_i, λ_i), where λ_i: $M \mapsto [0, 1]$ is a smooth support $\lambda_i \subseteq U_i$ and $\sum_i \lambda_i = 1$. On each U_i we define a chart Φ_i: $U_i \mapsto \mathbb{R}^n$, $\Phi_i = \Phi_U|_{U_i}$, where $U_i \subseteq U$ (for some U). Define $\widetilde{g}_i = \Phi_i^*(\langle , \rangle)$, which is a Riemannian metric on U_i and also $g_i := \lambda_i \widetilde{g}_i$. (So g_i is a global object, namely a symmetric 2-form on M.) Furthermore, we define $g := \sum_i g_i$.

1.) This is well defined (clear!)

2.) It defines a symmetric bilinear form on each tangent space $T_x M$ by $g|_x(v, w) = \sum_{i, \lambda_i(x) \neq 0} \lambda_i(x) \widetilde{g}_i(v, w)$.

3.) It is also positive definite, because $g|_x(v, v) = \sum_{i, \lambda_i(x) \neq 0} \lambda_i(x) \widetilde{g}_i(v, v) > 0 \Leftrightarrow v \neq 0$. ($\widetilde{g}_i(v, v) > 0 \Leftrightarrow v \neq 0$ and $\lambda_i(x) > 0$) □

Examples:

If (M, g) is a Riemannian manifold and $N \subseteq M$ a submanifold, then N has an induced Riemannian metric $h = g|_N$. This means $T_xN \subseteq T_xM \ \forall \ x \in N$. Furthermore we put $h_x = g|_x$. In particular, if $M \subseteq \mathbb{R}^n$ is a submanifold, then M has a Riemannian metric, which is defined by $v,\ w \in T_xM \subseteq \mathbb{R}^m$ (which is a vector subspace). We define the induced metric by $g|_x(v, w) = \langle v, w\rangle$, where $\langle , \rangle$ is the standard scalar product on $\mathbb{R}^m$.

Remark:

a.) Application: Theorem 1 follows by the Whitney embedding $M \hookrightarrow \mathbb{R}^N$ $(*)$ for arbitrary abstract M.

b.) The space of Riemannian metrics g on M is usually an infinite dimensional „space“. (It's not a vector space!)

c.) Nash embedding theory: Let (M, g) be a Riemannian manifold, then (M, g) can be obtained by the construction $(*)$. In other words: (M, g) is isometric to a submanifold of $\mathbb{R}^N$ for some $N \gg 0$. This submanifold is (M', h) with $M' \subseteq \mathbb{R}^N$ and $h = \langle\rangle|_{M'}$.

Definition:

Let (M, g) and (N, h) be two Riemannian manifolds and $\Phi\colon M \mapsto N$ be a (local) diffeomorphism. Then Φ is a (local) isometry, if and only if $g = \Phi^*h$. (This means, that $h_{\Phi(x)}(\mathrm{d}\Phi|_x(v), \mathrm{d}\Phi|_x(w)) = g_x(v, w)$ for all $x \in M$ and $v,\ w \in T_xM$.) The manifolds (M, g) and (N, h) are called isometric, if there is an isometry Φ: $(M, g) \mapsto (N, h)$.

Remark:

Not every manifold admits a metric of signature $(n, 1)$ (Lorentzian metric).

Example:

The S^2 has no metric of signature $(1, 1)$. (Hint: Proof, that a manifold with a metric of signature $(n, 1)$ has a **non-vanishing** vector field.)

Remark on (classical) notation:

$(\mathbb{R}^n, \langle , \rangle) = (\mathbb{R}^2, g)$ with $g = \sum_i \mathrm{d}x_i^2$

For $\mathbb{R}^2$ we have $g = \mathrm{d}x^2 + \mathrm{d}y^2$. $\mathrm{d}x$ is a differential form on $\mathbb{R}^2$ for example. So it is $g(v, w) = \mathrm{d}x(v) \cdot \mathrm{d}x(w) + \mathrm{d}y(v) \cdot \mathrm{d}y(w)$.

Example: Polar coordinates

We want to consider polar coordinates on $\mathbb{R}^2$.

$\Phi : \mathbb{R}^{>0} \times \mathbb{R} \mapsto \mathbb{R}^2$, $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$

Φ is the polar coordinates map. What is $h = \Phi^*(g)$, where $g = \mathrm{d}x^2 + \mathrm{d}y^2$? So we have to coordinate fields $\partial/\partial r$, $\partial/\partial\theta$ on $\mathbb{R}^{>0} \times \mathbb{R}$. We have to compute:

$$h\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = g\left(\frac{\partial}{\partial r}\Phi, \frac{\partial}{\partial r}\Phi\right) = g((\cos\theta, \sin\theta), (\cos\theta, \sin\theta)) = 1$$

$$h\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}\right) = g\left(\frac{\partial}{\partial r}\Phi, \frac{\partial}{\partial \theta}\Phi\right) = g((\cos\theta, \sin\theta), (-r\sin\theta, r\cos\theta)) = 0$$

$$h\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = g\left(\frac{\partial}{\partial \theta}\Phi, \frac{\partial}{\partial \theta}\Phi\right) = g((-r\sin\theta, r\cos\theta), (-r\sin\theta, r\cos\theta)) = r^2$$

So we get $h = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. (This metric is isometric to the canonical metric.)

Examples:

1.) Let $b > 0$ be any positive definite bilinear form on $\mathbb{R}^n$. Then b defines a Riemannian metric $(\mathbb{R}^n, g_b)$ on $\mathbb{R}^n$. But $(\mathbb{R}^n, g_{b_1})$ and $(\mathbb{R}^n, g_{b_2})$ are always isometric Riemannian manifolds.

2.) We consider the Torus $T^n = \mathbb{R}^n/\mathbb{R}^n = S^1 \times \ldots \times S^1$. π: $\mathbb{R}^n \mapsto T^n$ is the natural projection: $(v_1, \ldots, v_n) \mapsto (\cos(v_1), \sin(v_1), \cos(v_2), \sin(v_2), \ldots, \cos(v_n), \sin(v_n))$. Now let g_b be a Riemannian metric on $\mathbb{R}^n$, defined as in ①. π is a local diffeomorphism.

Lemma:

There exists a unique Riemannian metric h_b on T^n, such that the map π is a local isometry.

Proof:

Take $\overline{x} \in T^n$ and $x \in \mathbb{R}^n$, such that $\pi(x) = \overline{x}$. Define $h_{\overline{x}}(\overline{v}, \overline{w}) = y_b(\mathrm{d}\pi|_x^{-1}\overline{v}, \mathrm{d}\pi|_x^{-1}\overline{w})$ $(*)$ with $\overline{v}$, $\overline{w} \in T_{\overline{x}}T^n$. We have to show, that the definition of $h_{\overline{x}}$ is independent of the choice of x. Therefore, let x and y be, such that $\pi(x) = \pi(y) = \overline{x}$. Then there exists a $z \in \mathbb{Z}^n$, such that $y = x + z$.

$$\begin{array}{ccc} x & \longrightarrow & y \\ {\scriptstyle \mathrm{d}\pi|_x}\downarrow & \swarrow {\scriptstyle \mathrm{d}\pi|_y} & \\ \overline{x} & & \end{array}$$

We note, that $\pi \circ t_z = \pi$, where $t_z : x \mapsto x + z$. So we get $\mathrm{d}\pi|_x = \mathrm{d}\pi|_y \circ \mathrm{d}t_z|_x$. This implies $\mathrm{d}\pi_x^{-1}(\overline{v}) = (\mathrm{d}t_z|_x)^{-1} \circ \mathrm{d}\pi_y^{-1}(\overline{v})$, $\overline{v} = \mathrm{d}\pi_y^{-1}(\overline{v})$. So $(*)$ is independent of x of y (the choice of x).

Remark:

The proof works, because the translations t_z with $z \in \mathbb{Z}^n$, which cover the projection π: $\mathbb{R}^n \mapsto T^n$ $(\pi \circ t_z = \pi)$ are isometries of $(\mathbb{R}^n, g_b)$.

Question: Are the h_b metrics on T^n all isometric? The answer is no in general. (See upcoming exercise!)

Be (M, g) a Riemannian manifold. So (M, d_g) is a metric space, namely the distance function, which is compatible with the topology on M (metric space structure of a Riemannian manifold).

2.3 Riemannian conncetion on (M, g)

In $\mathbb{R}^n$ we can take the derivatives of curves and also on an arbitrary manifold. Furthermore we can take the derivatives of vector fields $X, Y \in \mathrm{Vect}(U)$, $U \overset{0}{\subseteq} \mathbb{R}^n$. $\mathrm{d}X$ is well defined on U, because X: $U \mapsto \mathbb{R}^n$ is a smooth map. $\mathrm{d}_Y X$ is $\in \mathrm{Vect}(U)$ and it holds, that $(\mathrm{d}_Y X)|_x = \mathrm{d}X|_x(Y(x))$. If X is fixed, $\mathrm{d}X$ ist a tensor field on U, namely $\mathrm{d}X \in \Gamma(U, T^*U \otimes TU)$. That is $\mathrm{d}X|_x \in \mathrm{Hom}(T_xU, T_xU) = \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$. Remember, that M is a submanifold of $\mathbb{R}^n$: $X \in \mathrm{Vect}(M)$, $M \subseteq \mathbb{R}^n$, $T_xM \subseteq \mathbb{R}^n$ $\forall\, x \in M$. Then it holds X: $M \mapsto \mathbb{R}^n$, such that $X(x) \in T_xM$. If c: $I \mapsto M$ is a curve, we can consider:

$$\left.\frac{\partial}{\partial t}\right|_{t=0} X(c(t)) \in \mathbb{R}^n$$

Let π_x: $\mathbb{R}^n \mapsto T_xM$ be the orthogonal projection onto $T_xM \subseteq \mathbb{R}^n$. If $c(0) = x$, define for $X \in \mathrm{Vect}(M)$:

$$\left.\frac{\mathrm{D}}{\mathrm{d}t}\right|_{t=0} X(c(t)) = \pi_x\left(\left.\frac{\partial}{\partial t}\right|_{t=0} X(c(t))\right) \in T_xM$$

Example:

We want to consider $M = S^2 \subseteq \mathbb{R}^3$ and we define $c(t) = (\cos(t), \sin(t), 0) \in S^2$. The tangent is $\dot{c}(t) = (-\sin(t), \cos(t)) \in T_{\cos(t),\sin(t),0}S^2$. The second derivative is:

$$\ddot{c}(t) = \frac{\partial}{\partial t}\dot{c}(t) = -(\cos(t), \sin(t), 0)$$

So $\mathrm{D}/\mathrm{d}t\dot{c}(t) = 0$, because $\dot{c}(t) \cdot \ddot{c}(t) = 0$.

Given two vector fields X, Y on the submanifold $M \subseteq \mathbb{R}^n$, $x \in M$ we define

$$(\nabla_X Y)|_X := \left.\frac{\mathrm{D}}{\mathrm{d}t}\right|_{t=0} Y(c(t))$$

where c: $I \mapsto M$ is a curve with $c(0) = X$ and $\dot{c}(0) = X(x)$.

Lemma:

$\nabla_X Y$ is a smooth vector field on M.

Proof:

Extend on a neighborhood $U \overset{0}{\subseteq} \mathbb{R}^n$, $x \in U$ the vector field $X|_{U\cap M}$, $Y|_{U\cap M}$ to vector fields $\widetilde{X}$, $\widetilde{Y}$: $U \mapsto \mathbb{R}^n$, such that $\widetilde{X}|_{U\cap M} = X|_{U\cap M}$ and $\widetilde{Y}|_{U\cap M} = Y|_{U\cap M}$. Given these extensions it follows, that

$$\left.\frac{\partial}{\partial t}\right|_{t=0} Y(c(t)) = \mathrm{d}_{\widetilde{X}(x)}\widetilde{Y} = (\mathrm{d}_{\widetilde{X}}\widetilde{Y})|_x$$

Thus, $(\nabla_X Y)|_U(y) = \pi_y(\mathrm{d}_{\widetilde{X}}\widetilde{Y}|_y)$. This shows then, that $\nabla_X Y$ is a smooth vector field on $U \cap M$. This shows, that $\nabla_X Y \in \mathrm{Vect}(M)$. □

2.3.1 Remarks on orthogonal projections

Let V be a k-dimensional subspace of $\mathbb{R}^n$. π: $\mathbb{R}^n \mapsto \mathbb{R}^n$ is called projection onto V, if

i.) $\mathrm{Image}(\pi) = \pi(\mathbb{R}^n) = V$

ii.) $\pi^2 = \pi$

The projection π is called orthogonal, if the kernel of π is orthogonal to the image of π ($= V$). This property is equivalent to $\pi^\intercal = \pi$, where $\pi^\intercal$ is the transpose of π.

$\langle \pi(v), w\rangle = \langle v, \pi^\intercal(w)\rangle = \langle v, \pi(w)\rangle = 0$ where $w \in \mathrm{Kern}(\pi)$

There exists a unique orthogonal projection onto V.

Lemma:

Let φ: $\mathbb{R}^k \mapsto \mathbb{R}^n$ be a projective linear map of rank k. Then the map $\pi = \varphi(\varphi^\intercal\varphi)^{-1}\varphi^\intercal$ is the orthogonal projection of $\mathbb{R}^n$ onto $V = \mathrm{Im}(\varphi)$.
The lemma shows, that the π_y: $\mathbb{R}^n \mapsto T_yM$ depend smoothly on y. (Think about it!) The map ∇: $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \mapsto \mathrm{Vect}(M)$ is called the **induced connection** on the submanifold $M \subseteq \mathbb{R}^n$.

Proposition:

The map ∇ satisfies

i.) ∇ is linear with respect to scalar multiplication with real numbers and addition of vector fields in both variables.

ii.) For all $f \in C^\infty(M)$ it holds, that $\nabla_X fY = (L_X f)Y + f\nabla_X Y$ (product rule).

iii.) For all $f \in C^\infty(M)$ it is $\nabla_{fX} Y = f\nabla_X Y$ (C^∞-linearity in the second variable).

Definition:

Let M be a manifold, then ∇: $\mathrm{Vect}(M) \times \mathrm{Vect}(M) \mapsto \mathrm{Vect}(M)$, $(X, Y) \mapsto \nabla_X Y$, which satisfies (i), (ii) and (iii) of the proposition, is called a (linear) connection on M. ($\nabla_X Y$ is also called the „covariant" derivative of X and Y with respect to ∇.)
We will see, that on each Riemannian manifold (M, g) there exists a uniquely defined canonical connection $\nabla = \nabla_g$, which is called the **Levi-Civita connection of g**.

2.3.2 Properties of connections

Definition:

A linear operator P: $\mathrm{Vect}(M) \mapsto \mathrm{Vect}(M)$ is called a local operator, if for all vector fields Y_1, $Y_2 \in \mathrm{Vect}(M)$ and for all open subsets $U \subseteq M$ the condition $Y_1|_U = Y_2|_U$ implies, that $P(Y_1)|_U = P(Y_2)|_U$.

Examples of local operators:

For $X \in \mathrm{Vect}(M)$ the Lie derivative $P_X := L_X$ with $Y \mapsto L_X Y = [X, Y]$ is a local operator.

Lemma:

Let ∇ be a connection on M. Then $Y \mapsto \nabla_X Y$ and $X \mapsto \nabla_X Y$ are local operators on M.

Proof:

∇_X: $Y \mapsto \nabla_X Y$ is a local operator. Let Y_1 and Y_2 be vector fields on M, which satisfy $Y_1|_U = Y_2|_U$ and let $x \in U$. Choose a test function $f \in C^\infty(M)$ with support contained in U and also $f \equiv 1$ on $U_1 \subseteq U$, $x \in U_1$. It follows, that $fY_1 = fY_2$ and $\nabla_X fY_1 = \nabla_X fY_2$. Compute at x as follows: $(\nabla_X fY_1)|_x = f\nabla_X Y_1|_x =$ and $f\nabla_X Y_2|_x = \nabla_X fY_2|_x$. (This is a result of the product rule.) Finally we have $f\nabla_X Y_1|_x = f\nabla_X Y_2|_x$.

2.4 Parallel transport

We are looking at a manifold M with ∇ as a connection on M. γ: $I \mapsto M$, $\gamma(0) = x$ be a curve. If $v \in T_x M$, there exists a unique vector field $X \in \mathrm{Vect}(\gamma)$, $X(0) = v$ with $\nabla_t X = 0$. P_t: $T_x M \mapsto T_{\gamma(t)} M$, $v \in T_x M \mapsto X(t)$ is then called parallel transport.

Proposition:

i.) The map P_t: $T_x M \mapsto T_{\gamma(t)} M$ is a linear isomorphism of tangent spaces.

ii.) If ∇ is the linear connection for a Riemannian metric g on M, then P_t: $(T_x M, g_x) \mapsto T_{\gamma(t)}, g_{\gamma(t)}$ is an isometry of metric linear spaces.

Proof:

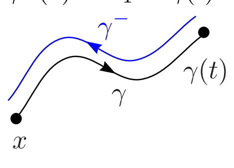
i.) P_t is a linear map. Let v_1, $v_2 \in T_x M$ and a,$b \in \mathbb{R}$. Furthermore let X_{v_1}, X_{v_2} be $\in \mathrm{Vect}(\gamma)$, $\nabla_t X_{v_1} = \nabla_t X_{v_2} = 0$ with $X_{v_1}(0) = v_1$ and $Y_{v_2}(0) = v_2$. Then we note:

$\nabla_t(aX_{v_1} + bX_{v_2}) = a\nabla_t X_{v_1} + b\nabla_t X_{v_2} = 0$ and $(aX_{v_1} + bX_{v_2})(0) = av_1 + bv_2$

This includes the following:

$P_t(av_1 + bv_2) = aX_{v_1}(t) + bX_{v_2}(t) = aP_t(v_1) + bP_t(v_2)$

So P_t is linear. To show, that P_t is an isomorphism, let us consider the curve $\gamma^-(s) = \gamma(t - s)$ with $\gamma^-(0) = x_1 = \gamma(t)$ and $\gamma^-(t) = \gamma(0) = x$.



If X is a vector field along γ, $X^-(s) = X(t - s)$ defines a vector field along γ. Suppose, X is parallel along γ, so $\nabla_t X = 0$. Then it follows, that X^- is parallel γ^-. (along γ: $0 = \nabla_t X = \nabla_{\dot\gamma} X$ with $X \in \mathrm{Vect}(M)$ and along γ^-: $\nabla_s X = \nabla_{\dot\gamma^-(s)} X = \nabla_{-\dot\gamma(t-s)} X = 0$) This shows, that the parallel transport of $w = X(t) = P_t(v)$ along γ^- is $X^-(t) = X(0) = v$. That is, if P_{-t} denotes parallel transport along γ^-: $T_{\gamma(t)} M \mapsto T_x M$, then $P_t \circ P_{-t} = \mathrm{id}_{T_x M}$. It also follows, that $O_t \circ P_{-t} = \mathrm{id}_{T_{\gamma(t)} M}$, what shows us, that P is bijective.

ii.) It is enough to proof, the following. Let X_1 and X_2 be parallel along γ. Then the function $g(X_1(t), X_2(t))$ is constant. For this we are allowed to compute

$$\frac{\partial}{\partial t} g(X_1(t), X_2(t)) = g((\nabla_t X_1)(t), X_2(t)) + g(X_1(t), (\nabla_t X_2)(t)) = 0$$

because ∇ is the Levi-Civita connection on M and the vector fields X_1 and X_2 are parallel. □

Application of (ii) would be the propagation of orthonormal frames along γ. Let $x = \gamma(0)$ and e_i be an orthonormal basis of (T_xM, g_x). Then the parallel transported vector fields $E_i(t)$, $E_i(0) = e_i$, $\nabla_t E_i = 0$ for an orthonormal basis at $(T_{\gamma(t)}M, g_{\gamma(t)})$ (orthonormal frame).

2.5 Geodesics

Definition:

We consider ∇ to be a connection on M. A curve γ: $I \mapsto M$ is called a geodetic for ∇, if $\nabla_t \dot\gamma = 0$.

Remark:

This is an analogue of the straight lines in $\mathbb{R}^n$ (curves, where the second derivatives vanished).
In local coordinates: Let x_i: $U \mapsto \mathbb{R}^n$ be a local coordinate system and $\gamma_i = x_i \circ \gamma$. Then we can compute:

$$\nabla_t \dot\gamma = \sum_l \left(\ddot\gamma_l + \sum_{i,j}^{l} {\Gamma_{ij}}^l \dot\gamma_i \dot\gamma_j \right) \frac{\partial}{\partial x_l}$$

So γ is a geodesic, if and only if the equation

$$\ddot\gamma_l + \sum_{i,j}^{l} {\Gamma_{ij}}^l \dot\gamma_i \dot\gamma_j = 0$$

holds for all $l = 1, \ldots, n$. This is a second order differential equation. As a consequence we note the following proposition:

Proposition:

Let $x \in M$, $v \in T_xM$. Then there exists an interval $I =]-\varepsilon, \varepsilon[$ and a curve γ: $I \mapsto M$ with $\gamma_v(0) = x$, $\dot\gamma_v(0) = v$, $\nabla_t \gamma = 0$.

Lemma:

Let γ: $I \mapsto M$ be a geodesic and f: $I \mapsto J$ a representation and $\widetilde\gamma$: $J \mapsto M$, $\widetilde\gamma(s) = \gamma(f^{-1}(s))$. Then $\widetilde\gamma$ is a geodesic, if and only if f is an affine reparametrization, $f(t) = at + b$ with a, $b \in \mathbb{R}$ and $a \neq 0$. (γ of course should be not constant.)

Proof:

We compute

$$\nabla_s \dot{\widetilde\gamma} = \frac{\partial}{\partial s} f^{-1} \dot\gamma + f^{-1} \nabla_t \dot\gamma = \left(\frac{\partial}{\partial s} f^{-1} \right) \dot\gamma$$

$\nabla_s \dot{\widetilde\gamma} = 0$ includes $\frac{\partial}{\partial s} f^{-1}) \dot\gamma = 0$. □

Lemma:

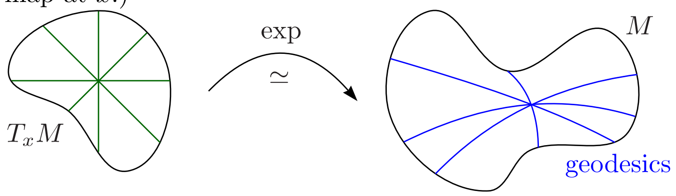
Let $x \in M$. Then there exists a neighbourhood U of $0_x \in T_xM$, $U \subseteq TM$ and an interval $I =]-\varepsilon, \varepsilon[$, such that the map $U \times I \mapsto M$, $(v, t) \mapsto \gamma_v(t)$ is well defined and smooth.

Proof:

That is the smooth dependence of solutions on the initial value. □

Proposition:

Given $x \in T_xM$, there exists a neighbourhood $U \subseteq TM$ with $0_x \in U$, such that the map exp: $v \mapsto \gamma_v(1)$ is well defined and smooth. (The map exp: $U \mapsto M$ is called the exponential map.)
A consequence of the proposition is the following. Let $x \in M$, then there exists $U \subset T_xM$ with $0_x \in U$, such that $\exp_x = \exp|_U \colon U \mapsto M$, $v \mapsto \exp_x(v) = \gamma_v(1)$ is well defined an smooth. ($\exp_x$ is called the exponential map at x.)



Remark:

Note, that for $v \in T_xM$ the curves $\exp_x(tv) = \gamma(t)$ are geodesics. For this note, that $\gamma_{sv}(t) = \gamma_v(st)$, because $\gamma(t) := \gamma_v(st)$ is a geodesic by our previous lemma. $\gamma(0) = \gamma_v(0) = x$ and $\dot\gamma(0) = s\dot\gamma_v(0) = sv$ implies $\gamma(t) = \gamma_{sv}(t)$.
Now we compute $\exp_x(tv) = \gamma_{tv}(1) = \gamma_v(t)$. Hence, the curve $t \mapsto \exp_x(tv)$ is the geodesic starting at x with tangent v.

Lemma:

The exponential map $\exp_x \colon U \mapsto M$ has the derivative $\mathrm{d}\exp_x|_0 \colon T_0T_xM = T_xM \mapsto T_xM$ and $\mathrm{d}\exp_x|_0 = \mathrm{id}_{T_xM}$.

Proof:

Let $v \in T_0T_xM = T_xM$.

$$\mathrm{d}\exp_x|_0(v) = \frac{\partial}{\partial t}\bigg|_{t=0} \exp_x(tv) = \frac{\partial}{\partial t}\bigg|_{t=0} \gamma_v(t) = \dot\gamma_v(0) = v \qquad \square$$

The consequence of this lemma is, that there exists a neighbourhood $U_1 \subseteq T_xM$, $0_x \in U_x$, such that $\exp_x$: $U_1 \mapsto M$ is a diffeomorphism of U_1 onto an open subset of M (**exponential coordinates in (M, ∇)**).

2.6 Gauß-Lemma

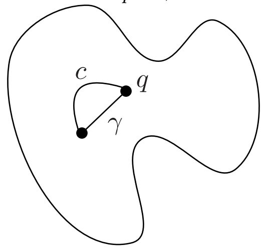
Let $p \in M$, $\exp_p \colon B(z, 0) \mapsto M$ with $B(\varepsilon, 0) \subset T_pM$. For $v \neq 0$ and $w \in T_pM$ it holds, that $g(\mathrm{d}\exp_p|_v(v), \mathrm{d}\exp|_p(w)) = \langle v, w\rangle$, where $\langle v, w\rangle = g|_p(v, w)$.

Proposition:

Let $q \in U = \exp_p(B(\varepsilon, 0))$. Then there exists a vector $v \in B(\varepsilon, 0)$ and a geodesic γ_v starting at p, such that $\gamma_v(1) = q$ and γ_v is the unique „distance“ minimizing curve joining p and q. (In particular, $L(\gamma_v|_{[0,1]}) = d(p, q)$.)

Proof:

We let $c \colon I = [0, 1] \mapsto M$ be a curve in M, such that $c(0) = p$ and $c(1) = q$. If $c(t)$ is $\in U$, we can define $\dot c^{\|}(t)$ as follows: Since **by assumption**, $\exp_p \colon B_\varepsilon(0) \mapsto U$ is a diffeomorphism, $\dot c(t) = \mathrm{d}\exp|_v(w)$ for a unique $v \in B_\varepsilon(0)$ and $w \in T_pM$, into its tangent direction $w = \alpha v + w_1$, where w_1 is orthogonal to v. Define $\dot c(t)^{\|} := \mathrm{d}\exp|_v(\alpha v)$.



We want to show, that $L(c) \geq L(\gamma)$.

$$L(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| = \int_0^1 |v|$$

Suppose c remains in U, then we compute the length of the curve:

$$L(c) = \int_0^1 g(\dot{c}(t), \dot{c}(t))^{\frac{1}{2}} \geq \int_0^1 g(\dot{c}(t)^{\|}, \dot{c}(t)^{\|})^{\frac{1}{2}}$$

$c(t) = \exp_p(\overline{c}(t))$ with $\overline{c} : I \mapsto B_\varepsilon(0)$ and $\overline{c}(t) = r(t)v + w_1$

The Gauss lemma now says:

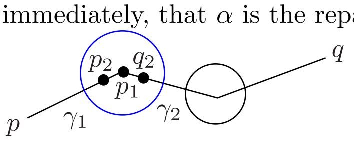
$$\int_0^1 g(\dot{c}(t)^{\|}, \dot{c}(t)^{\|})^{\frac{1}{2}} = \int_0^1 |\dot{r}(t)| \geq \int_0^1 \dot{r}(t) = r(1) - r(0) = |v| = L(\gamma)$$ □

Corollary:

Let (M, g) be a Riemannian manifold and p, $q \in M$. Furthermore be α the shortest curve between p and q. Then α is the representation for a geodesic.

Proof:

By the Gauss Lemma we know, that the geodesics are locally the only distance minimizing curves. It follows immediately, that α is the reparametrization of a broken geodesic, so α looks like something like that:



So we have to check, what happens near the neighbourhood of such a „break point“. If a broken geodesic is distance minimizing, then it is a geodesic. We take a geodesically convex neighbourhood of p_1. (Convention: If $\exp_p$: $B_\varepsilon(0) \mapsto M$ is a diffeomorphism with image U, then U will be called a **local neighbourhood** of p. We call U a **convex normal neighbourhood**, if it is a normal neighbourhood for all points $q \in U$.) The broken segment joining p_2 and q_2 is distance minimizing. Because q_2 is in a normal neighbourhood of p_2, this curve must be a geodesic. This implies, that α cannot be broken at p_1. (Heuristically: We could smoothen it up and make it shorter!) □

Summary:

Let (M, g) be a Riemannian manifold.

i.) Locally, geodesics are distance minimizing curves.

ii.) Every distance realizing curve α: $I \mapsto M$ with $L(\alpha) = d(p, q)$, $\alpha(0) = p$ and $\alpha(1) = q$ „is“ a geodesic.

Definition:

Every geodesic γ with $\gamma(0) = p$, $\dot{\gamma}(0) = v \in T_pM$ has a maximal interval of definition, such that γ: $I_{max} \mapsto M$ is defined. We call γ **complete**, **if** $\boldsymbol{I_{max}} = \mathbb{R}$. That is, γ can be extended to infinity (for all times).
We call (M, g) complete at $p \in M$, if all geodesics starting at p are complete. (M, g) is called **geodesically complete**, if it is complete at all points $p \in M$.

Remark:

(M, g) with its distance d is a metric space. (M, g) is called (metrically) complete, if the metric d is complete.

2.7 The Hopf Rinow theorem

Corollary:

Let (M, g) be complete. Then very two points p and a can be joined by a distance minimizing geodesic.

Key Lemma:

Let $p \in M$ be a point, such that M is geodesically complete at p. Then for all $q \in M$ there exists a distance realizing geodesic γ joining p and q.

Theorem:

Let (M, g) be a Riemannian manifold. Then the following are equivalent:

i.) M is geodesically complete.

i'.) M is geodesically complete for one $p \in M$.

ii.) M is metrically complete.

iii.) Every bounded and closed subset of M is compact.

(This result does not hold for non-Riemannian manifolds!)

Proof:

We now proof the Hopf Rinow theorem using the key lemma. Let us start with $(ii) \Rightarrow (i)$. Let γ: $I = [0, \varepsilon[$ be a geodesic. Therefore consider a sequence $t_i \mapsto \varepsilon$. It follows, that $\gamma(t_i)$ are Cauchy sequences with respect to d, because $d(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j|$. We may assume by (ii), that the $\gamma(t_i)$ converge to some point q in M. Choose **convex** normal coordinates around q, there exist a δ, such that $U = \exp_q(B_\delta(0))$. Now for $\gamma(t_i) \in B_\delta(0)$ it holds, that $|t_i - \varepsilon| < \delta/2$. We see, that there is an extension of our geodesic beyond ε, **because in our convex neighbourhood U, there exists a δ, such that every geodesic segment is defined at least up to length δ!** Hence, γ can be extended to $I = [0, \varepsilon + \delta/2($.

Corollary:

If M is compact, then (M, g) is complete.

Proof:

This follows from (iii) of the Hopf Rinow theorem, because M is Hausdorff.

2.8 Riemannian Curvature

Be ∇ be a connection on M:

$$R^\nabla(X,Y)Z = \nabla_[X,Y]Z - \nabla_X\nabla_Y Z + \nabla_Y \nabla_X Z \left(= (\nabla_{[X,Y]} - [\nabla_X, \nabla_Y])Z\right)$$

We have already seen, that R^∇ defines a $(1,3)$-tensor field, also called R^∇ on M. This tensor field is called the **curvature tensor** of ∇. First, let us look at the interpretation of the curvature definition. We define the second covariant derivative of $Z \in \mathrm{Vect}(M)$ as follows: For $Z \in \mathrm{Vect}(M)$ ∇Z, $X \mapsto \nabla_X Z$ is a (1,1)-tensor field.

$$\nabla_Y(\nabla_\bullet Z)(X)\nabla_Y\nabla_X Z - \nabla_{\nabla_Y X} Z =: \nabla^2_{Y,X} Z$$

This is a (1,2)-tensor for Z fixed.

Proposition:

It holds $R^\nabla(X,Y)Z = \nabla^2_{Y,X}Z - \nabla^2_{X,Y}Z$ for a torsion-free ∇.

Proof:

$$\nabla_Y X - \nabla_X Y = -[X,Y]$$ □

Example:

Be $(\mathbb{R}^n, D)$ and D the canonical connection on $\mathbb{R}^n$, $R^D = 0$. The connection D is flat.

2.8.1 Defined curvature notions

Define $\mathrm{Ric}^\nabla(X,Y) \mapsto \mathrm{Tr}(Z \mapsto R^\nabla(X,Z)Y)$. Ric^∇ defines a tensor of type (0,2). That is, Ric^∇ is a bilinear form on each tangent space, called the **Ricci-tensor of ∇** (**Ricci-curvature**).

Remark:

$R^\nabla(X,Y)$: $Z \mapsto R^\nabla(X,Y)Z$ is a linear operator of the tangent space(s)s, namely the **curvature operator**. It holds $R^\nabla(X,Y) = -R^\nabla(Y,X)$. So the curvature is a 2-form with values in the bundle $\mathrm{End}(TM) = TM^* \otimes TM$.

2.8.2 Symmetries of the curvature (Bianchi identities)

Proposition:

Let ∇ be a torsion free connection.

i.) First Bianchi identity: $R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$

So the sum over all cyclic permutations of the variables X, Y and Z vanishes.

ii.) Second Bianchi identity:

Let ∇R be the covariant derivative of R. (∇R is a (1,4) tensor.) It holds, that $(\mathcal{C}\nabla R)(X,Y,Z,U) = 0$, where $\mathcal{C}$ denotes the sum ober all cyclic permutations.

(For non-torsion-free connections the torsion appears in the upper formulas.)

Proof:

We will only proof the first identity. Since R^∇ is a tensor, it is enough, to proof (i) for vector fields, which commute. For such vector fields $R^\nabla(X,Y)Z = \nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z$. We want to calculate now the cyclic permutations of these R^∇:

$$\mathcal{C}R^\nabla(X,Y)Z = \mathcal{C}\nabla_Y\nabla_X Z - \mathcal{C}\nabla_X\nabla_Y Z = \mathcal{C}\nabla_Z\nabla_X Y - \mathcal{C}\nabla_Z\nabla_Y X = \mathcal{C}\nabla_Z[X,Y] = 0$$ □

Let (M,g) be a Riemannian manifold and ∇ the Levi-Civita connection of (M,g). We define a (0,4) tensor on M, namely $R(X,Y,Z,W) := g(R^\nabla(X,Y)Z,W)$ with $R := R^\nabla$. The (1,3)-tensor R is called the (Riemann) curvature tensor of (M,g).

Proposition:

i.) It holds $g(R(X,Y)Z,W) = -g(Z,R(X,Y)W)$.

ii.) Furthermore we have the symmetry $g(R(X,Y)Z,W) = g(R(Z,W)X,Y)$.

Proof:

We only want to proof (i). Use, that all brackets are zero. We use, that ∇ is a metric connection:

$$\begin{aligned} R(x,y,z,w) &= g(\nabla_Y\nabla_X Z - \nabla_X\nabla_Y Z, W) = \\ &= L_Y g(\nabla_X Z, W) - g(\nabla_X Z, \nabla_Y W) - L_X g(\nabla_Y Z, W) + g(\nabla_Y Z, \nabla_X W) = \\ &= \underbrace{L_{[Y,X]} g(Z,W)}_{=0} - g(Z, R(X,Y)W) \end{aligned}$$

□

2.8.3 Interpretation of the symmetries

The operator $R(X,Y)$ is skew with respect to the metric g by (i). Summarizing (iii) and (iv) we may interpret $R(x,y,z,w)$ as a symmetric form on $TM \wedge TM$, because $R(x \wedge y, z \wedge w) := R(x,y,z,w)$.

Proposition:

The Ricci tensor $\mathrm{Ric} = \mathrm{Ric}^\nabla$ is a symmetric bilinear form for a Riemannian connection ∇. Hence, it defines a symmetric linear operator $\widehat{\mathbf{Ric}}$ by the formula $\mathrm{Ric}(X,Y) = g(\widehat{\mathrm{Ric}}(X),Y)$. $\widehat{\mathrm{Ric}}$ is called the **Ricci operator**. The trace of $\widehat{\mathrm{Ric}}$ is called the **scalar curvature of (M,g)**.

$s: M \mapsto \mathbb{R}$, $s(p) = \mathrm{Trace}(\widehat{\mathrm{Ric}}|_p)$, $p \in M$

2.8.4 The sectional curvature of a Riemannian manifold

Be K: Grassmannian bundle of 2-planes on $M \mapsto \mathbb{R}$ for every $p \in M$ and every two-dimensional subspace $\pi \subseteq T_pM$. We define a number $K_p(\pi) = \mathbb{R}$, which is called the sectional curvature of the **plane $\boldsymbol{\pi}$**. Given w, $w \in T_pM$, which span a plane $\pi \subset T_pM$. Define $Q(v,w) = \langle v,v\rangle\langle w,w\rangle - \langle v,w\rangle^2$ with $\langle,\rangle = g|_{T_pM}$. $Q(v,w)$ is a positive number. Furthermore, Q is a quadratic form on $T_pM \wedge T_pM$. We define:

$$K(v,w) := \frac{R(v,w,v,w)}{Q(v,w)} = \frac{R(v\wedge w, v\wedge w)}{Q(v\wedge w)}$$

The number $K(v,w)$ only depends on the plane π, because if $\pi = \mathrm{Span}(v,w) = \mathrm{Span}(v^2,w^2)$, there exists $\alpha \in \mathbb{R}$, $\alpha \neq 0$ with $v \wedge w = \alpha v^2 \wedge w^2$.

Proposition:

The sectional curvature determines the curvature tensor R.

Proof:

R is a symmetric form on the space $T_xM \wedge T_xM$. Let F be the corresponding quadratic form. By the polarization is determined by F. Actually, we have, that $K(v\wedge w) = F(v\wedge w)/Q(v\wedge w)$. □

Examples:

1.) For $n = 1$, M is a one-dimensional manifold, $R \equiv 0$. (Heuristically: You can embed a line in curved way, but when you are inside it, you cannot see the curvature.)

2.) For $n = 2$, the sectional curvature is a function K: $M \mapsto \mathbb{R}$, $p \mapsto K_p(T_pM) = K(v,w)$, where v, w is a basis of T_pM.

3.) For $n = 3$ the Ricci tensor determines the curvature tensor.

Definition:

We say, that (M,g) has constant curvature, if the function K is constant. All sectional curvatures have the same value.

Exercise:

If M has constant curvature, it holds $R(X,Y)Z = C(g(X,Z)Y - g(Y,Z)X)$ with the constant C.

Example:

1.) (S^n, can) is of constant curvature $C = 1$.

2.) $(\mathbb{H}^2, 1/g\langle,\rangle)$ is of constant curvature $C = -1$.