String-Steilkurs Teil II: 2009
String Geometry and
Gauged Linear Sigma Models

I. Melnikov

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Albert-Einstein-Institut in Golm

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Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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Chapter 1

Gauged linear sigma model/stringy geometry

1.1 Motivation

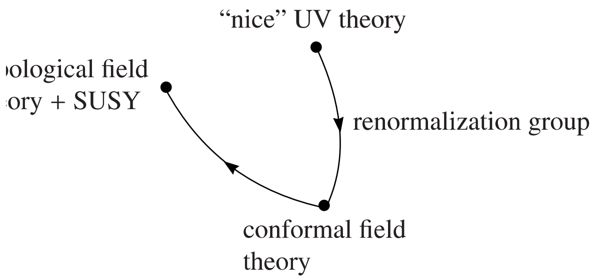
The nature of the string leads to some surprises for a particle physicists:

- A background which is bad from a particle point of view can be good for a string point of view. Orbifolds can be consistent string backgrounds. Strings do not have large backreactions although there are singularities.
- T-duality: The string theory on S^1_R is equivalent to a string theory on $S^1_{\alpha'/(2R)}$.
- Mirror symmetry: String theory on Calabi-Yau manifold CYM is equivalent to string theories on CYM^0, where M and M^0 are topologically distinct.

The non-perturbative aspects are for now unimportant; study of CFT!

1.1.1 Basic picture

For constructing these kinds of theories one has to construct CFT on these manifolds. Start with a nice UV theory and use RG to go to CFT. One can use topological field theory and SUSY to go between the two theories.



The rough plan will be to start with some overview of $d = 2$ $\mathcal{N} = (2,2)$ super conformal field theories. Furthermore we will consider Landau-Ginzburg models, gauged linear sigma models and topological field theory.

Why consider $\mathcal{N} = (2,2)$? It is a unitary, **compact** super conformal field theory. With $c = \bar{c} = q$ and $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ charges this leads to a vacuum of string with $d = 4$, $\mathcal{N} = 1$ SUGRA plus SYM with $G \supset E_6 \times E_8$. The algebraic structure is under control! Compact means that for every $H \geq 0$ there is a finite number of operators with weight $\leq H$. The left-moving $\mathcal{N} = 2$ super conformal algebra, which is a multiplet of holomorphic currents, has the following structure: There is a left handed U(1) current $j(z)$, the SUSY generators $G^+(z)$, $G^-(z)$, Virasoro algebra and the energy momentum tensor $T(z)$.

	$+1$	0	-1	$\mathrm{U}(1)_L/h$
$\mathrm{U}(1)_L$		$j(z)$		1
SUSY	$G^+(z)$		$G^-(z)$	3/2
Virasoro, $T(z)$				2

There is a non-obvious operator product expansion:

$$G^{\pm}(z)G^{\pm}(0) \sim 0\,, \quad j(z)j(0) \sim \frac{c}{3z^2}\,, \tag{1.1}$$

$$G^{+}(z)G^{-}(0) \sim \frac{2c}{3z^3} + \frac{2}{z^2}j(0) + \frac{2}{z}T(0) + \frac{1}{z}\partial j(0)\,. \tag{1.2}$$

Exercise: Obtain the "obvious" operator product expansions! Do a mode expansion in c (consider the NS sector):

$$T = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}\,, \quad J = \sum_{n\in\mathbb{Z}} J_n z^{-n-1}\,, \tag{1.3}$$

$$G^{\pm} = \sum_{r\in\mathbb{Z}\pm 1/2} G^{\pm}_r z^{-r-\frac{3}{2}}\,. \tag{1.4}$$

This allows us to describe nicely the action of the operators on the fields: $L_0|\phi\rangle = h_\phi|\phi\rangle$ and similarly $J_0|\phi\rangle = q_\phi|\phi\rangle$. Now to some definitions to characterize the very states $|\phi\rangle$ in the super conformal field theory.

- $|\phi\rangle$ is called **primary**, iff $L_n|\phi\rangle = J_n|\phi\rangle = G_r|\phi\rangle = 0$ for n, $r > 0$.
- $|\psi\rangle$ is called **chiral**, iff $G^{+}_{-1/2}|\psi\rangle = 0$.
- $|\psi\rangle$ is antichiral, iff $G^{-}_{-1/2}|\psi\rangle = 0$.

Consider $\mathcal{N} = 2$ super conformal algebra in a **unitary conformal field theory** [explained nicely in paper by Lerche, Vafa, Warner]. The consequence of this is that if we have a chiral state $|\phi\rangle$, then $h_\phi \geq q_\phi/2$ $(*)$. $|\phi\rangle$ is chiral primary, iff $h_\phi = q_\phi/2$. If $|\phi\rangle$ is antichiral, then $h_\phi \geq -q_\phi/2$. $|\phi\rangle$ is antichiral primary, iff $h_\phi = -q_\phi/2$. Furthermore, one can prove that $h_\phi \leq c/6$ and therefore a compact SCFT contains a finite number of (anti)chiral primaries. Exercise: Derive $\{G^{+}_{-1/2}, G^{-}_{1/2}\} = 2L_0 - J_0$ and use it to prove $(*)$.

1.2 The chiral rings

Consider both left and right movers. Let $\phi(z)$ and $\psi(z)$ be left and right chiral primaries, respectively. Let us look at the operator product expansion (follows from conformal invariance):

$$\phi(z)\psi(0) \underset{\substack{\chi\ \text{chiral} \\ q_\chi = q_\phi + q_\psi}}{\sim} z^{h_\chi - h_\phi - h_\psi}\,\overline{z}^{-\overline{h}_x - \overline{h}_\phi - \overline{h}_\psi}\chi(0)\,, \tag{1.5}$$

with $h_\chi - h_\phi - h_\psi = h_\chi - 1/2 q_\chi \geq 0$.

$$\lim_{z,\overline{z}\mapsto} \phi(z)\psi(0) = \chi(0)\,, \tag{1.6}$$

for some **chiral primary** χ. OPE gives the set of chiral primary operators with a ring structure. These rings are typically denoted by (c,c), (a,c), (a,a), and (c,a), if the operators are chiral (c) or antichiral (a). The two pairs (c,c), (a,c) are related to (a,a) and (c,a) by conjugation. It holds that

$$\phi_\alpha \cdot \phi_\beta = C_{\alpha\beta}{}^{\gamma}\phi_\gamma\,, \tag{1.7}$$

with the structure constants $C_{\alpha\beta}{}^{\gamma}$. These kinds of theories typically come in families and the structure constants vary, if one changes the parameters of the CFT.

1.3 Deformations of $\mathcal{N} = (2,2)$ super conformal field theories

Let $\Phi_i(z,\overline{z})$ be a **truly marginal** operator in some CFT. Define perturbed correlators $\langle \mathcal{O}_1(z_1)\ldots\mathcal{O}_k(z_k)\rangle_t$ in the following way:

$$\langle \mathcal{O}_1(z_1)\ldots\mathcal{O}_k(z_k)\rangle_t \equiv \left\langle \mathcal{O}^R_1(z_1)\ldots\mathcal{O}^R_k(z_k)\exp\left(-t^2\int \mathrm{d}^2z\,\Phi_i\right)\right\rangle\,. \tag{1.8}$$

The perturbation expansion in t has a finite radius of convergence. We have to make sure that the operators Φ_i really preserve conformal invariance. Φ_i is **truly marginal**, if the perturbed correlators are conformally covariant. The index R stands for "renormalized". Define a metric on the moduli space

$$G_{ij}(t) \equiv \langle \Phi_i(1)\Phi_j(0)\rangle_t \,, \tag{1.9}$$

which is the Zamolodchikov metric. In the process of renormalization one has to introduce contact terms; the curvature of the above metric shows up in these contact terms. Consider $\mathcal{N} = (2,2)$ SCFT with $\phi_\alpha \in$ (c,c) and $q_\alpha = \overline{q}_\alpha = 1$. Let $\psi_A \in$ (a,c) with $-q_A = q_A = 1$. Consider the following operators:

$$\Phi_\alpha \equiv \overline{G}^-_{-1/2}\overline{G}^-_{-1/2}\phi_\alpha \,, \quad \Psi_A = \overline{G}^-_{-1/2}G^+_{-1/2}\psi_A \,. \tag{1.10}$$

These have $h = \overline{h} = 1$ and $q = \overline{q} = 0$. A theorem by [Dixon] tells us that Φ_α and Ψ_A are truly marginal. The metric becomes block diagonal ($G_{ij} \mapsto G_{\alpha\overline{\alpha}}$, $G_{A\overline{A}}$) and the moduli space is $M_{(\mathrm{c,c})} \times \mathcal{M}_{(\mathrm{a,c})}$.

1.4 Geometric interlude

A **complex manifold** $\mathcal{M}$ admits consistent complex coordinates on each patch and between patches one can use holomorphic transition functions. The basic point is that the choice of transition functions preserving the diffeomorphism type of $\mathcal{M}$ is the choice of complex structure on $\mathcal{M}$. A complex Riemannian manifold $\mathcal{M}$ has a Hermitian metric $\mathrm{d}s^2 = g_{i\overline{j}}(z,\overline{z})\,\mathrm{d}z^i\,\mathrm{d}\overline{z}^{\overline{j}}$. Using $g_{i\overline{j}}$ we can build a Hermitian two-form $\omega = \mathrm{i}/2 g_{i\overline{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\overline{j}}$. The manifold is said to be Kähler, if this form is closed: $\mathrm{d}\omega = 0$. $[\omega] \in H^2_{\mathrm{dR}}(M)$ is called the Kähler class. Recall the de Rham Cohomology group:

$$H^k_{\mathrm{dR}} = \{\ker : d : \Omega^k \to \Omega^{k+1}\}/\{\mathrm{im} : d : \Omega^{k-1} \to \Omega^k\} \,. \tag{1.11}$$

Cohomology on complex manifolds:

$$\mathbb{C} \otimes \Omega^k = \oplus_{p,q}\Omega^{p,q} \,, \quad d = \partial + \overline{\partial} \,, \tag{1.12}$$

$$H^{p,q}_{\overline{\partial}}(\mathcal{N}) = \frac{\{\ker(\overline{\partial}) : \Omega^{p,q} \to \Omega^{p,q+1}\}}{\{\mathrm{im}\overline{\partial} : \Omega^{p,q-1} \to \Omega^{p,q}\}} \,. \tag{1.13}$$

On Kähler manifold:

$$H^k_j \otimes \mathbb{C} \simeq \oplus_{p+q=k} H^{p,q}_j(\mathcal{M}) \,, \tag{1.14}$$

$$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(\mathcal{M}) \equiv h^{p,q}(\mathcal{M}) \,. \tag{1.15}$$

Ricci curvature on Kähler manifold $R_{ii} = R_{\overline{j}\overline{j}} = 0$. From $R_{i\overline{j}}$ one can construct the Ricci form $\mathcal{R} \equiv \mathrm{i}/(2\pi)R_{i\overline{j}}\,\mathrm{d}z^i \wedge \mathrm{d}\overline{z}^{\overline{j}}$. One can show that the Ricci form is closed: $\mathrm{d}\mathcal{R} = 0$. This implies $[\mathcal{R}] \in H^2_{\mathrm{dR}}(\mathcal{M},\mathbb{Z})$. The class $[\mathcal{R}]$ is precisely the first Chern class $\mathcal{C}_1(T_M)$ of the manifold. $\mathcal{C}_1(T_M) \neq 0$ is an obstruction to finding a Ricci flat metric on $\mathcal{M}$. Definition: A **Calabi-Yau manifold** is a compact Kähler manifold with $\mathcal{C}_1(T_M) = 0$. This definition leads to **Yau's theorem**: For fixed complex structure and Kähler class, a Calabi-Yau manifold admits a **unique** Ricci-flat metric. **Deformations** of Ricci-flat metrics (studied by Tian, Todorov, Wilson) lead to the result that for a generic enough c-x-structure and Kähler class, the moduli space of Ricci-flat metrics is a smooth manifold $\mathcal{M}_K \times \mathcal{M}_{C-X}$ with

$$T\mathcal{M}_K \simeq H^{i,j}_{\mathrm{dR}}(M) \,, \quad T\mathcal{M}_{C-X} \simeq H^{d-1}_{\overline{\partial}}(M) \,, \quad d = \dim_{\mathbb{C}}(M) \,. \tag{1.16}$$

Description of manifolds in terms of cohomology groups of our space. Remark: To exclude degenerate possibilities

$$h^{R,0}(M) = \begin{cases} 1 & \text{if} \quad k = 0 \\ 0 & \text{otherwise} \end{cases} \,. \tag{1.17}$$

1.5 Example: Calabi-Yau compactification of type II string

We start with $\mathbb{R}^{1,q}$ and we want to compactify it to $\mathbb{R}^{1,3} \times M$ with $\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + g_{IJ}(\phi)\,\mathrm{d}\phi^I\,\mathrm{d}\phi^J$. World-sheet action:

$$S_{\text{string}} = S_{\text{H3}} + S_{\text{ghost}} + S_{\text{NLSM}}\,, \tag{1.18}$$

with

$$S_{\text{NLSM}} = \frac{1}{\pi\alpha'}\int \mathrm{d}^2 z\,[g_{IJ}(\phi) + \mathrm{i}B_{IJ}(\phi)]\partial\phi^I\overline{\partial}\phi^J + \text{fermions}\,. \tag{1.19}$$

The action has $\mathcal{N} = (2,2)$ supersymmetry, if the manifold M is Kähler and $\mathrm{d}B = 0$. Furthermore, it obeys a one-loop conformal invariance, if $R_{IJ} = 0$. Therefore, M has to be a Calabi-Yau manifold! Look at NLSM deformations: Choose a Ricci flat metric g and a closed B-field. The manifold has a complex structure: $\mathcal{M}_{CX}(M)$ with $\dim_{\mathbb{C}}\mathcal{M}_{C-X} = h^{2,1}(M)$. The **complexified** $[\omega + \mathrm{i}B]$: $\mathcal{M}_{\text{CK}}(M)$, $\dim_{\mathbb{C}}\mathcal{M}_{\text{CK}} = h^{1,1}(M)$. Chiral Ring analysis:

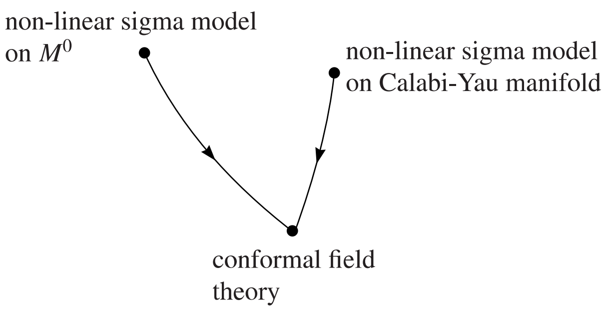
$$\mathcal{M}_{(\text{a,c})}(\text{NLSM}) = \mathcal{M}_{\text{CK}}(M)\,, \quad \mathcal{M}_{(\text{c,c})}(\text{NLSM}) = \mathcal{M}_{\text{CX}}(M)\,. \tag{1.20}$$

But from an abstract scientific point of view, what we call (a,c) and what (c,c), is **our** choice. Exercise: Show that $\mathcal{N} = 2$ SCA has a isomorphism that flips the **sign** of $\mathrm{U}(1)_L$ charge. There exists a mirror M^0, such that $h^{1,1}(M) = h^{2,1}(M^0)$ and $h^{2,1}(M) = h^{1,1}(M^0)$. **But**:

1.) $\mathcal{N} = (2,2)$ NLSM with Ric $= 0$ is not conformally invariant!

2.) No explicit Calabi-Yau metric is known.

3.) What is a nice description of moduli space?



Chapter 2

$\mathcal{N} = (2,2)$ Landau-Ginzburg models

2.1 $\mathcal{N} = (2,2)$ SUSY and superfields

We will work on a Minkowski world sheet and x^+, θ^+, $\overline{\theta}^+$ (right-movers) and x^-, θ^-, $\overline{\theta}^-$ (left-movers) be the coordinates of the superspace. Furthermore, we define $x^\pm = x^0 \pm x^1$ and $\partial_\pm = \partial_0 \pm \partial_1$. We are coming to the supercharges:

$$Q_+ = \frac{\partial}{\partial\theta^+} + \mathrm{i}\overline{\theta}^+\partial_+ \,, \quad \overline{Q}_+ = -\frac{\partial}{\partial\overline{\theta}^+} - \mathrm{i}\theta^+\partial_+ \,, \tag{2.1}$$

with Q_- and $\overline{Q}_-$ defined similarly. The bar denotes the U(1) charge and $\pm$ the right movers and the left movers, respectively. Furthermore

$$D_+ = \frac{\partial}{\partial\theta^+} - \mathrm{i}\overline{\theta}^+\partial_+ \,, \quad \overline{D}_+ = -\frac{\partial}{\partial\overline{\theta}^+} + \mathrm{i}\theta^+\partial_+ \,, \tag{2.2}$$

and D_-, $\overline{D}_-$ similarly defined. It holds that

$$\{D_\pm, \overline{D}_\pm\} = 2\mathrm{i}\partial_\pm \,, \quad \{Q_\pm, \overline{Q}_\pm\} = -2\mathrm{i}\partial_\pm \,, \tag{2.3}$$

whereas all other anticommutators vanish. A superfield $\Phi(x^\pm, \theta^\pm, \overline{\theta}\pm)$ is called chiral, if $\overline{D}_+\Phi = \overline{D}_-\Phi = 0$. An expansion of the superfield with respect to the superspace coordinates yields

$$\Phi = \phi(x) + \sqrt{2}\theta^+\psi_+ + \sqrt{2}\theta^-\psi_- - 2\theta^-\theta^+F + \ldots \,, \tag{2.4}$$

where the dots represent all higher components in $\theta^\pm$ and $\overline{\theta}^\pm$. A superfield is called antichiral, if $D_+\overline{\Phi} = D_-\overline{\Phi} = 0$. ϕ is a complex scalar, $\psi_\pm$ and $\overline{\psi}_\pm$ are Dirac fermions and f is a $c - x$ auxiliary field.

$$[\overline{Q}_+, \overline{\phi}^i] = \sqrt{2}\overline{\psi}^i_+ \,, \quad \{\overline{Q}_+, \psi^i_+\} = \mathrm{i}\sqrt{2}\partial_+\phi^i \,, \quad \overline{Q}_+\overline{\psi}^i_- = -\sqrt{2}\frac{\partial W}{\partial\phi^i} \,, \quad \ldots \tag{2.5}$$

whereas for the last one the equations of motion for $\overline{F}$ have been used.

2.2 Landau-Ginzburg action

Choice of a holomorphic polynomial $W(\Phi_1, \Phi_2, \ldots, \Phi_n)$. The beauty of the superspace formalism is that the Lagrangian can be wirtten as an integral over parts of the superspace:

$$\mathcal{L} = -\frac{1}{8}\int \mathrm{d}^4\theta\, \overline{\Phi}^i\Phi^i - \frac{1}{2}\int \mathrm{d}\theta^+\,\mathrm{d}\theta^-\, W(\Phi) + \text{c.c.} \,, \quad \mathrm{d}^4\theta = \mathrm{d}\overline{\theta}^+\,\mathrm{d}\overline{\theta}^-\,\mathrm{d}\theta^+\,\mathrm{d}\theta^- \,. \tag{2.6}$$

By expanding and integrating out the auxiliary field F one obtains:

$$\mathcal{L} = \partial_+\phi^i\partial_-\overline{\phi}^i + \overline{\psi}^i_+\mathrm{i}\partial_-\psi^i_+ + \overline{\psi}^i_-\mathrm{i}\partial_+\psi^i_- - \sum_i |W_i|^2 - \psi^i_+W_{ij}\psi^j_- - \overline{\psi}^i_-\overline{W}_{ij}\overline{\psi}^j_+ \,, \quad W_i = \frac{\partial W}{\partial\phi^i} \,. \tag{2.7}$$

This is super-renormalizable. Quantum corrections are suspected to alter the kinetic term, but not $W(\Phi)$. This tells us something about the supersymmetric vacuum. SUSY is unbroken, iff $W_i(\phi) = 0$ for all i. The set $Z = \{\phi \in \mathbb{C}^n | W_i(\phi) = 0\}$ is either non-compact or a collection of points. We will consider the second case and are only interested in the case, when $Z = \{0\} \in \mathbb{C}^n$. Anyway, $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ shall be preserved.

	θ^-	θ^+	Φ^i
$\mathrm{U}(1)_L$	1	0	$q_{L,i}$
$\mathrm{U}(1)_R$	0	1	$q_{R,i}$

The symmetries are broken by W, unless W is homogeneous of weight one (to cancel the transformation of θ): $W(\lambda^{q_1}\Phi, \ldots, \lambda^{q_n}\Phi_n) = \lambda W(\Phi)$. Then take the charges to simply be $q_{L,i} = q_{R,i} = q_i$. Assume $q_i > 0$ and **unique**.

2.3 Landau-Ginzburg chiral ring

W is quasi-homogeneous and therefore $Z = \{0\}$. In this situation

$$\int \mathrm{d}^2 z \, \mathrm{d}\theta^+ \, \mathrm{d}\theta^- \, W(\Phi) \,, \tag{2.8}$$

looks scale-invariant, if Φ^i has weight $(h_i, \overline{h}_i) = (g_i/2, g_i/2)$. The basic fields seem to be acting as elements in a chiral ring. The suggestion is that Φ^i **represents** an element in the (c,c) ring. Any $\Phi^{i_1} \ldots \Phi^{i_k}$ is **chiral**, but $W_1(\Phi) =$ descendent. (c,c) ring:

$$R_W = \frac{\mathbb{C}[\phi_1, \ldots, \phi_n]}{\langle W_1, W_2, \ldots, W_n \rangle} \,. \tag{2.9}$$

A check is that if R_W is finite dimensional, then $\dim Z = 0$ and vice versa.

2.4 Example 1

Consider a simple chiral field with the superpotential $W = 1/(k+2)\Phi^{k+2}$. The fact that W has charge 1 implies $q_\Phi = 1/(k+2)$.

1.) $k = -1$: $\partial W/\partial\phi = 0$, no SUSY

2.) $k = 0$: free massive Φ, Φ is a descendent.

3.) $k > 0$: $R_W = \{1, \phi, \phi^2, \ldots, \phi^k\}$, $\phi^a\phi^b = \phi^{a+b}$ for $a + b \leq k$ and 0 otherwise.

The highest weight state is ϕ^k and

$$h[\phi^k] = \frac{k}{2(k+2)} \stackrel{?}{=} \frac{c}{6} \Rightarrow c = 3(1-2q) \,. \tag{2.10}$$

For $q = 1/2$ the central charge c vanishes, just what one expects from a trivial theory. Check the proposal $c = 3\sum_i(1-2q_i)$.

- massive theory leads to $c = 0$
- $c_{\mathrm{UV}} > c_{\mathrm{IR}}$
- Consider the example $W = \Phi_1\Phi_2^{k+2} - 1/2\Phi_1^2$. Using our formula we expect $c \stackrel{?}{=} 3(1-2q_2)$. Integrating out Φ_1 leads to $\Phi_1 = \Phi_2^{k+2}$ and $W_{\mathrm{eff}} = \Phi_2^{2(k+2)}$.

There are two arguments for $c = 3\sum_i(1-2q_i)$.

- Partition function on a sphere for Landau-Ginzburg theory:

$$Z = \int [D\Phi]_y \exp(-\mathcal{L}_k - \mathcal{L}_W) \,, \quad Z[\lambda^2 g] = \lambda^{\frac{c}{6}} Z[g] \,. \tag{2.11}$$

Use the free measure $D[\Phi]_y$, which contributes $\lambda^{\sum_i 3}$.

$$\mathcal{L}_W[g\lambda^2] \sim \lambda \mathcal{L}_W[g] \,, \tag{2.12}$$

where the Jacobian contributes $\lambda^{-\sum_i 6q_i}$.

- This argumentation has to do with the $\overline{Q}_+$ cohomology and the secret $\mathcal{N}=2$ super conformal algebra [Witten, 9304026].

  Since $(\overline{Q}_+)^2 = 0$, define $\mathcal{H}^+ \equiv \{\mathcal{O} | \{\overline{Q}_+, Q\}_\pm = 0\}/\mathcal{O} \sim \mathcal{O} + \{\overline{Q}_+, \bullet\}$. The operator product expansion defines an algebraic structure on $\mathcal{H}^+$ (a holomorphic algebra).

  $$\partial_+ \mathcal{O} \sim \{Q_+, \{\overline{Q}_+, \mathcal{O}\}\} + \{\overline{Q}_+, \{Q_+, \mathcal{O}\}\}\,, \tag{2.13}$$

  hence, $\partial_+ \mathcal{O}$ is $\overline{Q}_+$ except for $\mathcal{O} \in \mathcal{H}^+$. Expect $\overline{Q}_+$ to make sense and to be robust with respect to renormalization group flow. However, this is not guaranteed (these are counter-examples).

Define

$$j = (q_i - 1)\psi_-^i \overline{\psi}_-^i + \mathrm{i} q_i \phi^i \partial_- \overline{\phi}^i\,, \quad G^- = \psi_-^i \partial_- \overline{\phi}^i\,, \quad G^+ = (1-q_i)\overline{\psi}_-^i \partial_- \phi^i - q_i \partial_- \overline{\psi}_-^i \phi^i\,, \tag{2.14}$$

$$T = \frac{1}{2}(\partial_- \phi^i \partial_- \overline{\phi}^i + \psi_-^i \mathrm{i}\partial_- \overline{\psi}_-^i) + \frac{\mathrm{i}}{4}\partial_- j\,. \tag{2.15}$$

- Exercise: Show that these satisfy $\{\overline{Q}_+, \bullet\} = 0$ with respect to EoM.
- Exercise: Using free operator product expansion show that j, $G^\pm$, T satisfy an $\mathcal{N}=2$ super conformal algebra with $c = 3\sum_i (1-2q_i)$.

2.5 Example 2

Consider a cubic superpotential with three chiral fields

$$W = \frac{1}{3}\phi_1^3 + \frac{1}{3}\phi_2^3 + \frac{1}{3}\phi_3^3 - \psi\phi_1\phi_2\phi_3\,, \tag{2.16}$$

with parameter ψ.

$$c = 3\sum_i \left(1 - 2\cdot\frac{1}{3}\right)\,, \quad c = 3\,. \tag{2.17}$$

- $\psi = 0$: We have three decoupled theories and the chiral ring is $R_W = \otimes_{i=1}^3 \mathbb{C}[\phi_i]/\langle\phi_i^2\rangle$.
- $\psi \neq 0$: $R_W = \mathbb{C}[\phi_1,\phi_2,\phi_3]/\langle\phi_1^2 = \psi\phi_2\phi_3, \phi_2^2 = \psi\phi_1\phi_3, \phi_3^2 = \psi\phi_1\phi_2\rangle$

  The chiral ring gets deformed, when we deforme the theory; we get from one algebraic structure to another one.
- 10 "naive" parameters, but $\mathrm{GL}(3,\mathbb{C})$ rotations $\widetilde{\Phi}^i = U^i{}_j \Phi^j$, which modify the kinetic term, minus 9 field redefinitions leads to one single parameter.
- $\psi = 1$: $W_i = 1/\phi_i(\phi_i^3 - \phi_1\phi_2\phi_3)$. Observe that $\phi_i = \xi$ solves $W_i = 0$ for any ξ. Hence, there is a non-compact direction and one expects a singularity in the CFT.

There is a formula to calculate the structure constants of the chiral ring:

$$\mu_1(\Phi)\cdot\mu_2(\Phi) \sim \frac{\mu_{1,2}(\Phi)}{1-\phi^3}\,. \tag{2.18}$$

2.6 Historical note

- Non-SUSY LG: Zamolodchikov 1986: A correct operator product expansion is the relevant definition of minimal models.
- $\mathcal{N} = (1,1)$ LG and discrete series: Kastor, Martinec, Sharter in 1988
- $\mathcal{N} = (2,2)$ Martinec, Vafa, Worner in 1989
- $\mathcal{N} = 2$ ADE discrete series (for supersymmetric): isolated, quasi-homogeneous singularities in $\mathbb{C}^2$

  $$W = \frac{1}{k+2}\Phi_1^{k+2} + \Phi_2^2\,, \quad \Phi_1^{\frac{k+2}{2}} + \Phi_1\Phi_2^2\,, \quad \Phi_1^4 + \Phi_2^3\,, \quad \Phi_1^3\Phi_2 + \Phi_2^3\,, \quad \Phi_1^5 + \Phi_2^3\,. \tag{2.19}$$

  Exercise: Evaluate the central charges by using the Landau-Ginzburg equation.

Two corrections

- Lecture 1: In unitary $\mathcal{N} = 2$ SCFT every operator satisfies $h \geq 1/2|q|$.
- Lecture 2: Equation of motion in superspace: $\overline{D}_+\overline{D}_-\overline{\Phi}^i = -2W_i$

Recapitulation

1.) Basic $\mathcal{N} = (2,2)$ SCFT, chiral rings, truly marginal operators ("moduli") $\mathcal{M}_{(\mathrm{a,c})} \times \mathcal{M}_{(\mathrm{c,c})}$, how to construct these? How to compute in them?

Point of View: RG flow to SCFT from simple UV theory: use $\mathcal{N} = (2,2)$ SUSY + $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$, R-symmetry

2.) NLSM: natural, but UV description is forced

- we do not know the metric
- deformation is pretty abstract

3.) Landau-Ginzburg models:

UV-Lagrangian determined: $W(\lambda^{q_1}\phi_1, \ldots, \lambda^{q_n}\phi_1) = \lambda W(\phi)$, holomorphic functions on $\mathbb{C}^n$

$$[\overline{Q}_+, \phi_i] = 0\,, \quad [\overline{Q}_-, \phi_i] = 0\,. \tag{2.20}$$

All derivatives of the superpotential W_i are "descendants". Guess: ϕ_i normalize to operators in (c,c) ring of SCFT with charges $q_{L_i} = q_{R_i} = q_i$ that generate the (c,c) ring

$$\text{(c,c) ring} = \frac{\mathbb{C}[\phi_1, \ldots, \phi_n]}{\langle W_1, \ldots, W_n\rangle}\,, \qquad \text{(a,c)} = 1\,. \tag{2.21}$$

Parameters: $W \mapsto W + \Delta W$, $q[\Delta W] = 1$, $\Delta W \in$ (c,c)

$$W = \frac{1}{3}(\phi_1^3 + \phi_2^3 + \phi_3^3)\,, \quad W_i = \phi_i^2\,, \quad q[\phi_i] = \frac{1}{3}\,. \tag{2.22}$$

$q = 0$	1
$q = 1/3$	ϕ_1, ϕ_2, ϕ_3
$q = 2/3$	~~ϕ_1^2~~, $\phi_1\phi_2$, $\phi_1\phi_3$, $\phi_2\phi_3$
$q = 1$	$\phi_1\phi_2\phi_3$
$q = 4/3$	0

ΔW with $q = 1$, 10 cubic monomials, 9 GL(3,$\mathbb{C}$) redefinitions: remaining 1 deformation

Free $\mathbb{C}^n \xrightarrow{W}$ geometry of "fat point" $0 \in \mathbb{C}^n$ (algebraic geometry: scheme)

Chapter 3

Gauge linear sigma model basics

3.1 $\mathcal{N} = (2,2)$ quantum electrodynamics in superspace

We start with n chiral fields Φ^i that are charged under the Abelian gauge symmetry $[\mathrm{U}(1)]^r$. These charges will be $Q_i^a \in \mathbb{Z}$ with $a = 1, \ldots, r$. We want to gauge certain symmetries of our theory. We will gauge the rotations of the fields in introducing a chiral superfield parameter Λ_a: $\Phi^i \mapsto \exp(Q_i^a \Lambda_a)\Phi^i$. (This has to be chosen like that, because we want to preserve chirality, Λ_a is chiral). The gauge fields come in **real multiplets** $V_a = \overline{V}_a$ with $V_a \mapsto V_a - 1/2(\Lambda_a + \overline{\Lambda}_a)$. Therefore, $\overline{\Phi}^i \exp(2Q_i^a V_a)\Phi^i$ is gauge invariant. The Φ^i will induce the gauge interactions in a sypersymmetric way. Then the question arises how to introduce field strengths:

$$\Sigma_a = \frac{1}{\sqrt{2}} D_+ D_- V_a \,, \quad \overline{\Sigma}_a = \frac{1}{\sqrt{2}} \overline{D}_- D_+ V_a \,, \tag{3.1}$$

whereas $\overline{D}_+ \Sigma_a = D_- \Sigma_a = 0$. Σ_a is **twisted chiral** and gauge invariant. An expansion in components yields the following structure:

$$\Sigma_a = \sigma_a(x) + \mathrm{i}\sqrt{2}(\theta^+ \lambda_{+,a} - \overline{\theta}^- \lambda_{-a}) + \sqrt{2}\theta^+ \overline{\theta}^- (D_a - \mathrm{i} f_{01,a}) + \ldots \,. \tag{3.2}$$

Herein, σ is a $c{-}x$ boson, $\overline{\lambda}_\pm$, $\lambda_\pm$ are Dirac fermions, D_a is an axial scalar auxiliary and $f_{01,a} = \partial_0 V_{1,a} - \partial_1 V_{0,a}$. We are now coming to the action in superspace:

$$\begin{aligned}\mathcal{L} = &\int \mathrm{d}^4\theta \left(-\frac{1}{4} \overline{\Phi}^i \exp(2Q_i^a V_a)\Phi^i + \frac{1}{4e_0^2} \overline{\Sigma}_a \Sigma_a \right) \\ &- \frac{1}{2} \Bigg\{ \int \underbrace{\mathrm{d}\theta^+ \,\mathrm{d}\theta^- \, W(\Phi)}_{\text{chiral superpotential}} + \frac{\mathrm{i}}{\sqrt{2}} \int \underbrace{\mathrm{d}\theta^+ \,\mathrm{d}\overline{\theta}^- \, \widetilde{W}(\Sigma)}_{\substack{\text{twisted chiral}\\ \text{superpotential}}} + \text{c.c.} \Bigg\} \,,\end{aligned} \tag{3.3}$$

whereas e_0 is the dimensionful gauge coupling.

0.) Sorry about Σ_a!

1.) There are two superpotentials, separate non-renormalizable terms.

2.) $W(\Phi)$ must be gauge invariant!

We would like to write it out in components: Integrate out the auxiliary fields; in **Wess-Zumino gauge** (potential gauge fixing) $\Lambda_a \mapsto \lambda_(x)$.

$$\widetilde{W} = \tau^a \Sigma_a \,, \quad \tau^a = \mathrm{i}\varrho^a + \frac{\theta^a}{2\pi} \,, \tag{3.4}$$

where ϱ^a is the Fayet-Iliopoulos term and θ^a the theta-angle. It's convenient to split up the Lagrangian in three terms:

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{Yuk}} - U(\phi, \sigma) \,, \tag{3.5}$$

with

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &-|D_\mu \phi^i|^2 + \overline{\psi}_+^i \mathrm{i} D_- \psi_+^i + \overline{\psi}_-^i \mathrm{i} D_+ \psi_-^i \\ &+ \frac{1}{e_0^2} \left[-|\partial_\mu v_a|^2 - \frac{1}{4} f_{\mu\nu,a} f^{\mu\nu,a} + \overline{\lambda}_{+,a} \mathrm{i}\partial_- \lambda_{+,a} + \overline{\lambda}_{-,a} \mathrm{i}\partial_+ \lambda_{-,a} \right] ,\end{aligned} \tag{3.6a}$$

$$\mathcal{L}_{\text{Yuk}} \overset{\text{w.r. to }\sqrt{2}}{\propto} \psi_+^i W_{ij} \psi_-^j + \psi_-^i Q_i^a \overline{\sigma}_a \psi_+^i + \psi_-^i Q_i^a \overline{\phi}^i \lambda_{+,a} + \text{c.c.}\,, \tag{3.6b}$$

and

$$U(\phi,\sigma) = 2\sum_i |\phi^a Q_i^a \sigma_a|^2 + \frac{e_0^2}{2}(Q_i^a|\phi^i|^2 - \varrho^a)^2 + |W_i|^2 - \frac{Q^a}{2\pi} f_{01,a}\,. \tag{3.6c}$$

3.1.1 $\mathbf{U(1)_L \times U(1)_R}$ symmetry

	θ^-	θ^+	Φ^i	V_a	Σ_a	ψ_+^i	ψ_-^i
$U(1)_L$	1	0	q_i	0	-1	q_i	$q_i - 1$
$U(1)_R$	0	1	q_i	0	$+1$	$q_i - 1$	q_i

For $r = 0$ we get back to the Landau-Ginzburg theory. We need $W(\Phi\lambda^a) = \lambda W(\Phi)$. Anyway, $\widetilde{W}$ must have the charge $(-1,1)$. Hence, the polynomial $\widetilde{W}$ is linear in Σ. Study the conservation of left-moving current by evaluating the one-loop contributions. One computes the graph and imposes the conservation equation

$$\partial_\mu J_L^\mu = \left(\sum_{\psi_-^i}(q_i-1)Q_i^a - \sum_{\psi_+^i} q_i Q_i^a\right)\frac{1}{2\pi}f_{01,a} = \left(\sum_i Q_i^a\right)\left(-\frac{1}{2\pi}f_{01,a}\right)\,. \tag{3.7}$$

The conservation of the left-handed current is ruined. The anomalies are proportional to the sum of all charges of the gauge group. The same study of the right-moving current reveals:

$$\partial_\mu J_R^\mu = -\partial_\mu J_L^\mu\,, \tag{3.8}$$

and because of that $J_L + J_R$ is a conserved charge. On the other hand, $J_L - J_R$ is conserved, iff

$$\boxed{\sum_i Q_i^a = 0\,, \forall\, a\,.} \tag{3.9}$$

Related is

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}\varrho_a = \frac{1}{2\pi}\sum_i Q_i^a\,, \tag{3.10}$$

hence, in general, the ϱ^a flow!

3.1.2 Example: the GLSM for $\mathbb{CP}^4$

Be $r = 1$, $n = 5$ and $Q_i = 1$. If W is polynomial and gauge invariant, it is reduced to a constant. Let us forget about quantum effects and just do a **classical analysis**. Set $f_{01} = 0$. The supersymmetric vacua are given by $M_\varrho \equiv \{\phi,\sigma | U(\phi,\sigma) = 0\}/\mathrm{U}(1)$. $U(\phi,\sigma)$ vanishes, iff the conditions

$$\sigma\phi^i = 0\,, \quad \sum_{i=1}^r |\phi_i|^2 = \varrho\,, \tag{3.11}$$

are separately fulfilled. For $\varrho < 0$ it holds that $U \geq e_0^2/2\varrho^2$ and classically, SUSY is broken. For $\varrho = 0$ all ϕ_i vanish. σ is then unconstrained; the U(1) symmetry is unkroken. For $\varrho > 0$ we have $\sigma = 0$ and

$$M_\varrho = \left\{\phi | \sum_i |\phi_i|^2 = \varrho\right\}/\mathrm{U}(1) = S^9/\mathrm{U}(1) \simeq \mathbb{CP}^4\,. \tag{3.12}$$

Since $\phi^i = 0 \notin M_{\varrho>0}$, U(1) is always Higgsed when $\varrho > 0$. However, there is no Higgs mechanism in $d = 2$. Therefore, one cannot break continuous symmetries and one cannot fix to a vacuum expectation value; one has to integrate over them. The light degrees of freedom are described by $\mathcal{N} = (2,2)$ NLSM with target space $\mathbb{CP}^4$. There are also massive degrees of freedom with $m^2 \sim e_0^2\varrho$ and their coupling to the light degrees of freedom is of the order e_0^2/ϱ.

NLSM to GLSM aside

$$\mathcal{L} = \int \mathrm{d}^4\theta \left(-\frac{1}{4}\overline{\Phi}^i \Phi^i \exp(2V) + \frac{1}{4e_0^2}\overline{\Sigma}\Sigma \right) + \frac{\mathrm{i}}{2\sqrt{2}} \int \mathrm{d}\theta^+ \,\mathrm{d}\overline{\theta}^- \,\Sigma \left(\mathrm{i}\varrho + \frac{\theta}{2\pi} \right) . \tag{3.13}$$

Secret: If one forgets about the theta angle the action can be rewritten as follows:

$$\mathcal{L} = \int \mathrm{d}^4\theta \left(-\frac{1}{4}\overline{\Phi}^i \Phi^i \exp(2V) + \frac{1}{4e_0^2}\overline{\Sigma}\Sigma \right) + \int \mathrm{d}^4\theta \,(\varrho V) . \tag{3.14}$$

If we want to do classical low energy physics, set $e_0 \mapsto \infty$. Then we arrive at

$$\mathcal{L} = \int \mathrm{d}^4\theta \left(-\frac{1}{4}\overline{\Phi}^i \Phi^i \exp(2V) + \frac{\varrho}{2} V \right) . \tag{3.15}$$

Solve the equations of motion and plug back in: First, fix the gauge symmetry: $\Phi^1 = 1$ assumes $\Phi_1 \neq 0$ and hence:

$$\mathcal{L} = \int \mathrm{d}^4\theta \left\{ -\frac{1}{4} R \exp(2V) + \frac{\varrho}{2} V \right\} , \quad R = 1 + |\Phi_2|^2 + \ldots + |\Phi_5|^2 , \tag{3.16a}$$

$$\Rightarrow -\frac{1}{2} R \exp(2V) + \frac{\varrho}{2} = 0 \Rightarrow \boxed{V = \frac{1}{2} \ln\left(\frac{\varrho}{R}\right) .} \tag{3.16b}$$

$$\mathcal{L}_{\phi_2 \ldots \phi_5} = \int \mathrm{d}^4\theta \left\{ -\frac{1}{4} R \cdot \frac{\varrho}{R} + \frac{\varrho}{4} \ln\left(\frac{\varrho}{R}\right) \right\} = \int \mathrm{d}^4\theta - \frac{\varrho}{4} \ln\left(1 + |\Phi_2|^2 + |\Phi_3|^2 + |\Phi_4|^2 + |\Phi_5|^2\right) . \tag{3.17}$$

Nonlinear sigma-model:

$$\int \mathrm{d}^4\theta \, K(\Phi, \overline{\Phi}) , \tag{3.18}$$

whereas Φ are coordinates on a patch for the curved manifold M. We have

$$K = \frac{\varrho}{4} \log\left(1 + |\phi_2|^2 + |\phi_3|^2 + |\phi_4|^2 + |\phi_5|^2\right) , \tag{3.19}$$

which is the Fubini-Study-Kähler potential on $\mathbb{CP}^4$. The notion of projective coordinates is a nice way to prescribe projective spaces and we will use it from now on:

$$[z_1, z_2, z_3, z_4, z_5] \qquad [z_1, \ldots, z_5] \sim [\lambda z_1, \ldots, \lambda z_5] , \quad \lambda \neq 0 , \quad \lambda \in \mathbb{C} . \tag{3.20}$$

For $z_1 \neq 0$:

$$\Big[1; \underbrace{\frac{z_2}{z_1}, \frac{z_3}{z_1}, \frac{z_4}{z_1}, \frac{z_5}{z_1}}_{\mathbb{C}^4}\Big] , \quad \mathbb{CP}^4 = \frac{\mathbb{C}^5 \setminus \{0\}}{\mathbb{C}^*} . \tag{3.21}$$

3.1.3 Example: Quintec in $\mathbb{CP}^4$

To fix problems we introduce a new field Φ_0 with $Q_0 = -5$. Hence

$$W = \Phi_0 P_5(\Phi_1, \ldots, \Phi_5) , \quad \text{e.g. } P_5 = \Phi_1^5 + \Phi_2^5 + \ldots + \Phi_5^5 . \tag{3.22}$$

Now we want to look again on the classical vacua. $\phi^i \sigma = 0 \,\forall\, i$:

$$-5|\Phi_0|^2 + \sum_{i=1}^{5} |\Phi_i|^2 = \varrho , \tag{3.23a}$$

$$\frac{\partial W}{\partial \Phi^0} = P_5 = 0 , \tag{3.23b}$$

$$\frac{\partial W}{\partial \Phi^i} = \Phi^0 \frac{\partial P_5}{\partial \Phi^i} = 0 . \tag{3.23c}$$

For $\varrho > 0$ it follows from (3.23a) that not all Φ_i are zero. So, $\Phi_0 = 0$ (from (3.23c)) and $\sigma = 0$ (from 1).

$$M_{\varrho > 0} = \{P_5 = 0\} \subset \mathbb{CP}^4 . \tag{3.24}$$

$M_{\varrho>0}$ is the most famous Calabi-Yau manifold! Exercise: What happens when $\varrho < 0$? What is $M_\varrho < 0$ and what is necessary there?

3.1.4 Example: GLSM for quintic in $\mathbb{CP}^4$, $\mathbb{Q}_i = (-5, 1, 1, 1, 1, 1)$

Classical vacuum moduli space: $\phi^i\sigma = 0$ for all i leads to: ...

Let us now consider the case $\varrho < 0$. We can actually solve equation (2): $\phi_0 \neq 0$. Equation (1) implies again $\sigma = 0$ and equation (4) tells us that $\phi^i = 0$. Then we can solve equation (2) explicitly for ϕ_0:

$$\phi_0 = \sqrt{\frac{-\varrho}{5}} \exp(\mathrm{i}\alpha) \,, \tag{3.25}$$

whereas we set the phase $\alpha = 1$ by gauge fixing. What are the low energy massless degrees of freedom: ϕ^i for $i = 1, \ldots, 5$? They interact with

$$W = \sqrt{\frac{-\varrho}{5}} P_5(\phi) \,. \tag{3.26}$$

Before we can conclude that this Landau-Ginzburg theory we have to remember the following. Since $Q_0 = -5$ there is a remaining gauge symmetry: $\mathbb{Z}_5 \subset \mathrm{U}(1)$. At low energy for $\varrho < 0$ we get LG/$\mathbb{Z}_5$-orbifold. This description only emerges in the limit $\varrho \mapsto -\infty$. We have the picture of the GLSM moduli space: $\varrho = 0$ looks like a mess, since σ is now unconstrained and the theory looks singular and sick in this regime, at least classically. Before we will investigate quantum corrections, we will however consider something different.

3.2 Deformations: $\varrho \gg 0$

One can show that $[\overline{Q}_+, \sigma] = 0$, $[Q_-, \sigma] = 0$. When we try to match UV physics with the IR, we saw that σ is a candidate for (a,c) ring. **Recall**: On Calabi-Yau NLSM the (a,c) ring elements with the particular charges $q_L = -1$, $q_R = +1$ gave us deformations. These are exactly the charges of σ. The corresponding deformation in the Lagrangian is:

$$\int \mathrm{d}\theta^+ \, \mathrm{d}\overline{\theta}^- \; \tau^a \Sigma_a \,. \tag{3.27}$$

We conclude that it makes sense to identify the parameter τ^a with the complexified Kähler class. ϱ^a corresponds to the Kähler class and θ^a to the B-field. This is valid for $\varrho \mapsto \infty$ and will then receive corrections. Let $\mathcal{O} = \phi_0 \mathcal{M}_5(\phi_i\phi_5)$, where $\mathcal{M}_5$ is a quintic monomial. $\mathcal{O}$ is gauge invariant and

$$[\overline{G}_+, \mathcal{O}] = 0 \,, \quad [\overline{G}_-, \mathcal{O}] = 0 \,, \quad q_L(\mathcal{O}) = q_R(\mathcal{O}) = 1 \,. \tag{3.28}$$

Hence, $\mathcal{O} \in$ (c,c) riong element with $q_L = q_R = 1$.

$$\mathcal{O} \mapsto \int \mathrm{d}\theta^+ \, \mathrm{d}\theta^- \; \mathcal{O}\psi_{\mathcal{O}} \,. \tag{3.29}$$

The number of $\mathcal{O}$ is 126. GL(5,$\mathbb{C}$) redefinitions eliminate 25 and there remain 101 parameters for the (c,c) ring. Similarly, the complex structure deformation $h^{2,1}$(quintic) $= 101$.

Remarks

1.) This construction generalizes to give all Calabi-Yau complete intersections in toric varieties. This is the largest family of Calabi-Yau constructions (comes with natural mirror construction).

2.) Often not full (a,c)/(c,c) rings are visible in the natural GLSM variables.

3.) References:

- Big mirror book, edited by Hori et al.
- Witten: 9301042
- Morrison and Plesser: 9412236

Chapter 4

Quantum Coulomb Vacua

Recall the scalar potential for fields ϕ, σ:

$$U(\varphi,\sigma) = 2\sigma_a\overline{\sigma}_b Q_i^a Q_i^b |\phi^i|^2 + \frac{e_0^2}{2}(Q_i^a|\phi^i|^2 - \varrho^2)^2\,. \tag{4.1}$$

Consider a region in field space, where $|\sigma| \gg e_0$. Then, the ϕ^i get very large masses and can be integrated out. This leads to a one-loop effective superpotential for Σ_a:

$$\widetilde{W}_{\text{eff}} = -\frac{1}{4\pi\sqrt{2}}\sum_{a=1}^{r}\Sigma_a \log\left[\prod_{i=1}^{n}\left(\frac{\Sigma_b Q_i^b}{\mu\exp(1)}\right)^{Q_i^a}\frac{1}{q_a}\right]. \tag{4.2}$$

where μ is a renormalization scale. q_a is a convenient variable and is defined as $\exp(2\pi\mathrm{i}\tau^a) = \exp(2-\pi\varrho^a+\mathrm{i}\theta^a)$.

Comments

1.) The result is believed to be exact ['t Hooft anomaly matches ϱ, holomorphy, large field behavior]

2.) σ_a have potential $\sim \left|\frac{\partial\widetilde{W}_{\text{eff}}}{\partial\sigma^a}\right|^2$ SUSY-potential: $\frac{\partial\widetilde{W}_{\text{eff}}}{\partial\sigma_a}0 = 0$ (SUSY ground state configurations)

$$\frac{\partial W_{\text{eff}}}{\partial\sigma_a} = 0 \Leftrightarrow \prod_{i=1}^{n}\left(\frac{Q_i^b\sigma_b}{\mu}\right)^{Q_i^a} = q_a\,, \quad a = 1,\ldots,5\,. \tag{4.3}$$

3.) Only valid when $|\sigma/\mu| \gg 1$.

4.0.1 Example: Quantum Correction for $\mathbb{CP}^{n-1}$

Classical

Coulomb vacua:

$$\left(\frac{\sigma}{\mu}\right)^n = \exp(-2\pi\varrho + \mathrm{i}\theta)\,. \tag{4.4}$$

Quantum mechanically: Linie rechts $\mathbb{CP}^{n-1}$, ϱ, RG flow nach links, mitte: not a special points, links: reliable Coulomb vacua

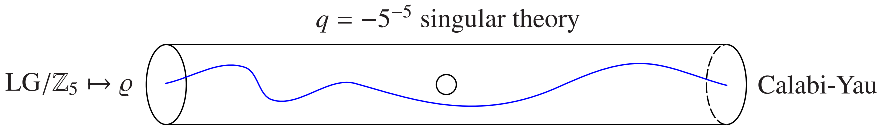
$$\mathrm{Tr}(-1)^F = \chi(\mathbb{CP}^{n-1}) = n\,. \tag{4.5}$$

4.0.2 Example: quintic $\subset \mathbb{CP}^4$

The Coulomb vacua are given by a solution to

$$\left(-\frac{5\sigma}{\mu}\right)^{-5}\left(\frac{\sigma}{\mu}\right)^{5} = q\,. \tag{4.6}$$

Because of Calabi-Yau condition q/μ drops out and it results $1 + 5^5 q \neq 0$. Then there are no Coulomb vacua. For $1 + 5^5 q = 0$ then σ can be anything. It happens at a fixed value of θ, because q is showing up here.



4.0.3 Singular loci in $\mathcal{M}_{(a,c)} \times \mathcal{M}_{(c,c)}$

Non-compact σ direction for $q = (-5)^{-5}$ results in a singular CFT at $g = -5^{-5}$ in $\mathcal{M}_{(a,c)}$. Another singular **limit** point is $\varrho \mapsto \infty$. This is a decompactification; it is okay we understand it. This kind of limit point allows us to use SUGRA (when the Calabi-Yau is very big and curvature is very small). Additional $\mathcal{M}_{(a,c)}$ singularities can arise at other limit points: Aspinwdl, Plesser: 0909.0252.
$\mathcal{M}_{(a,c)}$: We describe the space as coefficients in $P_5(\phi_1, \ldots, \phi_5)$ modulo $\mathrm{GL}(5, \mathbb{C})$ redefinition. $\mathcal{M}_{(a,c)}$ contains singular points: Choose P_5 such that $\partial P_5/\partial\phi^i = 0$ at some $p \in \mathbb{CP}^4$. ϕ_0 gives a non-compact direction!

4.0.4 Example

Let us choose the polynomial as $P_5 = \phi_1^5 + \ldots + \phi_5^5 - 5\alpha\phi_1\phi_2\phi_3\phi_4\phi_5$. Exercise: Show that $\{P_5 = 0\}$ is a singular hypersurface, iff $\alpha^5 = 1$. Set $\alpha = 1$:

$$\frac{\partial P_5}{\partial\phi^i} = 0 \Leftrightarrow \phi_1^4 = \phi_2\phi_3\phi_4\phi_5 \text{ and permutations}\,. \tag{4.7}$$

The solution requires $\phi_1 \neq 0$. ($\phi_i = 0$ implies a point that is not in $\mathbb{CP}^4$.)

$$\phi_2^5 = \phi_3^5 = \phi_4^5 = \phi_5^5 = 1 = \phi_2\phi_3\phi_4\phi_5\,, \tag{4.8}$$

and as a result of that there are $5^3 = 125$ singular points. Each of these points is an example for a **conifold** singularity. Near a singular point the hypersurface looks like $xy - zw = 0$ in $\mathbb{C}^4$.

4.1 A Hiat of Mirror Symmetry

$M = \{P_5 = 5\} \subset \mathbb{CP}^4$ has a mirror: $M^0 = \{\widehat{P}_5 = 0\} \subset \mathbb{CP}^4/\mathbb{Z}_5^3$. (We need to do certein resolution of singularities.) Let us first of all describe the singular space $\mathbb{CP}^4/\mathbb{Z}_5^3$. Projective coordinates:

$$\begin{aligned}[z_1 : z_2 : z_3 : z_4 : z_5] &\sim [\lambda z_1 : \lambda z_2 : \lambda z_3 : \lambda z : 4 : \lambda z_5] \sim [\zeta_5 z_1 : z_2; z_3 : z_4 : \zeta_5^4 z_5] \\ &\sim [z : 1 : \zeta_5 z : 2 : z : 3 : z : 4 : \zeta_5^4 z_5] \sim [z_1 : z_2 : \zeta_5 z_3 : z_4 : \zeta_5^4 z_5]\,,\end{aligned} \tag{4.9}$$

with

$$\zeta_5 \equiv \exp\left(\frac{2\pi\mathrm{i}}{5}\right)\,, \tag{4.10}$$

whereas $z_i = 0$ is excluded: $\lambda \in \mathbb{C}^*$. Remark on the check of mirror proposal: $\dim \mathcal{M}_{(a,c)}(M) = 1$ (our parameter q). The mirror symmetry claims $\mathcal{M}_{(c,c)}(M) = \mathcal{M}_{(c,c)}(M^0)$. Is additionally $\dim \mathcal{M}_{(c,c)} = 1$? We count monomials in $\widehat{P}_5(z)$ covariant unter $\mathbb{C}^*$ [charge 5] and invariant unter $\mathbb{Z}_5^3$. Doing the first leads us to the 126 quintic monomials. Exercise: Show that there are only six monomials! The only field redefinitions that are consistent with the above symmetries is $Z_i \mapsto t_i Z_i$ with $t_i \in \mathbb{C}^*$. Hence, there are five remaining.

$$\widehat{P}_5 = z_1^5 + \ldots + z_5^5 - 5\psi z_1 z_2 z_3 z_4 z_5\,, \tag{4.11}$$

whereas ψ is the parameter for $\mathcal{M}_{(c,c)}(M^0)$. The mirror symmetry gives the **mirror map** $\mathcal{M}_{(c,c)}(M^0) \mapsto \mathcal{M}_{(a,c)}(M)$. It is particularly easy to write it down in GLSM coordinates: $q = (-5\psi)^{-5}$ and this is a fact! Singular Loci: $1 + 5^5 q = 0$ in $\mathcal{M}_{(a,c)}(M)$ and $1 - \psi^5 = 0$ in $\mathcal{M}_{(c,c)}(M^0)$.

Chapter 5

Introduction to topological field theory

5.1 Basic picture

For doing quantum field theory on a compact manifold $\mathcal{M}$ we typically need a choice of the metric on M.

$$\frac{\delta}{\delta g^{\mu\nu}(y)}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n) = \langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)T_{\mu\nu}(g)\rangle \neq 0\,. \tag{5.1}$$

The left-hand side vanishes in topological field theory. In the Schwarz-type topological field theory the action is independent of the metric, for example:

$$S = \frac{k}{4\pi}\mathrm{Tr}\int\limits_{M_3}\left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right)\,. \tag{5.2}$$

Caveat: Regularization requires a choice of g. The only left-over effect is choice of topological dara of TM_3.

5.1.1 Cohomological topological field theory

This is a quantum field theory with a nilpotent fermionic scalar operator Q, $Q^2 = 0$ such that $T_{\mu\nu} = \{Q, \mathcal{G}_{\mu\nu}\}$, whereas $\mathcal{C}_{\mu\nu}$ is some operator of the theory. A natural way is to consider the cohomology of Q, namely the set $\mathcal{H} = \{\mathcal{O}|\{\mathcal{Q},\mathcal{O}\} = 0\}/\mathcal{O}\sim\mathcal{O}+\{Q,X\}$. We project the theory to $\mathcal{H}$. Let us consider variations of correlators with respect to the metric:

$$\frac{\delta}{\delta g^{\mu\nu}}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = \langle \mathcal{O}_1\ldots\mathcal{O}_n\{Q,\mathcal{G}_{\mu\nu}\}\rangle = \pm\langle\{Q,\mathcal{O}_1\ldots\mathcal{O}_n\}\mathcal{G}_{\mu\nu}\rangle = 0\,. \tag{5.3}$$

Furthermore

$$\frac{\partial}{\partial x^\mu}\langle \mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)\rangle = 0\,, \tag{5.4}$$

since $\partial_\mu\mathcal{O} = -\{P_\mu,\mathcal{O}\}$, $P_\mu = \{Q, G_\mu\}$. Any Q-exact operator will decouple from the correlator of Q-closes operators (observables).

5.2 A few properties of cohomological topological field theory

Descent: Given $\mathcal{O}(x)$ an observable: $\mathrm{d}\mathcal{O}(x) = -\mathrm{d}x^\mu[P_\mu,\mathcal{O}]$. Let $\mathcal{O}^{(1)}\equiv \mathrm{d}x^\mu[G_\mu,\theta]$. $\{Q,\mathcal{O}^{(1)}\} = \pm\mathrm{d}\theta(x)$. This implies that

$$\int\limits_{\Gamma_1}\mathcal{O}^{(1)}\,, \tag{5.5}$$

is Q-closed for any 1-cycle $\Gamma\in M$. By repeating this procedure, one can obtain a whole tower of operators: $\mathcal{O}$, $\mathcal{O}^{(1)}$, $\mathcal{O}^{(2)}$, …, $\mathcal{O}^{(\dim M)}$.

$$\Delta S = \int\limits_M \mathcal{O}^{(\dim M)}\psi_{\mathcal{O}}\,, \tag{5.6}$$

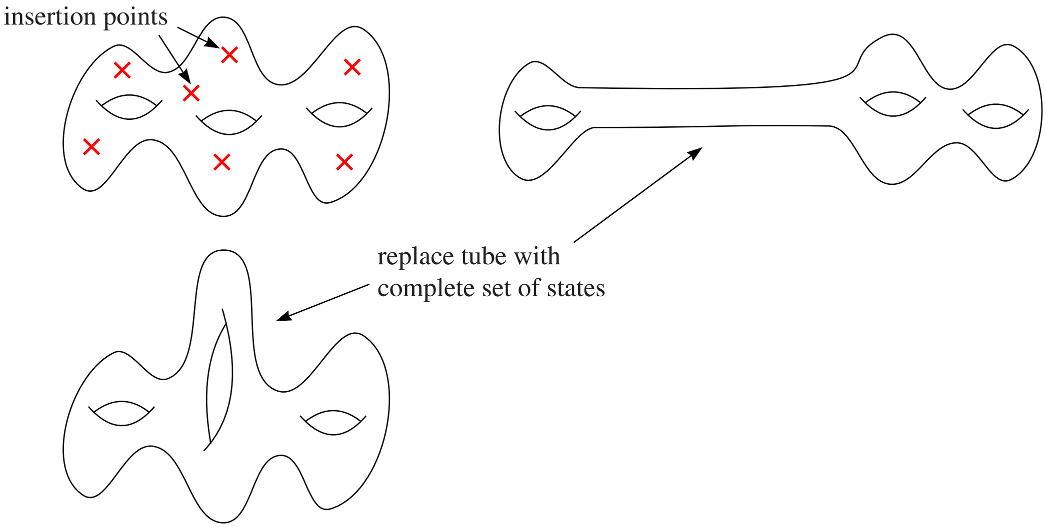
with parameter $\psi_{\mathcal{O}}$.

5.2.1 Factorization

From now on M will be a Riemann surface Σ_g. Let $\{\mathcal{O}_i\}$ be a complete set of local observables in $\mathcal{H}$.

$$\mathbb{1} = *|i\rangle\eta^{ij}\langle j|\,, \quad \eta_{ij} = \langle\mathcal{O}_i\mathcal{O}_j\rangle_{j=0}\,, \tag{5.7}$$

where η_{ij} is the **topological metric** (**not** the Zamdodchikov metric). We consider a Riemann surface decorated with insertion points of operators. There are two ways to make a Riemann surface degenerated:



a.) pull it apart and force it to a tube:

$$\langle\mathcal{O}_1\ldots\mathcal{O}_k\rangle_g = \langle\mathcal{O}_1\ldots\mathcal{O}_{k'}\mathcal{O}_i\rangle_{g'}\eta^{ij}\langle\mathcal{O}_j\mathcal{O}_{k'+1}\ldots\mathcal{O}_k\rangle_{g-g'}\,. \tag{5.8}$$

b.) add a noise:

$$\langle\mathcal{O}_1\ldots\mathcal{O}_k\rangle_g = \eta^{ij}\langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_1\ldots\mathcal{O}_k\rangle_{g-1}\,. \tag{5.9}$$

Exercise: Show that all correlators of local observables can be reduced to η_{ij} and $C_{ijk} = \langle\mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle_{g=0}$.

5.2.2 Localization

This is heuristic for simplifying **the path integral**. Given some symmetry G that has a free action on configuration space $\mathcal{E}$ it holds that

$$\int\limits_{\mathcal{E}}(\mathscr{D}\phi)\,\exp(-S)\mathcal{O} = \mathrm{Vol}(G)\int\limits_{\mathcal{E}/G}(\mathscr{D}\phi)\exp(-S)\mathcal{O}\,. \tag{5.10}$$

Since the action is free we will not need over the isometry directions. We pick some coordinate (which is the Grassmann number) and integrate ober it. Q generates fermionic symmetry:

$$\mathrm{Vol}(G_F) = \int \mathrm{d}\theta\,1 = 0\,. \tag{5.11}$$

If Q interacts freely on the configuration space, the correlator will vanish. The topological field theory path integral localizes onto configurations annihilated by Q.

5.3 Twisting

A $\mathcal{N} = (2,2)$ $D = 2$ quantum field theory produces a cohomological field theory as follows. Let us use Euclidian signature. The Lorentz group $\mathrm{SO}(2) = \mathrm{U}(1)$; charges are measured by corresponding Lorentz current J_T. The idea of Witten was: Modify the coupling to gravity by mixing J_T with J_L and J_R. Replace J_T by following linear combination:

- A-twist: $J_T \mapsto J_T - \frac{1}{2}(J_R + J_L)$: $Q_T = \overline{Q}_+ + Q_-$

  Both $\overline{Q}_+$ and Q_- become world-sheet scalars.
- B-twist: $J_T \mapsto J_T - \frac{1}{2}(J_R - J_L)$: $Q_T = \overline{Q}_+ + \overline{Q}_-$

$Q_T^2 = 0$. However, it is not good to twist by anomalous symmetries. One will have a Lorentz symmetry that is anomalous!

5.3.1 Example: B-Twisted Landau-Ginzburg model

We rescale the B-twisted charge for convenince:

$$Q_T = \frac{1}{\sqrt{2}}(\overline{Q}_+ + \overline{Q}_-)\,. \tag{5.12}$$

$\overline{\psi}^i_\pm$ become world-sheet scalars and ψ^i_+, ψ^i_- become (0,1)- and (1,0)-forms on Σ_g. We introduce following convenient linear combinations:

$$\mathcal{O}^i = \frac{1}{2}(\overline{\psi}^i_+ + \overline{\psi}^i_-)\,, \quad \chi^i = \frac{1}{2}(\overline{\psi}^i_- - \overline{\psi}^i_+)\,, \quad \varrho^i = \frac{1}{2}(\psi^i_- \mathrm{d}z + \psi^i_+ \mathrm{d}\overline{z})\,. \tag{5.13}$$

The ϕ^i remain scalars on the world-sheet. ϱ^i is a fermionic one-form on the world-sheet.

$$\{Q_T, \phi\} = 0\,, \quad \{Q_I, \overline{\phi}^i\} = 2\theta^i\,, \quad \{Q_T, \mathcal{O}^i\} = 0\,, \quad \{Q_T, \varrho^i\} = -\mathrm{d}\phi^i\,, \quad \{Q_T, \chi^i\} = -W_i\,. \tag{5.14}$$

We fix some background metric and the action takes the following form:

$$S = \int_\Sigma \left(\mathrm{d}\phi^i \wedge *\mathrm{d}\overline{\phi}^i + 2\varrho^i \wedge *\mathrm{d}\theta^i + 2\mathrm{i}\varrho^i \wedge \mathrm{d}\chi + + * (W_i \overline{W}_i + 2\chi^i \overline{W}_{ij}\theta^j)\right)\,. \tag{5.15}$$

$*$ is the Hodge star on Σ_g for background metric.

$$S = \int_\Sigma \left(2\varrho^i \wedge \mathrm{d}\chi^i - \mathrm{i}\varrho^i \wedge \varrho^j W_{ij}\right) - \left\{ Q_T, \int_\Sigma [*\chi^i \overline{W}_i + \varrho^i \wedge *\mathrm{d}\overline{\phi}^i] \right\}\,. \tag{5.16}$$

Any correlators should be independent of the metric. Local observables

$$f(\phi) \in \frac{\mathbb{C}[\phi_1, \ldots, \phi_n]}{\langle W_1, \ldots, W_n \rangle}\,, \tag{5.17}$$

which is nothing but the representation of the chiral ring.
Be ϕ^i the one zero-mode for each ϕ. θ^i, χ^i has each one zero mode. ϱ^i has $2g$ zero modes. What we get is a multi-dimensional contour integral:

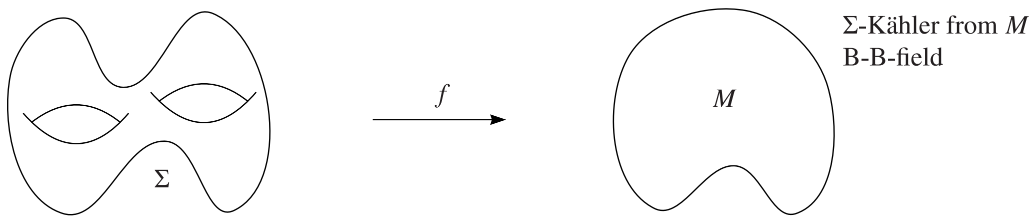
$$\langle f_1(\phi) f_2(\phi) f_3(\phi) \rangle_{g=0} = \left(\frac{1}{2\pi\mathrm{i}}\right)^n \int_\Gamma f_1 f_2 f_3 \frac{\mathrm{d}\phi_1 \wedge \ldots \wedge \mathrm{d}\phi_n}{W_1 W_2 \ldots W_n}\,. \tag{5.18}$$

Γ is a real cycle in $\mathbb{C}^r$: $\Gamma_i = \{\phi || W_i|^2 = \epsilon > 0\}$ and oriented by $\mathrm{d}(\arg W_1) \wedge \mathrm{d}(\arg W_2) \wedge \ldots \mathrm{d}(\arg W_1) \geq 0$. A **simple case** is $n = 1$ and $W = 1/(k+2)\phi^{k+2}$. $\mathcal{O} = \phi^a$, $0 \leq a \leq k$, $W = \phi^{k+1}$:

$$\langle \phi^a \cdot \phi^b \cdot \phi^c \rangle = \frac{1}{2\pi\mathrm{i}} \oint_{\text{contour around } \phi=0} \phi^a \phi^b \phi^c \frac{\mathrm{d}\phi}{\phi^{k+1}} = \delta_{a+h+c,k}\,. \tag{5.19}$$

We ware extracting the (c,c) chiral ring structure.

5.3.2 Example: A twisted NLSM with Calabi-Yau manifold target space



$$S = \int_\Sigma \Phi^*(J + \mathrm{i}B) + \{Q_T, \bullet\}\,. \tag{5.20}$$

Observables (local):

$$\omega_{i_1 \ldots i_q \overline{j}_1 \ldots \overline{j}_q}(\phi, \overline{\phi}) \psi^{i_1}_- \cdot \ldots \psi^{i_q}_- \overline{\psi}^{\overline{j}_1}_+ \cdot \ldots \overline{\psi}^{\overline{j}_q}_+\,. \tag{5.21}$$

Localization: $\partial_{\bar{z}}\phi^i = 0$ (holomorphic maps aka world-sheet instantons). Correlators are given by the sum over the world-sheet instantons $\mathcal{C}$:

$$\langle \mathcal{O}[\omega_1]\mathcal{O}[\omega_2]\mathcal{O}[\omega_3]\rangle = \sum \exp\left\{\int\limits_C \mathrm{i}(B+\mathrm{i}J)\right\}\int \langle \ldots \rangle\,. \tag{5.22}$$

For every instanton there is a contribution from the Euclidian action from the instanton. This is a semi-classical factor. However, there is a nasty integration to be done, namely an integral over the moduli space of the instanton. These instanton moduli spaces are typically non-compact. Therefore, one has to introduce a compactification.