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Renormalization Group Flows in 2D: Perturbative and Non-Perturbative Aspects

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Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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Consider the following action in conformal quantum field theory

$$S = S_{\mathrm{CFT}} + \sum_i \lambda_B^i \int \mathrm{d}^2 w \, \phi_i(w, \overline{w}) \,, \tag{1.1}$$

with λ_B^i being bare coupling constants and $\{\phi_i\}$ some set of linear independent local scaling operators of spin 0 and scaling dimension Δ_i.

$$\begin{aligned} \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle_\lambda &= \left\langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \exp\left(\sum_i \lambda_B^i \int \mathrm{d}^2 w \, \phi_i(w) \right) \right\rangle_{\mathrm{CFT}} = \\ &= \langle \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle_{\mathrm{CFT}} + \sum_i \lambda_B^i \int \mathrm{d}^2 w \, \langle \phi_i(w) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle \\ &\quad + \frac{1}{2!} \sum_{i,j} \lambda_B^i \lambda_B^j \iint \langle \phi_i(w_1) \phi_j(w_2) \mathcal{O}_1(z_1) \ldots \mathcal{O}_n(z_n) \rangle + \ldots . \end{aligned} \tag{1.2}$$

In this equation one has both UV and IR divergences, which have to be regularized. We introduce a UV regulator ε, not allowing two insertion points to come closer than ε: $|w_i - w_j| > \varepsilon$. An infrared regulator L is introduced such that $L > |w_i - w_j|$. There are two kinds of UV divergences:

1.) Some $\phi_i(w_i)$ come together with all other operators staying away. This can be dealt with by introducing renormalized coupling constants.

2.) Some $\phi_i(w_i)$ approach one of the observables $\mathcal{O}_j(z_j, \overline{z}_j)$. This can be dealt with by introducing renormalized observables.

Consider

$$\int \ldots \int \mathrm{d}^2 w_1 \ldots \mathrm{d}^2 w_n \, \phi_{i_1}(w_1) \phi_{i_2}(w_2) \ldots \phi_{i_n}(w_n) \prod_{i<j}^n \theta(|w_i - w_j| - \varepsilon) \theta(L - |w_i - w_j|) \,, \tag{1.3}$$

understood in operator sense, that is inside correlation functions with arbitrary insertions. In order to study the UV divergencies it is convenient to use the operator product expansion:

$$\phi_{i_1}(w_1) \ldots \phi_{i_n}(w_n) = \sum_A C^A_{i_1 \ldots i_n}(w_1, \ldots, w_{n-1}) \Phi_A(w_n) \,, \tag{1.4}$$

whereas ϕ_A is a complete basis of local operators. An important property of two-dimensional CFT is that the operator product expansion converges; this was proven by Lüscher. Substituting this into the perturbation theory integrals, we obtain:

$$\sum_A C^A_{i_1\dots i_n}(\varepsilon, L) \int \mathrm{d}^2 w\, \phi_A(w)\,, \tag{1.5}$$

with

$$C^A_{i_1\dots i_n}(\varepsilon, L) = \int \dots \int \mathrm{d}^2 w_1 \dots \mathrm{d}^2 w_{n-1}\, C^A_{i_1\dots i_n}(w_1, \dots, w_{n-1}) \prod_{i<j}^{n} \theta(|w_i - w_j| - \varepsilon)\theta(L - |w_i - w_j|)\,. \tag{1.6}$$

In the limit $\varepsilon \mapsto 0$ the coefficients $C^A_{i_1\dots i_n}(\varepsilon, L)$ diverge. This behavior comes from regions, where insertion points coincide. Therefore, one needs a counterterm proportional to Φ_A in the Lagrangian. The quantities $C^A_{i_1\dots i_n}$ behave like $C\varepsilon^{y_{i_1}+\dots+y_{i_k}-y_s} L^{y_s+y_{i_{k+1}}+\dots+y_{i_n}-y_A}$ (where $y_i = 2 - \Delta_i$ are the anomalous dimensions), when $(\phi_i, \dots, \phi_{i_k})$ come close together, whereas the rest $\phi_{i_{k+1}}, \dots, \phi_{i_n}$ stay away. Important to consider is $y_{i_1} + \dots + y_{i_k} \leq y_s$ $(*)$, whereas logarithmic divergencies appear, if the equality holds.

1.) When all $y_{i_k} > 0$ (all relevant), the divergences occur, when $y_s > 0$. If, in addition, the fields ϕ_i are OPE closed and there are finitely many of them, the theory is power-counting renormalizable. For large enough k there are no divergences and the theory is **super-renormalizable**.

2.) When at least one $y_i = 0$ and the rest $y_j \geq 0$, there are divergences at every k and the theory is called **asymptotically free**. (Then, the theory is renormalizable, but not super-renormalizable.)

3.) When at least one $y_{i_k} < 0$ (irrelevant), then the left-hand siede of $(*)$ can be made arbitrarily small and y_s can be arbitrarily small as well. This case is power-counting non-renormalizable.

We consider the following example:

$$S = S_{\text{CFT}} + \sum_i l^{-y_i} \lambda_i \int \mathrm{d}^2 w\, \phi_i(w) + S_{\text{counter-term}}\,, \tag{1.7}$$

where l is the renormalization scale and the λ^i are dimensionless couplings. Assume $y_i > 0$ and that the ϕ_i are primaries only (since $L_{-1}\overline{L}_{-1}\phi_i$ are irrelevant). At the second order:

$$\frac{1}{2!} \sum_{i,j} l^{-y_i-y_j} \lambda^i \lambda^j \iint \mathrm{d}^2 w_1\, \mathrm{d}^2 w_2\, \phi_i(w_1)\phi_j(w_2)\theta(|w_1 - w_2| - \varepsilon)\theta(L - |w_1 - w_2|)\,, \tag{1.8}$$

with the OPE

$$\phi_i(w_1)\phi_j(w_2) = \sum_k {c^k}_{ij} |w_1 - w_2|^{\Delta_k - \Delta_i - \Delta_j} \phi_k(w_2) + \text{descendants}\,. \tag{1.9}$$

Substitute this OPE:

$$S^{(2)}_{\text{div}} = -\pi {\sum_{i,j,k}}' \frac{{c^k}_{ij}}{y_i + y_j - y_k} \left(\frac{\varepsilon}{l}\right)^{y_i+y_j-y_k} l^{-y_k} \lambda^i \lambda^j \int \mathrm{d}^2 w\, \phi_k(w)\,, \tag{1.10}$$

where $\sum'$ means that $y_i + y_j - y_k \leq 0$. If $y_i + y_j - y_k < 0$, it holds that $S^{(2)}_{\text{counter-term}} = -S^{(2)}_{\text{div}}$ in the **minimal subtraction scheme**.

$$\lambda^k_B = l^{-y_k} \left[\lambda^k + \pi {\sum_{i,j}}' \frac{{c^k}_{ij}}{y_i + y_j - y_k} \left(\frac{\varepsilon}{l}\right)^{y_i+y_j-y_k} l^{-y_i-y_j} \lambda^i \lambda^j \right]\,. \tag{1.11}$$

Exercise:

$$l \frac{\mathrm{d}\lambda^k_B}{\mathrm{d}l} = 0 \Rightarrow l \frac{\mathrm{d}\lambda^k}{\mathrm{d}l} = \beta^k(\lambda_i) = y_k \lambda^k\,. \tag{1.12}$$

Hence, the β function is linear to the order λ^2. When $y_k = y_i + y_j$, which is the resonant case, the divergent part is given by

$$S^{(2)}_{\text{div}} = -\pi \sum_{i,j} {c^k}_{ij} \ln\left(\frac{\varepsilon}{l}\right) l^{-y_k} \lambda^i \lambda^j \int \mathrm{d}^2 w\, \phi_k(w)\,, \quad S^{(2)}_{\text{counter-term}} = -S^{(2)}_{\text{div}}\,. \tag{1.13}$$

Then, the β function is

$$\boxed{\beta^k = y_k \lambda^k + \pi c^k{}_{ij} \lambda^i \lambda^j\,,} \tag{1.14}$$

which is nonlinear. Let us now discuss the scheme dependence. Renormalization schemes are related by finite counterterms. Coupling constants redefinition: $\lambda^k \mapsto \widetilde{\lambda}^k(\lambda)$, $\widetilde{\lambda}^l = \lambda^l + \sum_{i,j} c^k{}_{ij}\lambda^i\lambda^j + \ldots$. The β functions transform as vector fields under a change of coordinates:

$$\widetilde{\beta}^k = l\frac{\mathrm{d}\widetilde{\lambda}^k}{\mathrm{d}l} = \sum_i \frac{\partial\widetilde{\lambda}^k}{\partial\lambda^i} l\frac{\mathrm{d}\lambda^i}{\mathrm{d}l}\,, \quad l\frac{\mathrm{d}\lambda^i}{\mathrm{d}l} = \beta^i(\lambda(\widetilde{\lambda}))\,. \tag{1.15}$$

The linear term stays intact, but not the second order terms:

$$\widetilde{\beta}^k = y_k\widetilde{\lambda}^k + \sum_{i,j}\widetilde{\lambda}^i\widetilde{\lambda}^j\widetilde{\mathcal{C}}^k{}_{ij} + \mathcal{O}(\lambda^3)\,, \quad \widetilde{\mathcal{C}}^k{}_{ij} = \mathcal{C}^k{}_{ij} + c^k{}_{ij}(y_i + y_j - y_k)\,. \tag{1.16}$$

For $y_i + y_j - y_k \neq 0$ one can eliminate the $\mathcal{O}(\lambda^2)$ term completely in an **appropriate scheme**. $y_i + y_j - y_k = 0$ is the resonant case and this term is universal. More generally, $C^k{}_{i_1 i_2 \ldots i_n}\lambda^{i_1}\ldots\lambda^{i_n}$ is resonant, if $y_k = y_{i_1} + \ldots + y_{i_n}$. A theorem by Poincaré tells us that if there are no resonances, there exists a formal change of coordinates, such that the β-functions are linear (minimal subtraction). There is another helpful theorem by Pincaré-Dulac: There exist formal coordinates, such that the β-functions contain only resonant terms.

Comments

1.) Not all of the higher-order resonances are universal (scheme dependent).

2.) The above theorems do not apply in the case when there are couplings with $y_i = 0$.

3.) The coupling constant redefinitions are **formal** power series.

In QCD there is a so-called 't Hooft renormalization scheme, where the β function is only a cubic polynomial (by a formal coupling constant redefinition). However, this scheme did not lead to the many breakthroughs in QCD.

4.) In minimal models $\Delta_i \in \mathbb{Q}$. Therefore, it is not hard to construct examples with resonances.

At the order λ^3 there is a region, where three operators come together.

$$\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3) = \sum_j c_j(z_1,z_2,z_3)\phi_j(z_3,\overline{z}_3) + \text{descendents}\,, \tag{1.17}$$

where the coefficients can be competed through four-point functions. The ϕ_i's are assumed to be self-conjugate and the two-point functions are normalized in the standard way

$$\langle\lambda\phi_i\phi_j\rangle \sim \frac{\delta_{ij}}{|z|^{2\Delta}}\,. \tag{1.18}$$

$$C_4(z_1,z_2,z_3) = \lim_{z_4\mapsto\infty}|z_4|^{yh_j}\langle\phi_1(z_1,\overline{z}_1)\phi_2(z_2,\overline{z}_2)\phi_3(z_3,\overline{z}_3)\phi_4(z_4,\overline{z}_4)\rangle\,. \tag{1.19}$$

In principle the four-point functions are fixed by c, Δ_i, and $C^k{}_{ij}$. The global conformal group is SL(2,$\mathbb{C}$), which is a symmetry of every correlation function. The anharmonic ratio

$$\eta = \frac{z_{12}z_{34}}{z_{13}z_{14}}\,, \tag{1.20}$$

is independent under all transformations. Then, the four-point function can be written as

$$C_4(z_1,z_2,z_3) = |z_{13}|^{2(h_y-h_1-h_2-h_3)} f(\eta,\overline{\eta})Y_{1,2,3,4}(\eta,\overline{\eta})\,, \tag{1.21}$$

$$Y_{1,2,3,4}(\eta,\overline{\eta}) = \sum_m C_{12}{}^m C_{m3}{}^4 F^m_{1,2,3,4}(\eta)\widetilde{F}^m_{1,2,3,4}(\eta)\,. \tag{1.22}$$

$Y_{1,2,3,4}$ is associated to the Feynman diagram: $f(\eta,\overline{\eta})$ is a function, which is not important now. In perturbation theory, higher order universal terms in a β-function can be obtained in terms of certain integrals of conformal blocks over the anharmonic ratios η_1, …, η_k.

1.1 RG flows between nearby fixed points (minimal models)

Unitary minimal models $\mathcal{M}_m$ labeled by $m \in \mathbb{Z}_+$ with central charge

$$c_m = 1 - \frac{6}{m(m+1)}\,, \tag{1.23}$$

contain $m(m-1)/2$ spinless primaries $\Phi_{(k,l)}$ with $k = 1, \ldots, m-1$ and $l = 1, \ldots, m$.

$$\Phi_{(k,l)} = \Phi_{(m-k,m+1-l)}\,, \quad \Phi_{(m-1,m)} = \Phi_{(1,1)} = \mathbb{1}\,. \tag{1.24}$$

The explicit formula for the conformal weight is given by

$$h_{(k,l)} = \overline{h}_{(k,l)} = \frac{[(m+1)k - ml]^2 - 1}{4m(m+1)}\,. \tag{1.25}$$

For $m \gg 1$ the fields $\Phi_{(n,n+2)}$, $\Phi_{(n+2,n)}$ have conformal weights near 1. Hence, they are near marginal operators. For example, the OPE for $\Phi_{(1,3)}$ is given by

$$\Phi_{(1,3)} \times \Phi_{(1,3)} \sim \mathbb{1} + \phi_{(1,3)} + \phi_{(1,5)}\,, \tag{1.26}$$

whereas $\phi_{1,5}$ becomes strongly irrelevant for $m \gg 1$. Therefore, if one perturbes a given minimal model by $\Phi_{(1,3)}$ one essentially has a one-coupling flow.

$$h_{1,3} = 1 - \frac{8}{m+1} \equiv h\,. \tag{1.27}$$

We like to exploit the largeness of m (or the smallness of $1/m$). Using this parameter we will get a handle of the infrared fixpoint.

- $\beta_{\min} = 4/(m+1)\lambda$ is strictly linear.

  $$l\frac{\mathrm{d}\lambda}{\mathrm{d}l} = \frac{4}{m+1}\lambda\,. \tag{1.28}$$

  λ exponentially increases as one approaches the infrared. Hence, the infrared fixpoint is at $\lambda = \infty$ in this renormalization group scheme.

- Moreover, the perturbative corrections to correlators calculated in the minimal scheme are singular as $m \mapsto \infty$. Consider for example a two-point function

  $$\langle \phi_{1,3}(x)\phi_{1,3}(0)\rangle = \langle \phi_{1,3}(x)\phi_{1,3}(0)\rangle_{\mathrm{CFT}} + \lambda_0 \int \mathrm{d}^2x_1\, \langle \phi_{1,3}(x_1)\phi_{1,3}(x)\phi_{1,3}(0)\rangle_{\mathrm{CFT}} + \ldots\,, \tag{1.29}$$

  $$C^{(1,3)}_{(1,3)(1,3)} \equiv C\,, \quad y_{1,3} = \frac{4}{m+1} \equiv y\,. \tag{1.30}$$

  We arrive at the following integral:

  $$\lambda_0 C \int \mathrm{d}^2x_1\, \frac{1}{x^{2h}x_1^{2h}(x-x_1)^{2h}} = \lambda_0 C|x|^{y-4h}\widetilde{I}(h)\,, \tag{1.31}$$

  with

  $$\widetilde{I}(h) = \pi\frac{\Gamma(2h-1)\Gamma^2(1-h)}{\Gamma(2-2h)\Gamma^2(h)} = \frac{4\pi}{y}(1 + \mathcal{O}(y^3))\,, \tag{1.32}$$

  which is singular in $y = 4/(m+1)$.

- Add a finite counterterm that cancels this term with the pole in y:

  $$\lambda_0 = l^{-y}\left(\lambda - \frac{\pi C}{y}\lambda^2\right)\,. \tag{1.33}$$

  Exercise: Differentiate both sides with $l\mathrm{d}/\mathrm{d}l$ and extract the β-function. One then arrives at the expression (non-singular scheme):

  $$\boxed{\beta^{\mathrm{ns}}(\lambda) = y\lambda + \pi C\lambda^2\,.} \tag{1.34}$$

There is an infrared fixpoint at

$$\lambda^* = -\frac{y}{\pi C} \approx -\frac{\sqrt{3}}{\pi m} \ll 1 \Rightarrow C = \frac{4}{\sqrt{3}} + \mathcal{O}\left(\frac{1}{m}\right) . \tag{1.35}$$

Now, the fixpoint is a some finite small value for the coupling.

$$\begin{aligned}\int \phi_{1,3} &= \frac{\partial}{\partial \lambda} \exp\left(\int l^{-y}\left[\lambda - \frac{\pi C}{y}\lambda^2\right]\phi_{1,3}\right) = \\ &= \int l^{-y}\left(1 - \frac{2\pi C}{y}\lambda\right)\phi_{1,3} \exp\left(\int l^{-y}\left[\lambda - \frac{\pi C}{y}\lambda^2\right]\phi_{1,3}\right) .\end{aligned} \tag{1.36}$$

One then obtains

$$[\phi_{1,3}] = l^{-y}\left(1 - \frac{2\pi C}{y}\lambda\right)\phi_{1,3} , \tag{1.37}$$

which is a wave function renormalization. Putting this into the correlator, one has

$$\langle[\phi_{1,3}](x)[\phi_{1,3}](0)\rangle_\lambda = \frac{l^{-2y}}{x^{4h}}\left\{1 + 4\pi C\lambda\left(\frac{\left(\frac{x}{l}\right)^y - 1}{y}\right) + \mathcal{O}(\lambda^2)\right\} . \tag{1.38}$$

This is nonsingular as $y \mapsto 0$. $\langle[\phi_{1,3}](l)[\phi_{(1,3)}](0)\rangle = l^{-y} + \mathcal{O}(\lambda^2)$ does not depend on λ at the leading order.

$$\widetilde{\lambda} = \lambda - \frac{\pi C}{y}\lambda^2 + \mathcal{O}(\lambda^3) , \tag{1.39}$$

whereas $\widetilde{\lambda}$ is the minimal coupling constant and λ is the coupling constant in the nonsingular scheme. Exercise: Assume that $\beta(\lambda) = y\lambda + \pi C\lambda^2$ and $\beta(\widetilde{\lambda}) = y\widetilde{\lambda}$. Find

$$\widetilde{\lambda}(\lambda) = \frac{y\lambda}{y + \pi C\lambda} , \tag{1.40}$$

and hence, the fix point $\lambda = -\pi C/y$ is mapped to $\widetilde{\lambda} = \infty$.

If one perturbs a minimal model $\mathcal{M}_m$ (for $m \gg 1$) by $\phi_{1,3}$, it will flow to another minimal model $\mathcal{M}_{m-1}$, this will be discussed more later.

1.2 Infrared divergences

$$\langle\phi\rangle \sim \lambda_0^n \int \frac{|x|^{2n-1}\,\mathrm{d}|x|}{|x|^{2h(n+1)}} \sim \lambda_0^n L^{2(n-h(n+1))} , \tag{1.41}$$

is infrared divergent for $n \geq 2/y - 1$. As y decreases, this pushes the order further and further, at which infrared divergences appear. $L \sim \lambda_0^{-1/y}$ means that we introduce a cut-off at the correlation length.

$$\langle\phi\rangle \sim \lambda_0^{\frac{2-y}{y}} = \lambda_0^{\frac{2}{y}-1} . \tag{1.42}$$

Infrared divergences lead to non-perturbative effects which reveal themselves as nonanalytic (in the coupling) contributions to correlation functions.

$$\lambda_{\mathrm{fp}} \sim \frac{1}{m} , \quad y \sim \frac{1}{m} , \quad \lambda_{\mathrm{fp}}^{\frac{2}{y}-1} \sim y^{\frac{2}{y}} , \tag{1.43}$$

which is very small. Theorem (by Guida and Magnoli): There is an infrared finite scheme to compete operator product expansion coefficients, which are analytic in the couplings. All non-perturbative effects can be put into one-point functions, for example

$$\langle\mathcal{O}_1(x)\mathcal{O}_2(0)\rangle = \sum_n C_{12}^n(x,\lambda)\langle\mathcal{O}(n)\rangle_\lambda , \tag{1.44}$$

whereas $C_{12}(x,\lambda)$ can be computed perturbatively and $\langle\mathcal{O}(n)\rangle_\lambda$ contributes non-perturbatively.

Consider the action

$$S = \frac{g}{2}\int \mathrm{d}^2x\,(\partial_\mu\phi\partial^\mu\phi + m^2\phi^2) , \tag{1.45}$$

with perturbation $m^2\phi^2$. The UV fix point is a massless free boson:

1.) non-unitary: $\langle\phi(x_1)\phi(x_2)\rangle \sim -\ln|x_1-x_2|^2$

2.) ϕ^2 is not a primary.

The non-perturbative propagator is a Bessel function:

$$\langle\phi(x)\phi(y)\rangle = \frac{1}{2\pi g}K_0(m|x-y|)\,, \quad K_0(x) = \int\limits_0^\infty \frac{\cos(xt)}{\sqrt{t^2-1}}\,, \quad x>0\,. \tag{1.46}$$

As $x \mapsto 0$ the Bessel function has a logarithmic singularity:

$$K_0(x) \sim -\ln\left(\frac{x}{2}\right) - C\,. \tag{1.47}$$

We can use this exact propagator to obtain the non-perturbative one-point function:

$$\langle[\phi^2]\rangle = \lim_{x\mapsto 0}\langle\phi(x)\phi(0)\rangle - \left\{-\ln\left(\frac{x}{2}\right)\right\}\frac{1}{2\pi g} = -\frac{\ln\left(\frac{m\gamma}{2}\right)}{2\pi g}\,, \tag{1.48}$$

which is non-analytic in m^2 and the coupling constant g.

1.3 The uses of conformal perturbation theory

- proving the integrability of a given deformation
- compute correlation functions by computing the OPE coefficients perturbatively and supplementing them with non-perturbative one-point functions
- solve models with a small dimensionless parameter (for example $1/m$ expansion)

Chapter 2

Integrable flows

We ware looking for integrals of motion in conformal field theories. For any $\Phi(z)$ that is $\overline{\partial}\Phi = 0$. Integrating a circle

$$Q_\phi = \oint_C \mathrm{d}z\, \Phi(z)\,, \tag{2.1}$$

the conserved charge Q_ϕ does not change by going from slice to slice (Cauchy's theorem.) In any conformal field theory one can build an infinite tower of the composits of $T(z)$, which are all holomorphic. In general, one can define an action by using the rule

$$(L_n\phi)(z,\overline{z}) = \frac{1}{2\pi\mathrm{i}} \oint_z \mathrm{d}w\, (w-z)^{n+1} T(w)\Phi(z,\overline{z})\,. \tag{2.2}$$

$$L_{-1}\phi = \partial\phi\,, \quad (L_{-n}\mathbb{1})(z) = \frac{1}{(n-2)!}\partial^{n-2}T(z)\,, \tag{2.3}$$

$$(L_{-2}L_{-2}\mathbb{1})(z) =: T^2(z) :\overset{\mathrm{def}}{=} \oint_z \frac{\mathrm{d}w}{2\pi\mathrm{i}}\, \frac{T(w)T(z)}{w-z}\,. \tag{2.4}$$

$\Lambda = \oplus_{s=0}^{\infty}\Lambda_s$ is the Virasoro vacuum module $L_{-n_2}L_{-n_1}|0\rangle$, whereas $\Lambda_0\Lambda_s = s\Lambda_s$. We discard the total derivatives $L_{-1}(\bullet)$.

$$\widehat{\Lambda} = \Lambda / L_{-1}\Lambda\,, \quad \widehat{\chi}(q) = \sum_{s=0}^{\infty} q^s \dim \widehat{\lambda}_s = q + (1-q)\prod_{m=2}^{\infty}(1-q^m)^{-1} = 1 + q^2 + q^4 + 2q^6 + \ldots\,, \tag{2.5}$$

if there are no null vectors. $\widehat{\Lambda}_s$ contains spin-s holomorphic fields which are no total derivatives. $s = 1$, $L_{-1}|0\rangle = 0$:

$\widehat{\Lambda}_0$	$\{\mathbb{1}\}$
$\widehat{\lambda}_1$	$\emptyset$
$\widehat{\Lambda}_2$	$\{L_{-2}\mathbb{1}\}$, $T(z)$
$\widehat{\Lambda}_3$	$\emptyset$
$\widehat{\Lambda}_4$	$\{L_{-2}^2\mathbb{1}\}$, $: T^z :$
$\widehat{\Lambda}_6$	$\{(L_{-2})^3\mathbb{1}, (L_{-3})^2\mathbb{1}\}$, $: T^3 :$, $: \partial T \partial T :$

In general $\{T_s^{(k)}\}$ is a basis in $\widehat{\Lambda}_s$.

Consider a conformal deformation!

$$S = S_{\mathrm{CFT}} + \lambda_0 \int \mathrm{d}^2x\, \Phi(x)\,, \tag{2.6}$$

with conformal weights $0 < h = \overline{h} < 1$.

$$\partial_{\overline{z}} T_s = \partial_z \Theta_{s-2}\,, \quad Q_{s-1} = \oint [\mathrm{d}z\, T_s + \mathrm{d}\overline{z}\, \Theta_{s-1}]\,. \tag{2.7}$$

Since theory preserves rotation invariance the counterterms will be of the same spin.

$$[T_s^{(k)}] \equiv T_s^{(k)}\,, \quad \overline{\partial} T_s^{(k)} = \lambda_0 (R_{s-1}^{(k)})^1 + \ldots + \lambda_0^n (R_{s-2}^{(k)})^n + \ldots\,. \tag{2.8}$$

$(R_{s-1}^{(k)})^l$ are some local renormalized fields of spin $s-1$ and $\dim(\lambda_0) = 2-2h$. $(R_{s-1}^{(k)})^n$ have conformal weights $(s-n(1-h), 1-n(1-h))$. Since the weights become negative for large enough n and the UV conformal field theory is unitary, the series stops. We assume that in the OPE $\Phi\Phi$ there are no other relevant operators but Φ itself and $\mathbb{1}$. Hence, $(R_{s-1}^{(k)})^l$ are bulit from Φ^r-operators arising in the $\Phi\Phi$ OPE and their descendants. This one must have conformal weigth $h_r = 1-n(1-h)$ and $y_r = ny$ is the resonance condition. By our assumptions on the OPE of $\Phi\Phi$ $h_r = 0$ can only happen, if $h = 1 - 1/N$ for some $N \in \mathbb{Z}_+$. $\lambda_0^N (R_{s-1}^{(k)})^N$ can then appear with R from $:T:$. Let us now study the simplest situation

$$\boxed{\overline{\partial} T_s^{(k)} = \lambda_0 (R_{s-1}^{(k)})^1\,.} \tag{2.9}$$

Compute the first order correction

$$\partial_{\overline{z}} \int \mathrm{d}^2 w\, T_s^{(k)}(z)\Phi(w,\overline{w})\,. \tag{2.10}$$

Since T is holomorphic in the conformal theory, there exists a Laurent expansion:

$$T_s^{(k)}(z)\Phi(w,\overline{w}) = \sum_{n=0}^{\infty} (z-w)^{n-s} (\mathcal{L}_{s,-n}^{(k)}\Phi)(w,\overline{w})\,. \tag{2.11}$$

The derivatives should be understood in a distributional sense:

$$\partial_{\overline{z}} \left(\frac{1}{z}\right) = \pi\delta^{(2)}(z)\,, \quad \partial_{\overline{z}}(z-w)^{-n-1} = \frac{(-1)^n}{n!}\partial_z^n \delta^{(2)}(z-w)\,. \tag{2.12}$$

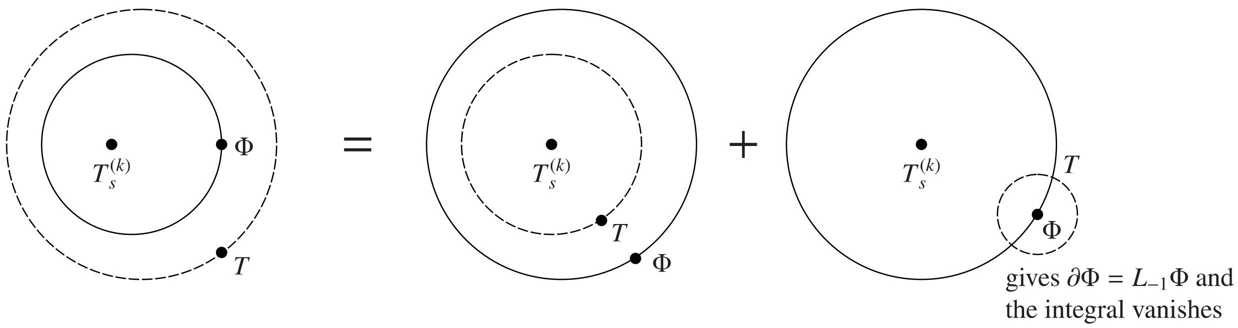
Substitution of the OPE, differentiation of the powers $(z-w)^{n-s}$ and taking the integral yields:

$$\lambda_0 \overline{\partial}_{\overline{z}} \int \mathrm{d}^2 w\, T_s^{(k)}(z)\Phi(w,\overline{w}) = \lambda_0 \sum_{n=0}^{s-1} \frac{\pi(-1)^{s-n-1}}{(s-n-1)!}\partial_z^{s-n-1}(\mathcal{L}_{s,-n}^{(k)}\Phi)(z,\overline{z})\,, \tag{2.13}$$

which is an expansion in terms of local operators. This can be rewritten as

$$\frac{\lambda_0}{2\mathrm{i}} \oint_z \mathrm{d}w\, \Phi(w,\overline{z}) T_s^{(k)}(z)\,. \tag{2.14}$$

The previous formula defines $\partial_{\overline{z}}$: $\widehat{\Lambda}_s \mapsto \Phi_{s-1}$, where Φ_{s-1} ist the space of spin-$(s-1)$-fields built on the primary Φ. The metric has an important property, namely $\partial_{\overline{z}} L_{-1}\Lambda = L_{-1}\partial_{\overline{z}}\Lambda$. The right-hand side can be defined in taking to contours:



D_n: $\Lambda \mapsto \Phi$

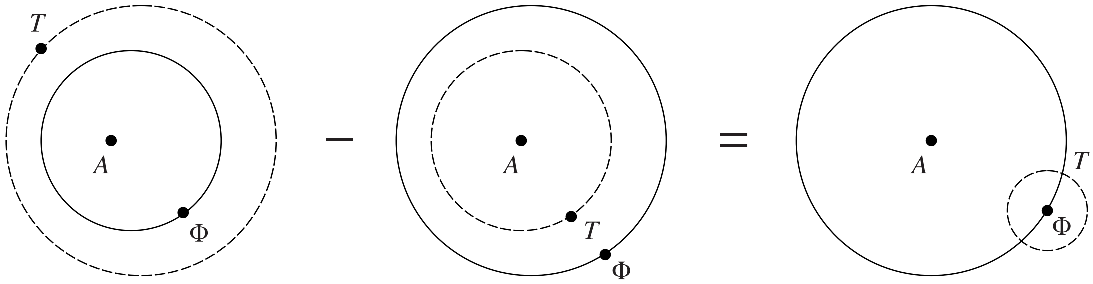
$$D_n A(z,\overline{z}) = \frac{1}{2\mathrm{i}} \oint_z \mathrm{d}w\, (w-z)^n \Phi(w,\overline{z}) A(z,\overline{z})\,, \quad D_0 \equiv \partial_{\overline{z}}\,. \tag{2.15}$$

An important property of these operators is

$$\boxed{[L_n, D_m] = -[(1-h)(n+1)+m]D_{n+m}\,.} \tag{2.16}$$

Is is computed with a very similar contour deformation, but we have to throw in an extra factor when integrating Φ.

$$[L_n, D_m]A(z,\overline{z}) = \oint\limits_z \frac{\mathrm{d}\xi}{2\mathrm{i}}(\xi - z)^m \oint\limits_\xi \frac{\mathrm{d}w}{2\pi\mathrm{i}}(w-z)^{n+1}\,. \tag{2.17}$$



$$T(w)\Phi(\xi,\overline{z}) = \frac{h\Phi(\xi,\overline{z})}{(w-\xi)^2} + \frac{\partial\Phi(\xi,\overline{z})}{w-\xi} + \text{non-singular terms}\,. \tag{2.18}$$

$$\oint \frac{\mathrm{d}\xi}{2\mathrm{i}}(\xi-z)^{m+n}[h(n+1)-(n+1+m)]\Phi(\xi,\overline{z})A(z,\overline{z}) = [(h-1)(n+1)-m]D_{n+m}A\,, \tag{2.19}$$

by definition of D_n. Taylor formula:

$$\boxed{D_{-n-1}\mathbb{1} = \frac{1}{n!}L_{-1}^n\Phi(z,\overline{z})\,.} \tag{2.20}$$

These ingredients lead us to

$$\begin{aligned}\partial_{\overline{z}}T &= \lambda_0 D_0 L_{-2}\mathbb{1} = \lambda_0(-[L_{-2},D_0]\mathbb{1} + L_{-2}D_0\mathbb{1}) = \lambda_0(h-1)D_{-2}\mathbb{1} = \\ &= \lambda_0(h-1)L_{-1}\Phi = \lambda_0(j-1)\partial\Phi\,.\end{aligned} \tag{2.21}$$

Hence

$$\partial_{\overline{z}}T = \partial^{\Theta}\,, \quad \Theta = (h-1)\lambda_0\Phi\,. \tag{2.22}$$

The usual form of the conservation of the stress-energy tensor is

$$\partial^z T_{zz} + \partial^{\overline{z}}T_{\overline{z}z} = 0\,. \tag{2.23}$$

So we have computed the leading order contribution to the trace of the stress-energy tensor:

$$T_\mu{}^\mu \sim (y\lambda_0)\Phi = \beta\Phi\,. \tag{2.24}$$

So the stress-energy tensor is still conserved, but it is no-longer conformal. Therefore it now has a non-vanishing trace. On the other hand, for the composit $:T^2:$ it holds that

$$\begin{aligned}\partial_{\overline{z}}:T^2: &= \lambda_0 D_0 L_{-2}L_{-2}\mathbb{1} = \lambda_0[D_0,L_{-2}]L_{-2}\mathbb{1} + \lambda_0 L_{-2}D_0L_{-2}\mathbb{1} = \\ &= \lambda_0(h-1)\left(\frac{h-3}{3!}L_{-1}^3\Phi + 2L_{-2}L_{-1}\Phi\right)\,.\end{aligned} \tag{2.25}$$

Generically, since $L_{-3}\Phi$ is not a partial derivative the spin current will no longer be conserved in the deformed theory.

$$\left(L_{-3} - \frac{2}{h+2}L_{-1}L_{-2} + \frac{1}{(h+1)(h+2)}L_{-1}^3\right)\Phi_{1,3} = 0\,. \tag{2.26}$$

In this case it holds that $\partial_{\overline{z}}:T^2:=\partial_z\Theta_2$ with

$$\Theta_2 = \frac{\lambda_0(h-1)}{h+2}\left(2hL_{-2} + \frac{(h-2)(h-1)(h+3)}{6(h+1)}L_{-1}^2\right)\Phi_{1,3}\,. \tag{2.27}$$

One gets a spin-4 higher conserved current and a spin-3 conserved charge. One can argument that there have to be higher conserved currents for $\Phi_{1,3}$. Therefore, define the projector Π_s: $\Phi_s \mapsto \widehat{\Phi_s} = \Phi_s/L_{-1}\Phi_{s-1}$ and $B_s = \Pi_{s^0}D_0$: $\widehat{\Lambda}_s \mapsto \widehat{\Phi}_{s-1}$. If $\dim\operatorname{Ker}(B_s) \geq 1$ then there are spin s currents. If $\dim\widehat{\Lambda}_s > \dim\widehat{\Phi}_{s-1}$, at least one non-trivial integral exists. For $\Phi_{1,3}$ one is lucky; this happens for $s-1 = 1$, 3, 5, and 7.
Theorem (S. Parker): In any local scattering theory in $(1+1)$ dimensions with at least two local higher spin $(s \geq 2)$ conserved charges there is no particle production; the final set of momenta is equal to the initial set and the $n \to n$ S-matrix factorizes into a product of $2 \to 2$ S-matrices.

$$\text{trivial theory} \xleftarrow[\substack{\text{massive theory}\\ \text{factorized S-matrix regime}}]{\Phi_{1,3}} \mathcal{M}_m \xrightarrow{\Phi_{1,3}} \mathcal{M}_{m-1}\,. \tag{2.28}$$

2.1 C-theorem (Zamolodchikov)

There exists a function $C = C(\lambda^i, \mu)$ that monotonically decreases along the RG-flows and whose value at conformal fixed points equals the Virasoro central charge. He proved this result by constructing explicitly the function C and showed that it is of the form

$$\mu\frac{\partial C}{\partial \mu} = -\beta^i g_{ij}\beta^j\,, \tag{2.29}$$

with some positive definite metric g_{ij}. $\mu = l^{-1}$ is a renormalization momentum scale.

$$C_\Lambda = 4\pi^2\left(x^\nu x^\mu x^\alpha x^\beta - x^2 g^{\mu\nu}x^\alpha x^\beta - \frac{1}{2}x^2 x^\mu g^{\nu\alpha}x^\beta\right)\langle T_{\mu\nu}(z)T_{\alpha\beta}(z)\rangle|_{|x|=\Lambda^{-1}}\,, \tag{2.30}$$

where Λ is some arbitrary momentum scale.

$$g_{ij}^\Lambda = 6\pi^2\Lambda^{-4}\langle\phi_i(x)\phi_j(0)\rangle|_{|x|=\Lambda^{-1}}\,. \tag{2.31}$$

The ϕ^i are the fields that couple with the coupling constant λ^i. The two-point functions are fully renormalized at the scale μ. On can then prove that independently of Λ the formula

$$\mu\frac{\partial C_\Lambda}{\partial \mu} = -\beta^i g_{ij}\beta^j\,, \tag{2.32}$$

holds. The **action principle** (Schwinger) introduces the operators ϕ^i in a formal way. It says that the renormlized operators ϕ_i are defined such that for any correlation function $\langle\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle$

$$\frac{\partial}{\partial\lambda^i}\langle\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle = \int \mathrm{d}^2z\,\langle\phi_i(z)\mathcal{O}(z_1)\ldots\mathcal{O}(z_n)\rangle_c\,, \tag{2.33}$$

whereas the label "c" means "connected". When the action principle holds

$$\Theta = T_\mu{}^\mu(z,\bar{z}) = \beta^i\phi_i(z,\bar{z})\,, \tag{2.34}$$

up to contact terms. Look at the Callan-Symanzik equation:

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle_c &= \int \mathrm{d}^2z\,\langle T_\mu{}^\mu(z,\bar{z})\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle = \\ &= \beta^i\partial_i\langle\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle_c + \langle\Gamma\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle \\ &\qquad + \langle\mathcal{O}_1(z_1)\Gamma\mathcal{O}_2(z_2)\ldots\mathcal{O}_n(z_n)\rangle + \ldots = \\ &= \int \mathrm{d}^2z\,\langle\beta^i\phi_i(z,\bar{z})\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle_c + \langle\Gamma\mathcal{O}_1(z_1)\ldots\mathcal{O}_n(z_n)\rangle + \ldots\,.\end{aligned} \tag{2.35}$$

$\Gamma\mathcal{O}_i$ means the RG mixings of the observables $\mathcal{O}_i$. $T_\mu{}^\mu = \beta^i\phi_i$ up to contact terms which in particular take care of the mixings $\Gamma\mathcal{O}_i$.

$$\langle T_{\alpha\beta}(x)T_{\mu\nu}(0)\rangle_c = \mu^4 F_{\alpha\beta\mu\nu}(|x|\mu)\,,\quad \langle\phi_i(x)\phi_j(0)\rangle = \mu^4 F_{ij}(|x|\mu)\,. \tag{2.36}$$

Proof

Define

$$C^\Lambda = -\int \mathrm{d}^2x\,G_\Lambda(x)\langle\Theta(x)\Theta(0)\rangle_c\,,\quad G_\Lambda(x) = 3\pi x^2\theta(1-\Lambda|x|)\,. \tag{2.37}$$

Notice, using the scaling in x and μ that $C^\Lambda = C(\mu/\Lambda, \lambda^i)$

$$\begin{aligned}\mu\frac{\partial C^\Lambda}{\partial\mu} &= -\Lambda\frac{\partial C}{\partial\Lambda} = \int \mathrm{d}^2x\,\Lambda\frac{\partial G_\Lambda(x)}{\partial\Lambda}\langle\Theta(x)\Theta(0)\rangle_c = \int \mathrm{d}^2x\,\Lambda 3\pi x^2\delta(1-\Lambda|x|)|x|\langle\Theta(x)\Theta(0)\rangle_c = \\ &= 6\pi^2\Lambda^{-4}\langle\Theta(x)\Theta(0)\rangle_c|_{|x|=\Lambda^{-1}} = 6\pi^2\Lambda^{-4}\langle\beta^i\phi_i(x)\beta^j\phi_j(0)\rangle|_{|x|=\Lambda^{-1}} = -g_{ij}^\Lambda\beta^i\beta^j\,,\end{aligned} \tag{2.38}$$

where the last step follows that at finite separation $|x| = \lambda^{-1}$ one can use $\Theta = \beta^i\phi_i$. At a conformal fixed point

$$\langle T(z)T(0)\rangle = \frac{c}{2z^4}\,. \tag{2.39}$$

The Ward identities $\partial^\mu T_{\mu\nu} = 0$ impose contact terms in the correlation functions. In particular they imply that the correlation function itself is a pure contact term

$$\Theta(x)\Theta(0)\rangle = -\frac{c}{12\pi}\partial^2\delta(x)\,. \tag{2.40}$$

Using $\overline{\partial}(1/z) = \pi\delta^{(2)}(z)$ and the conservation equations $\partial^\mu T_{\mu\nu} = 0$ ($\overline{\partial}T + \partial\widetilde{\Theta} = \text{const.}$) imply

$$\overline{\partial}\langle TT\rangle = \overline{\partial}\frac{c}{z^4} = c\partial^3\delta\,, \quad \langle\partial\Theta\overline{\partial}T\rangle \sim c\overline{\partial}\partial^3\delta\,, \quad \langle\partial\Theta\partial\Theta\rangle \sim c\overline{\partial}\partial^3\delta\,, \quad \langle\Theta\Theta\rangle \sim c\overline{\partial}\partial\delta\,. \tag{2.41}$$

In general, in quantum field theory contact terms are scheme dependent. On is quite free to change to correlation functions by contact terms.

$$\partial^\alpha\langle T_{\alpha\beta}T_{\mu\nu}\rangle = 0\,, \tag{2.42}$$

assuming $\langle T_\mu{}^\mu\rangle = 0$. The only contact term that will fulfill the Ward identities for both T is the combination

$$(\partial_\alpha\partial_\beta - \partial^2\delta_{\alpha\beta})(\partial_\mu\partial_\nu - \partial^2\delta_{\mu\nu})\delta^{(2)}(z)\,. \tag{2.43}$$

It respects the Ward identities. Other ones will have more derivatives on the δ function. There is still an ambuigity. There could be some higher derivative terms in $\langle\Theta(x)\Theta(0)\rangle$. However, by integrating it with the weight function $G_\Lambda \sim x^2$ at the origin they will not contribute any more. It holds that $C^\Lambda = C$, which can be checked by substituting

$$\langle\Theta\Theta\rangle = -\frac{c}{12\pi}\partial^2\delta\,. \tag{2.44}$$

Bonus: C^Λ can be alternatively expressed in terms of two-point functions $\langle T_{\alpha\beta}T_{\mu\nu}\rangle$ taken at distance $|x| = \Lambda^{-1}$. The quantity $-3x^2\langle\Theta(x)\Theta(0)\rangle$ is just a total derivative:

$$\boxed{-3x^2\langle\Theta(x)\Theta(0)\rangle = \partial^\mu\left[(2x^\nu x^\alpha x^\beta - 2x^2x^\nu g^{\alpha\beta} - x^2 g^{\nu\alpha}x^\beta)\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle_c\right]\,,} \tag{2.45}$$

which can be proved by using the Ward identities and tensor algebra in two dimensions. An important point is that there are many C-functions.

$$\int G_\Lambda(x)\langle T_{\mu\nu}(x)T^{\mu\nu}(0)\rangle\,. \tag{2.46}$$

Fruthermore $C_\Lambda(\Lambda/\mu, \lambda^i) \to C(\lambda^i) = C_\mu(1, \lambda^i)$. If the function is such that is satisfies the RG equation

$$\beta^i\partial_i C = \mu\frac{\partial C}{\partial\mu}\,, \tag{2.47}$$

then is follows from the formula we have just proven:

$$\beta^i\partial_i C = -\beta^i g_{ij}\beta^j\,. \tag{2.48}$$

2.2 The C-theorem in the flows between $\mathcal{M}_m$ and $\mathcal{M}_{m-1}$

$$C_m = 1 - \frac{6}{m(m+1)}\,, \quad y = y_{(0,3)} = \frac{4}{m+1}\,, \quad C^{(1,3)}_{(1,3),(1,3)} = \frac{y}{\sqrt{3}} + \mathcal{O}\left(\frac{1}{m}\right)\,. \tag{2.49}$$

Its OPE closes on itself up to an irrelevant operator.

$$\beta \equiv \beta^{\text{ns}} = \frac{4}{m+1}\lambda + \pi C\lambda^2\,, \quad \lambda^* = -\frac{\sqrt{3}}{\pi m} + \mathcal{O}\left(\frac{1}{m^2}\right)\,. \tag{2.50}$$

We have a one-coupling flow that is described by the equation

$$\beta(\lambda)\frac{\mathrm{d}C}{\mathrm{d}\lambda} = -\beta^2 g_{\lambda\lambda}\,. \tag{2.51}$$

$$\langle\phi_{1,3}(x)\phi_{1,3}(0)\rangle|_{|x|=l} = l^{-4} + \mathcal{O}(\lambda^2)\,, \quad g_{\lambda\lambda} = 6\pi^2 + \mathcal{O}(\lambda^2)\,. \tag{2.52}$$

One then obtains

$$\frac{\mathrm{d}C}{\mathrm{d}\lambda} = -6\pi^2\beta \Rightarrow \Delta C = C_{\mathrm{uv}} - C_{\mathrm{IR}} = -6\pi^2 \int_0^{\lambda^*} \beta(\lambda)\,\mathrm{d}\lambda = -\frac{12}{m^3}\,, \tag{2.53a}$$

$$\Rightarrow C_m - C_{m-1} = -\frac{12}{m(m^2-1)} \approx -\frac{12}{m^3} + \dots\,. \tag{2.53b}$$

The formula gives us the correct shift of the central charge of the nearby models.
Conformal quantum field theory implies $T_\mu{}^\mu = 0$ (up to contact terms). Scale invariance means that $T_\mu{}^\mu = \partial^\mu J_\mu$ (up to contact terms). Hence, one can define a conserved dilatation current in the following way:

$$D_\mu = x^\nu T_{\mu\nu} - J_\mu\,, \tag{2.54}$$

with $\partial^\mu D_\mu = 0$ and $T_\mu{}^\mu = \beta^i\phi_i$. If

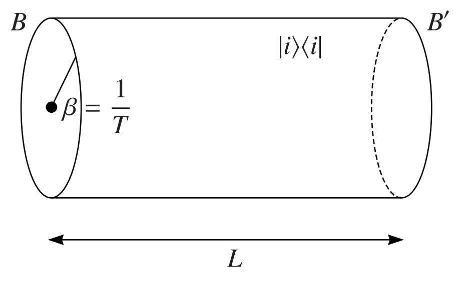
$$\mu\frac{\partial C}{\partial\mu} = \beta^i\partial_i C\,, \tag{2.55}$$

then $\beta^i\partial_i C = -\beta^i g_{ij}\beta^j$. For a scale invariant theory $\beta^i \neq 0$ only for total derivative operators. Hence $\beta^i\partial_i C = 0$ and the right-hand side has to vanish $\langle\beta^i\phi_i\beta^i\phi_i\rangle$. From that $\beta^i\phi_i = 0$ and the scale invariance of the theory is confirmed.

$$T_{\mu\nu} \mapsto T_{\mu\nu} + O_{\mu\nu}\,, \quad O_{\mu\nu} = (\partial_\mu\partial_\nu - \delta_{\mu\nu}\partial^2)\theta\,. \tag{2.56}$$

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle &= \beta^i\partial_i\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle + \langle(T_{\mu\nu}+O_{\mu\nu})(x)T_{\alpha\beta}\rangle + \langle T_{\mu\nu}(x)(T_{\alpha\beta}(0)+O_{\alpha\beta}(0))\rangle = \\ &= (\beta^i\partial_i + \beta^{p_i}\partial_{p_i})\langle T_{\mu\nu}(x)T_{\alpha\beta}(0)\rangle\,.\end{aligned} \tag{2.57}$$

If the spectrum of scaling dimensions is discrete, then one can redefine $T_{\alpha\beta}$ by adding some of the improvement terms $\mathcal{O}_{\mu\nu}$ (which are trivially conserved currents) so that under the change of scale there are no mixings. Scale invariant theories with discrete spectrum of scaling dimensions are conformally invariant. $\langle\phi_i\phi_j\rangle/\overline{Z}$ can happen in non-compact σ-models with non-homogeneous target manifold.
Consider a conformal theory in a half-plane with some boundary conditions. Although the theory is conformal at the bulk, it can happen that the boundary conditions break conformal invariance. Then there is an RG flow on the space of couplings parameterizing the space of boundary conditions. Such flows are called boundary RG flows. The central charge of these flows stay intact. However, there is another quantity that behaves like the central charge that monotonically decreases under the boundary flows: the so-called boundary entropy. Affleck and Ludwig:



The Hamiltonian becomes in this direction

$$H_x = 2\pi\beta\left[L_0 + \overline{L_0} - \frac{c}{12\beta^2}\right]\,. \tag{2.58}$$

At infinity everything but the vacuum state will be suppressed. The two boundaries decouple.

$$\lim_{L\mapsto\infty} Z_{BB'}(L) \sim \exp\left(\frac{\pi c L}{6\beta}\right) Z\cdot Z'\,, \tag{2.59}$$

with Z and Z' being the boundary partition functions; it depends only on the boundary condition. The product has to be a positive number, if H_x is Hermitian (unitary theory). One denotes this number as $Z \equiv g$

and calls it the "non-integer ground state degeneracy". $\ln(g) = S$ is the boundary entropy at critical points. $g = \langle\beta|0\rangle$.

Let us consider a stationary (time-independent) background: $\mathcal{M} \times \mathbb{R}^1$ (with $\mathcal{M}$ being the space and $\mathbb{R}^1$ time). The time-direction is describes by a free bosonic field x_0, which describes the time-coordinate of the string. Hence, we consider $G \otimes \{x_0\}$ with G describes by a CFT.

$$S = -\frac{1}{4\pi}\int \mathrm{d}^2 z\, \partial X_0 \partial X_0 \,. \tag{2.60}$$

We define the theory not by the functional integral but the propagator (with different sign):

$$\langle x_0(z) x_0(0)\rangle = \frac{1}{2}\ln |z|^2 \,. \tag{2.61}$$

The non-unitarity of a free boson comes from the UV and not the IR because of the minus sign. Normal-ordered exponents $:\exp(\mathrm{i}\omega X_0):$ are primary fields of $\Delta = \omega$. Since the theory is not unitary one has negative (complex) conformal weights: $\omega \in \mathbb{C}$. A scalar string state has a vertex operator $c\overline{c}V : \exp(\omega X_0) :$, whereas c and $\overline{c}$ are ghosts and V is an operator in G of scaling dimension Δ. We want this to satisfy the Virasoro constraints (level matching condition), which gives us the mass-shell $\omega^2 + \Delta = 2$, hence $\omega^2 = 2 - \Delta$. When $\Delta > 2$, which means that V is irrelevant, $\omega \in \mathrm{i}\mathbb{R}$, while for $\Delta < 2$, where V is relevant, $\omega \in \mathbb{R}$ and the exponents are real. What does that mean? The fact that there appear imaginary ω means that one has the usual plain wave scattering states; all massive states correspond to irrelevant operators. When one has a relevant operator, one has to address it with a real exponent, which means that the solutions exponentially increase/descrease in spacetime, which means that some instability emerges here. This instability corresponds to a tachyonic particle; the vacuum wants to go somewhere else. To summarize: Relevant operators in the spatial part of the field theory correspond to tachyons (instabilities) in the string spectrum. The theory can be deformed by a relevant operator, which gives rise to some RG flow. On the string side there is a time-dependent process describing the evolution triggered by the instability. One can ask, what happens, if the system is perturbed a little bit and starts to evolve. Is there any relationship between the last two processes? To find another vacuum one has to go off-shell, that means off the Virasoro constraints and break conformal invariance. One **conjecture** can be stated as follows: The end of the dynamic process describing the tachyonic condensation is $\mathcal{C}' \otimes X_0$, where $\mathcal{C}'$ is the end point of the RG flow triggered by V. If one starts rolling down the potential on the string side and once the theory is stationary again, the theory is of this form. For several relevant operators one can have different RG flows and the theory will settle at different stationary points. One serious problem is the central charge. We have stated that this has to decrease with the RG flow, hence $c' < c$, unless the C-theorem breaks down (but let us assume that this does not happen). However, the total central charge in string theory has to stay critical. The equation describing the two processes, namely the RG flow and the string background evolution are very different in nature. Nevertheless it turns out that there are some situations, where the two processes can be matched quite nicely. The mechanism we want to talk about was proposed by Friedman, Headrick and Lawrence in hep-th/0510126. For the solution of both problems, dilaton effects are essential (for their mechanism).

1.) The dilaton gives a damping force (friction) in the space-time evolution equations.
2.) The dilaton profile in the X_0-direction changes the central charge of $\{X_0\}$ and that compensates for the loss of central charge in the spatial theory.

Whatever the spatial theory (compact) G is, the tachyons are very light, which means that the anomalous dimensions are very small: $2 - 2h_i \ll 1$ (near marginal operators). Hence, $\omega_i \ll 1$ and the solutions are slowly rolling. In quantum field theory:

$$S_{\mathrm{QFT}} = S_G + \int \mathrm{d}^2 z\, \lambda^i \phi_i(z) + \frac{1}{4\pi\alpha'}\Phi R_2 \,, \quad \Phi = \mathbb{1}\dot{\Phi}(\lambda^i) \,. \tag{2.62}$$

Be $\overline{\beta}^i$ and $\overline{\beta}^\phi$ the β-functions. One can describe the sigma-model for the time-dependent process trigerred by these operators. Let us consider the string theory sigma-model action:

$$S_{\mathrm{str}} = S_{\mathcal{C}} + \int \mathrm{d}^2 z \left\{ \frac{G_{00}}{2\pi\alpha'}\partial X_0 \overline{\partial} X_0 + \lambda^i(X_0)\phi_i + \frac{1}{4\pi\alpha'}\Phi(X_0) R_2 \right\} , \tag{2.63}$$

where the last term is the dilaton term. The true coupling **constants** can be obtained by expanding the operators in some basis of operators. One can, for example, expand in the exponentials

$$\lambda^i(X_0) = \int : \exp(\mathrm{i}\omega X_0) : \lambda^i_\omega \, \mathrm{d}\omega \,, \tag{2.64}$$

where the λ^i_ω are the true coupling constants. One can introduce generating functions

$$\lambda^i(t) = \int \exp(\mathrm{i}\omega t)\lambda^i_\omega \,\mathrm{d}\omega\,, \tag{2.65}$$

and similarly for $\phi(t)$. One can think of $\lambda^i(t)$ as time-dependent coupling constants. In general, one can think of the spatial theory as being split as follows:

$$\mathcal{C} \otimes (\text{spectator CFT}) \otimes \{X_0\} \otimes \text{ghosts}\,, \tag{2.66}$$

where the spectator CFT does not contribute to time-dependent processes, but can modify the on-shell relations (contributes some central charges).

The brackground charge model is a simple modification of the free theory:

$$S = \frac{1}{4\pi\alpha'}\int \mathrm{d}^2 z\,\sqrt{g}\,\{\eta\partial_\alpha X \partial^\alpha X + \alpha' R_2 Q X\}\ , \quad \eta = \pm 1\,, \tag{2.67}$$

whereas Q is the background charge, R_2 the world-sheet curvature and the last term is the linear dilaton term.

$$\langle : \exp(\mathrm{i}q_i X) : \ldots : \exp(\mathrm{i}q_n X) : \ldots\rangle = \delta(q_1 + \ldots + q_n + Q)\,, \tag{2.68}$$

which is why Q is called background charge. The central charge follows from

$$T(z) = -\frac{\eta}{\alpha'} : \partial X \partial X : + Q\partial^2 X\,, \tag{2.69}$$

and the central charge is shifted: $c = 1 + 6\alpha'\eta Q^2$. The β functions for the free fields were computed with the answers

$$(\alpha')^{-1}\beta^i = (\alpha')^{-1}\overline{\beta}^i + \frac{1}{2}\ddot{\lambda}^i + \frac{1}{2}\Gamma^i{}_{jk}\dot{\lambda}^j\dot{\lambda}^k - \dot{\phi}\dot{\lambda}^i\,, \tag{2.70a}$$

$$(\alpha')^{-1}\beta^G = -4\pi g_{ij}\dot{\lambda}^i\dot{\lambda}^j + 2\ddot{\phi}\,, \tag{2.70b}$$

$$(\alpha')^{-1}\beta^\Phi = V - \dot{\Phi}^2 + \frac{1}{2}\ddot{\Phi}\,. \tag{2.70c}$$

whereas the dots denote time derivatives: $\dot{\lambda}^i = \mathrm{d}\lambda^i(t)/\mathrm{d}t$. $\overline{\beta}$ is the beta-function of the model S_{QFT}. $\Gamma^i{}_{jk}$ are Christoffel symbols for the Zam-V metric g_{ij} of the space of couplings λ^i. V is given by

$$V = (\alpha')^{-1}\left(\overline{\beta}^\Phi + \frac{1}{6}(c_{\mathrm{aux}} + 1)\right)\,, \tag{2.71}$$

and is part of the central charge (not taking into account the dilaton). $\overline{\beta}^\Phi = c/6$ at fixed points. c_{aux} includes ghosts. From $T_\mu{}^\mu = \beta^i\phi_i$ comes the contribution $-(\alpha')^{-1}\partial\overline{\partial}\Phi(X_0)$, $R_2 = -2\partial\overline{\partial}N(z,\overline{z})$, $g = \Lambda^2\,\mathrm{d}^2 z$.

$$-(\alpha')^{-1}\partial\overline{\partial}\Phi = -(\alpha')^{-1}(\dot{\Phi}\partial\overline{\partial}X_0 + \ddot{\Phi}\partial X_0\overline{\partial}X_0)\,. \tag{2.72}$$

The equation of motion follows from the variation of the action S_{str}:

$$\partial\overline{\partial}X_0 = -\pi\alpha'\left(\dot{\lambda}^i(X_0)\Phi_i + \frac{1}{4\pi\alpha'}\dot{\Phi}R_2\right)\,. \tag{2.73}$$

$$T_\mu{}^\mu = -(\alpha')^{-1}\partial\overline{\partial}\Phi = \pi\dot{\Phi}\dot{\lambda}^i\Phi_i + \frac{1}{4\pi}\dot{\Phi}^2 R_2 - (\alpha')^{-1}\ddot{\Phi}\partial X_0\overline{\partial}X_0\,. \tag{2.74}$$

Note: One can introduce

$$H = \frac{1}{\alpha'}\left(\beta^\Phi - \frac{1}{4}\beta^G\right) = V - \dot{\Phi}^2 + \pi^2 g_{ij}\dot{\lambda}^i\dot{\lambda}^j\,. \tag{2.75}$$

1.) From the first equation it follows that if $\dot{\lambda}^i = 0$ then $\overline{\beta}(\lambda^i) = 0$. So, the evolution stops only at the fixed points of the RG flow.

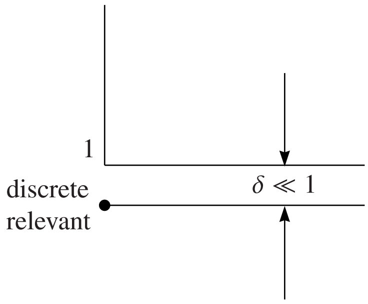
2.) From the second equation it follows that one cannot decouple the dilaton from the evolution. From $\dot{\lambda}^i = 0$ follows $\dot{\Phi} \neq 0$.

3.) $\dot{\Phi} < 0$ means damping and $\dot{\Phi} > 0$ means antidamping. Since $\ddot{\Phi} > 0$ follows that once antidamping is there it will stay and cannot reverse into damping. Therefore, to have a nice evolution, $\dot{\Phi} \leq 0$ at all times.

4.) From $H = 0$ it follows that $\lambda^i(t)$ cannot stop in the region, when $V < 0$.

For $V < 0$ if $\dim\mathcal{M} < c$ (matter central charge, no background charges taken into account) the situation if subcritical (includes two-dimensional strings), for $V > 0$ $\dim\mathcal{M} > c$ the situation is supercritical and for $V = 0$ it is critical.

2.3 Tachyonic D1-branes in 2D non-critical string theory



RG: D1 $\mapsto \widetilde{\text{D1}}$ (non-tachyonic) The time-dependence never stops. The energy by rolling down the potential goes into radiation. Time-dependent process: Evolve into $\widetilde{\text{D1}}$ plus radiation escaping to ∞.