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Gravity and Hydrodynamics:
the Fluid-Gravity Correspondence

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Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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The Fluid-Gravity Correspondence

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- 0712.2456
- 0803.2526
- 0809.4272
- review: 0905.4352

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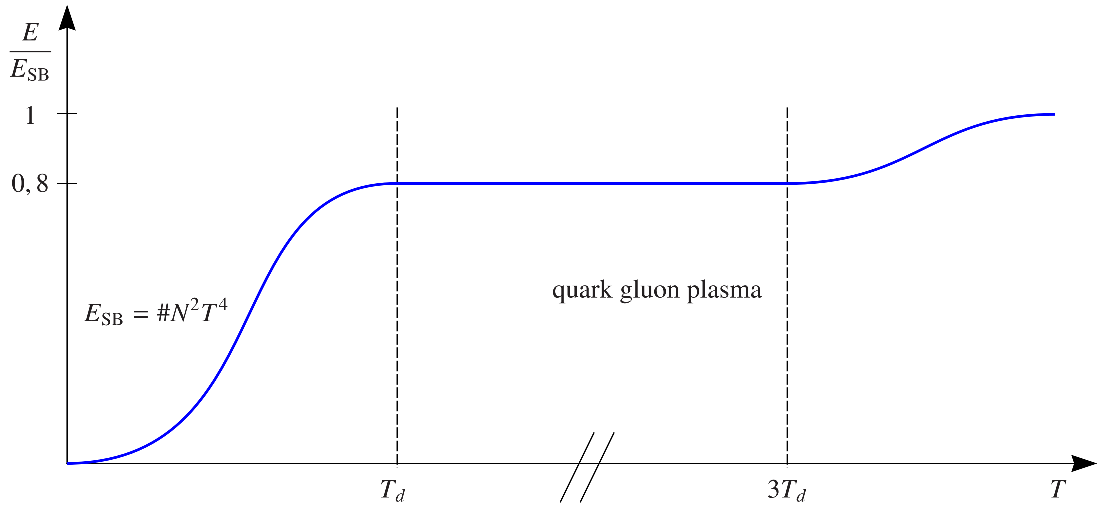
- Extensions: charged fluids, forced fluids, non-relativistic fluids

1.3 Motivation

The claim we want to make is that fluid dynamics is the effective long-wavelength description of an interacting quantum field theory (in local equilibrium). One could use this, for example, to study the physics of QCD just above deconfinement.



One can now ask, what kinds of properties this state of matter has. The quark gluon plasma behaves as a nearly ideal fluid; it has almost no dissipation. If perturbation theory was applicable in this regime, one would find a strongly dissipating fluid.

$$\frac{\text{Shear viscosity}}{\text{entropy density}} = \frac{\eta}{S} \approx (0,1 - 0,2)\,\frac{\hbar}{k_B}\,. \tag{1.1}$$

This answer comes from a fit to a hydrodynamical system, since we do not understand the microscopics, in this case QCD. This value is far below the values for any other fluid. Liquid helium comes close to it, but it is still a factor of ten below this value. From perturbation theory it follows that

$$\frac{\eta}{S} = \frac{1}{\lambda^2 \ln(\lambda)}\,. \tag{1.2}$$

For $\lambda \mapsto 0$, where the theory is dissipation-free, the shear viscosity also vanishes, but this behavior cannot be seen in perturbation theory. From black hole physics using the AdS-CFT correspondence one obtains

$$\frac{\eta}{S} = \frac{\hbar}{k_B}\frac{1}{4\pi} \approx 0,08\frac{\hbar}{k_B}\,. \tag{1.3}$$

This calculation was done in a field theory with infinitely large coupling. The calculation was done by evaluating the following correlation function

$$\frac{\eta}{S} = \lim_{\omega \mapsto 0} \frac{1}{\omega} \langle T_{\mu\nu}(\omega) T_{\mu'\nu'}(0)\rangle\,. \tag{1.4}$$

The bound

$$\frac{\eta}{S} \geq \frac{\hbar}{k_B}\frac{1}{4\pi}\,, \tag{1.5}$$

has been proposed. Another calculation on DB-branes reveals

$$\frac{E}{E_{\text{SB}}} = \frac{3}{4}\,, \tag{1.6}$$

at strong coupling.

Something we do not unterstand is

- turbulence
- regularity of Navier-Stokes evolution (usually for non-relativistic in-compressible systems)

Systems of interest are relativistic and compressible ones.

1.4 Gravity motivations

One important question is the following: What is the set of regular black hole solutions in $d > 4$? In four dimensions this is the Kerr-Newman black hole (S^2 topology horizon), which is completely specified by mass M, angular momentum J and charge Q. However, in five dimensions this is not true.

1.5 Summary of the story

1.) We want to consider fluid dynamics as an effective field theory. In order to do this one has to identify

  - the variables (thermodynamic)
  - and the operators at given order.

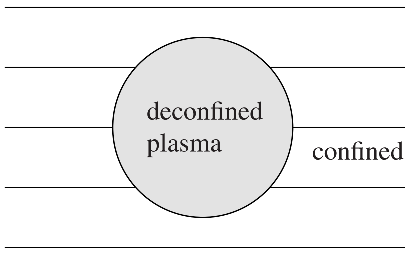
  We boil this down to determining a finite set of transport coefficients (η, ζ, etc.), which characterize the fluid. The language to describe the fluid is just statistical mechanics. The transport coefficients distinguish one fluid from another one. On the other hand we are interested in causality issues in relativistic fluids and we want to go beyond first order fluid dynamics.

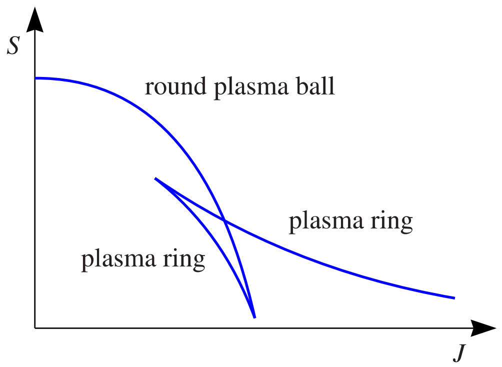
2.) We want to apply this to gravity using the AdS/CFT correspondence. We will see that Einstein's equations will lead us to the fluid equations by considering inhomogeneous dynamical asymptotically AdS black holes. We will be interested in the entropy current for fluids from gravitational perspective. As a next step, generalizations will come into play, namely charged or forced fluids. Finally we want to compare known stationary solutions in AdS.

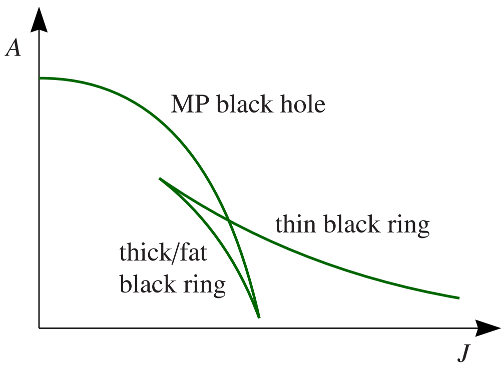
Consider Scherk-Schwarz compactifications of $\mathcal{N} = 4$ SYM on $\mathbb{R}^{2,1} \times S^1$ with anti-periodic boundary conditions for fermions: The ground state of the theory is a AdS_5 soliton:

$$\mathrm{d}s^2 = r^2 \left(1 - \frac{r_+^4}{r^4}\right) \mathrm{d}\chi^2 + \frac{\mathrm{d}^2}{r^2 \left(1 - \frac{r_+^4}{r^4}\right)} + r^2(-\mathrm{d}t^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2)\,, \tag{1.7}$$

$$\mathrm{d}s^2_{\mathrm{Schw}} = -r^2 \left(1 - \frac{r_+^4}{r^4}\right) \mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2 \left(1 - \frac{r_+^4}{r^4}\right)} + r^2(\mathrm{d}\chi^2 + \mathrm{d}x_1^2 + \mathrm{d}x_2^2)\,. \tag{1.8}$$



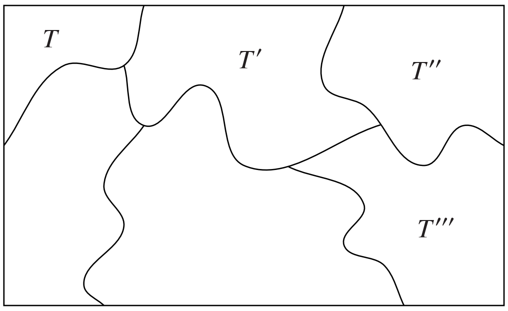




Chapter 2

Fluid dynamics

Fluid dynamics is an effective field theory which is valid at long wavelengths. The length scale in the system is the mean free path l_{mfp}. The system be in thermal equilibrium.



We want to use extensive thermodynamical variables ε, q_I, s etc. and intensive variables T, μ_I, P etc.

$$\Xi = \mathrm{Tr}\left\{\exp\left(-\beta(H+\mu_I q_I)\right)\right\} \,. \tag{2.1}$$

Hydrodynamics is valid, if the scales of variation of the thermodynamic variables L is much bigger than l_{mfp}. Hydrodynamics is conservative with respect to the dynamics of the system; hence, the dynamical **equations** are simply conservation laws.

$$\boxed{\nabla_\mu T^{\mu\nu} = 0\,, \quad \nabla_\mu J_I^\mu = 0\,,} \tag{2.2}$$

which means that the energy-momentum tensor and the charged current are conserved. From that, the non-relativistic continuity equation follows:

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{j} = 0\,. \tag{2.3}$$

The system is completely specified by giving $T_{\mu\nu}$, $J_{\mu,I}$ as functions of hydrodynamic variables There are both extensive and intensive hydrodynamic variables:

i.) Extensive variables: ε, q_I

ii.) (or) Intensive variables: T, μ_I, P

They are not independent; the equation of state relates intensive and extensive variables. What we furthermore need, are energy and charge fluxes ε_i, $q_{i,I}$, the velocity u^μ. We will use the metric $g^{\mu\nu}$ such that $g^{\mu\nu}u_\mu u_\nu = -1$. For an ideal fluid one has

$$(T_{\mu\nu})_{\mathrm{ideal}} = \varepsilon u_\mu u_\nu + P(g_{\mu\nu}+u_\mu u_\nu)\,, \quad (J_{\mu,I})_{\mathrm{ideal}} = q_I u_\mu \,. \tag{2.4}$$

The equations are written in terms of extensive variables, since the pressure P can be expressed by the energy ε by using the equation of state. We define a spatial projector by $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$, for which it holds that

$$P^{\mu\nu}u_\mu = 0\,, \quad P^{\mu\varrho}P_{\varrho\nu} = P^\mu{}_\nu\,, \quad P^\mu{}_\mu = d-1\,. \tag{2.5}$$

Then, we can write

$$(T^{\mu\nu})_{\mathrm{ideal}} = \varepsilon u^\mu u^\nu + P P^{\mu\nu}\,. \tag{2.6}$$

$\nabla_\mu T^{\mu\nu} = 0$ are d equations for d variables. In particular, an ideal fluid does not dissipate, because there is no friction. The entropy current measures entropy production: $J_S^\mu = su^\mu$, whereas s is the entropy density. Since there is no friction, the entropy current is conserved: $\nabla_\mu J_S^\mu = 0$.

We account for dissipation like follows:

$$T^{\mu\nu} = (T^{\mu\nu})_{\text{ideal}} + \Pi^{\mu\nu}\,, \quad J_I^\mu = (J_I^\mu)_{\text{ideal}} + \gamma_I^\mu\,, \tag{2.7}$$

whereas $\Pi^{\mu\nu}$ and γ_I^μ are the dissipative parts. The phenomenological method to determine $\Pi^{\mu\nu}$, γ_I^μ is to use the first law and to require that there exists an entropy current J_S^μ with $\nabla_\mu J_S^\mu \geq 0$. As mentioned, hydrodynamics is valid, if $L \gg l_{\text{mfp}}$. This implies that one can use an effective field theory to organize terms appearing in $T^{\mu\nu}$ and J_I^μ. Look at a domain of local equilibrium at temperature T with $\partial T/T \ll 1$. l_{mfp} will be a function of coupling constants divided by temperature: $l_{\text{mfp}} \sim f(\text{coupling})/T$. Because of that we can think of fluid dynamics as a **gradient** expansion, because derivatives with respect to the equilibrium value shall be small. $\Pi^{\mu\nu}$ and γ_I^μ should be built out of derivatives of (ε, P, q_I) and u^μ. This is completely analogous to constructing infrared effective field theories (for example the chiral Lagrangian für QCD).

- At zeroth order, which is equivalent to an ideal fluid, there are no derivatives.
- At first order one should allow all terms (subject to **symmetry**) with one derivative ($\nabla_\mu u_\nu$, $\nabla_\mu \varepsilon$, $\nabla_\mu P$,etc.)

  At any given order one is free to use the lower order equations of motion.

In order to define the velocity field for dissipative fluids we need a definition. We will use the Landau frame, where in the local inertial frame we measure the total energy density.

$$\Pi^{\mu\nu} u_\mu = 0\,, \quad \gamma_I^\mu u_\mu = 0\,. \tag{2.8}$$

We would like to decompose velocity gradients:

$$\nabla^\mu u^\nu = -a^\mu u^\nu + \sigma^{\mu\nu} + \frac{1}{d-1}\theta P^{\mu\nu} + \omega^{\mu\nu}\,, \tag{2.9a}$$

with

$$\theta = \nabla_\mu u^\mu = P^{\mu\nu}\nabla_\mu u_\nu\,, \quad a^\mu = u^\nu \nabla_\nu u^\mu\,, \tag{2.9b}$$

$$\sigma^{\mu\nu} = \nabla^{(\mu}u^{\nu)} + u^{(\mu}a^{\nu)} - \frac{1}{d-1}\theta P^{\mu\nu} = P^{\mu\alpha}P^{\nu\alpha}\nabla_{(\alpha}u_{\beta)} - \frac{1}{d-1}\theta P^{\mu\nu}\,, \tag{2.9c}$$

and

$$\omega^{\mu\nu} = -(\nabla^{[\mu}u^{\nu]} + u^{[\mu}a^{\nu]}) = -P^{\mu\alpha}P^{\nu\beta}\nabla_{[\alpha}u_{\beta]}\,. \tag{2.9d}$$

The first order contributions to $\Pi^{\mu\nu}$ come from $\sigma^{\mu\nu}$ and θ. There are no contributions from $\nabla_\mu \varepsilon$ or $\nabla_\mu P$ for conservation equations at zeroth order give:

$$u_\nu (\nabla_\mu T^{\mu\nu})_{\text{ideal}} = (\varepsilon + P)\nabla_\mu u^\mu + u^\mu \nabla_\mu \varepsilon = 0\,, \tag{2.10a}$$

$$P_{\nu\alpha}(\nabla_\mu T^{\mu\nu})_{\text{ideal}} = P_\alpha{}^\mu \nabla_\mu P + (\varepsilon + P) P_{\nu\alpha} u^\mu \nabla_\mu u^\nu = 0\,. \tag{2.10b}$$

One can use the decomposition

$$\Pi^{\mu\nu}_{(1)} = -2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}\,, \tag{2.11}$$

where η is the shear viscosity and ζ the bulk viscosity. Furthermore

$$\gamma_I^\mu = -\widetilde{\kappa}_{IJ} P^{\mu\nu}\nabla_\nu g_J - \widetilde{\gamma}_I P^{\mu\nu}\nabla_\mu \varepsilon - v_I l^\mu\,, \tag{2.12}$$

whereas $l^\mu = \varepsilon_{\alpha\beta\gamma}{}^\mu u^\alpha \nabla^\beta u^\gamma$ and hence the third term exists only in four dimensions. The transport coefficient v_I is related to the global anomaly coefficient (Sont Surowka). This above equation can also be written in the form

$$\gamma_I^\mu = -\kappa_{IJ} P^{\mu\nu}\nabla_\nu\left(\frac{\mu_J}{T}\right) - \gamma_I P^{\mu\nu}\nabla_\nu T - v_I l^\mu\,, \tag{2.13}$$

with parameters κ_{IJ}, γ_I and v_I. At first order the entropy current is given by

$$\nabla_\mu J_S^\mu = \frac{2\eta}{T}\sigma_{\alpha\beta}\sigma^{\alpha\beta}\,. \tag{2.14}$$

2.1 Causality in relativistic hydrodynamics

Let us look at the stress tensor:

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu + (\varepsilon + P)P^{\mu\nu} - 2\eta\sigma^{\mu\nu} - \zeta\theta P^{\mu\nu}\,. \tag{2.15}$$

The equations $\nabla_\mu T^{\mu\nu} = 0$ are **parabolic** in all frames.

2.1.1 Isreal-Muller-Stewart formalism

Let us look at **diffusion** to have a simple example. The continuity equation is given by

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot \mathbf{j} = 0\,. \tag{2.16}$$

Furthermore, we have Fick's law $\mathbf{j} = -D\nabla\varrho$. Plugging this into the continuity equation leads to

$$\frac{\partial \varrho}{\partial t} = -D\boldsymbol{\nabla}^2\varrho = 0\,, \tag{2.17}$$

which is a parabolic equation. Modify Fick's law to make the current relax exponentially:

$$\mathbf{j} = -D\boldsymbol{\nabla}\varrho - \tau_\pi \frac{\partial \mathbf{j}}{\partial t} \Rightarrow \frac{\partial \varrho}{\partial t} - D\boldsymbol{\nabla}^2\varrho - \tau_\pi \boldsymbol{\nabla} \cdot (\partial_t \mathbf{j}) \approx \frac{\partial \varrho}{\partial t} - D\boldsymbol{\nabla}^2\varrho + \tau_\pi \frac{\partial^2 \varrho}{\partial t^2} = 0\,, \tag{2.18}$$

whereas we now have a hyperbolic system. The signal propagation has now a bounded velocity

$$v_{\text{prop}} = \sqrt{\frac{D}{\tau_\pi}}\,. \tag{2.19}$$

Chapter 3

Conformal fluids

Consider a Weyl rescaling of background metric

$$g_{\mu\nu} = \exp(2\phi)\widetilde{g}_{\mu\nu} \,. \tag{3.1}$$

Normalization of velocity field implies $u^\mu = \exp(-\phi)\widetilde{u}^\mu$. The conformal weight of any operator $Q = \exp(-\omega\phi)\widetilde{Q}$ ist ω. Conformal invariance implies the tracelessness of the stress tensor: $T_\mu{}^\mu = 0$. Furthermore, it follows that $T^{\mu\nu} = \exp(-(d+2)\phi)\widetilde{T}^{\mu\nu}$. From $T_\mu{}^\mu = 0$ $((T^{\mu\nu})_{\text{ideal}} = \varepsilon u^\mu u\nu + PP^{\mu\nu})$ the equations of state follows: $\varepsilon = (d-1)P$. Find the weights of thermodynamic variables: $T = \exp(-\phi)\widetilde{T}$. Using the equation of state and the Stefan-Boltzmann scaling law $\varepsilon \sim T^d$ one obtains

$$(T^{\mu\nu})_{\text{ideal}} = \alpha T^d(du^\mu u^\nu + g^{\mu\nu}) \,. \tag{3.2}$$

θ scales inhomogeneously

$$\theta = \exp(-\phi)(\widetilde{\theta} + (d-1)\widetilde{u}^\mu\widetilde{\nabla}_\mu\phi) \,, \tag{3.3}$$

and

$$\sigma^{\mu\nu} = \exp(-3\phi)\widetilde{\sigma}^{\mu\nu} \,, \quad l^\mu = \exp(-2\phi)\widetilde{l}^\mu \,. \tag{3.4}$$

A conformal fluid has $\zeta = 0$:

$$T^{\mu\nu} = \alpha T^d(g^{\mu\nu} + du^\mu u^\nu) - 2\eta\sigma^{\mu\nu} \,, \quad J_I^\mu = q_I u^\mu - \kappa_{IJ} P^{\mu\nu}\nabla_\nu\left(\frac{\mu_J}{T}\right) \,, \tag{3.5}$$

with η being of the general form $\eta \sim T^{d-1}f(\mu_I/T)$. Anyway, the Gibbs-Duhem relation holds:

$$P + \varepsilon = sT + q_I\mu_I \,. \tag{3.6}$$

We will now go to the second order, from which two derivative terms will emerge. A conformal invariant fluid on a manifold B_d does not case about metric data, but rather on the conformal structure of B_d. Denote the conformal class (B_d, C) (with some structure C) and define a derivation on B_d. Define a Weyl connection ∇^{Weyl}. For each representative $g \in C$ there exists a connection $\mathcal{A}$ such that

$$\nabla_\alpha^{\text{Weyl}} g_{\mu\nu} = 2\mathscr{A}_\alpha g_{\mu\nu} \,. \tag{3.7}$$

Define a new derivative operator

$$\mathscr{D} = \nabla^{\text{Weyl}} + \omega\mathscr{A} \,. \tag{3.8}$$

Let us consider a tensor $Q^{\mu\cdots}{}_{\nu\ldots}$ with weight ω under Weyl rescalings: $Q^{\mu\cdots}{}_{\nu\ldots} = \exp(-\omega\phi)\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots}$.

$$\begin{aligned}\mathscr{D}_\lambda\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots} = \nabla_\lambda\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots} + \omega\mathscr{A}_\lambda\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots} + (g_{\lambda\alpha}\mathscr{A}^\mu - \delta^\mu{}_\lambda\mathscr{A}_\alpha - \delta^\mu{}_\alpha\mathscr{A}_\lambda)\widetilde{Q}^{\alpha\cdots}{}_{\lambda\ldots} \\ - (g_{\lambda\nu}\mathscr{A}^\alpha - \delta^\alpha{}_\lambda\mathscr{A}_\nu - \delta^\alpha{}_\nu\mathscr{A}_\lambda)\widetilde{Q}^{\mu\cdots}{}_{\alpha\ldots} \,.\end{aligned} \tag{3.9}$$

It holds that

$$\mathscr{D}_\lambda\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots} = \exp(-\omega\phi)\widetilde{\mathscr{D}}_\lambda\widetilde{Q}^{\mu\cdots}{}_{\nu\ldots} \,, \tag{3.10}$$

and the question now arises, what $\mathscr{A}$ is. $\mathscr{D}$ is metric compatible: $\mathscr{D}_\lambda g_{\mu\nu} = 0$. We are going to fix $\mathscr{A}_\alpha$ for fluids using the velocity field u^μ. The Weyl covariant derivative of u^μ is required to be transverse and traceless:

$$u^\alpha \mathscr{D}_\alpha u^\mu = 0\,, \quad \mathscr{D}_\alpha u^\alpha = 0 \Rightarrow \mathscr{A}_\mu = u^\lambda \nabla_\lambda u_\mu - \frac{1}{d-1} u_\mu \nabla^\lambda u_\lambda = a_\mu - \frac{1}{d-1}\theta u_\mu\,. \tag{3.11}$$

At first order in the derivatives one can write $\sigma^{\mu\nu} = \mathscr{D}^{(\mu}u^{\nu)}$ and $\omega^{\mu\nu} = -\mathscr{D}^{[\mu}u^{\nu]}$. They are homogeneous with $\omega = 3$. Furthermore $\mathscr{D}_\mu T^{\mu\nu} = \nabla_\mu T^{\mu\nu} = 0$. Let us go to the second order in derivatives:

$$\mathscr{D}_\mu \mathscr{D}_\nu u^\lambda = \mathscr{D}_\mu \sigma^\lambda{}_\nu + \mathscr{D}_\mu \omega_\nu{}^\lambda\,, \quad \omega = 1\,, \tag{3.12}$$

$$\mathscr{D}_\lambda \sigma_{\mu\nu}\,, \quad \mathscr{D}_\lambda \omega_{\mu\nu}\,, \quad \sigma_\alpha{}^\mu \sigma^{\alpha\nu}\,, \quad \omega^\mu\ \omega^{\nu\alpha}\,, \quad \sigma^\mu{}_\alpha \omega^{\nu\alpha}\,, \quad \omega = -1\,. \tag{3.13}$$

At second order we can have contributions from the background curvature. The only two derivative operator built from curvature tensors, which is homogeneous under Weyl rescalings is $C_{\mu\nu\varrho\sigma}u^\mu u^\varrho$.

- First order: $\sigma^{\mu\nu}$
- Second order: $\tau_1^{\mu\nu} = 2u^\alpha \mathscr{D}_\alpha \sigma_{\mu\nu}$, $\tau_2^{\mu\nu} = C^\mu{}_\alpha{}^\nu{}_\beta u^\alpha u^\beta$, $\tau_3^{\mu\nu} = 4\sigma^{\alpha\langle\mu}\sigma^{\nu\rangle}$,
  $\tau_4^{\mu\nu} = 2\sigma^{\alpha\langle\mu}\omega^{\nu\rangle}{}_\alpha$, $\tau_5^{\mu\nu} = \omega^{\alpha\langle\mu}\omega^{\nu\rangle}{}_\alpha$

$$\Pi^{\mu\nu} = -2\eta\sigma^{\mu\nu} + \tau_\pi \eta \tau_1^{\mu\nu} + \kappa \tau_2^{\mu\nu} + \lambda_1 \tau_3^{\mu\nu} + \lambda_2 \tau_4^{\mu\nu} + \lambda_3 \tau_5^{\mu\nu}\,. \tag{3.14}$$

For $\mathcal{N} = 4$ SYM-theory one obtains:

$$\eta = \frac{N^2}{8\pi}(\pi T)^3\,, \quad \frac{\eta}{S} = \frac{1}{4\pi}\,, \quad \alpha = \frac{\pi^2 N^2}{8}\,. \tag{3.15}$$

Now to the transport coefficients:

$$\tau_\pi = \frac{2-\ln(2)}{2\pi T}\,, \quad \kappa = \frac{\eta}{\pi T}\,, \quad \lambda_1 = \frac{\eta}{2\pi T}\,, \quad \lambda_2 = \frac{\eta \ln(2)}{\pi T}\,, \quad \lambda_3 = 0\,. \tag{3.16}$$

Consider now thermal $\mathcal{N} = 4$ SYM at $\lambda = \infty$ and $N \mapsto \infty$. According to AdS/CFT this theory is dual to a planar Schwarzschild-AdS_5 black hole at Hawking temperature T. Study the linearized gravitational perturbation. This has a spectrum of gravity quasi-normal modes with dispersion relations:

- Sound channel: $\omega = v_s k - \mathrm{i}k^2\Gamma_s + \ldots$ with $v_s = 1/\sqrt{d-1}$ ($= 1/\sqrt{3}$ for $\mathcal{N} = 4$)

  Herein, $\Gamma_s = \eta/(\varepsilon + P)$. The mode propagates with speed of sound v_s and as it propagates, it loses energy to the medium.
- Shear channel: $\omega = -\mathrm{i}Dk^2 + \ldots$

3.1 AdS/CFT correspondence

String theory on asymptotic AdS spacetimes is dual to a non-gravitational quantum field theory (gauge theory).

$$\text{Strings on } \mathrm{AdS}_5 \times S^5 \overset{\text{dual}}{=} \mathcal{N} = 4 \text{ SYM SU}(N)\,, \tag{3.17}$$

$$\text{Strings on } \mathrm{AdS}_5 \times T^{1,1} \overset{\text{dual}}{=} \mathcal{N} = 1 \text{ SCFT SU}(N) \times \text{SU}(N)\,. \tag{3.18}$$

An infinite class of $\mathcal{N} = 1$ SCFTs obtained from D3-branes at the tip of a CY_3 cone with base of the cone being a 5-dimensional Sasaki-Einstein manifold $\mathrm{AdS}_5 \times X_5$. The nice thing about AdS/CFT is that there is a universal sector, which describes the dynamics of Einstein's equations with negative cosmological constant. At strong coupling λ in the planar limit ($N \mapsto \infty$), one can replace string theory by classical supergravity. The strong coupling controls the spectrum of string theory and N controls the string interactions. String states (oscillators) has energy $\sim \lambda^{1/4}$. Hence, in the limit $\lambda \mapsto \infty$ and $N \mapsto \infty$ classical supergravity on $\mathrm{AdS}_5 \times X_5$ can be done. Since X_5 is compact this can be reduced to supergravity with Kaluza-Klein modes on AdS_5, whose low-energy effective action is given by

$$S_{\text{sugra}} = \frac{1}{16\pi G_5}\int \sqrt{-g}(R - 2\Lambda + \ldots)\,. \tag{3.19}$$

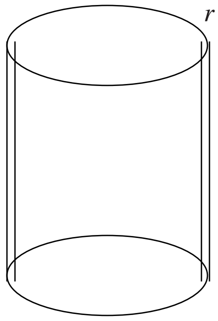
The decoupled sector of the theory is Einstein-Hilbert plus cosmological constant. The dual field theory statement is that the decoupled sector corresponds to the stress tensor dynamics. The distinction between different choices of X_5 is buried in G_5.

$$\frac{1}{16\pi G_5} = \frac{\mathrm{Vol}(X_5)}{16\pi G_{10}}\,, \quad G_{10} = (2\pi)^7 l_p^8 = (2\pi)^7 l_s^8 g_s^2\,, \tag{3.20}$$

$$\frac{1}{16\pi G_5} = c_{\mathrm{SCFT}}\,. \tag{3.21}$$

Hence, $c_{\mathcal{N}=4} \propto N^2$.

For calculational purposes it is useful not allow the AdS spacetime to have its boundary at infinity, but to cut it off by a finite radius.



For this cut-off surface we can calculate the extrinsic curvature for the surface at $r = \Lambda_c$:

$$K_{\mu\nu} = g_{\mu\varrho}\nabla^{\varrho} n_{\nu}\,, \tag{3.22}$$

where n^μ is a unit normal vector. The boundary stress tensor is given by

$$T^{\mu\nu} = \lim_{\Lambda_c \mapsto \infty} \frac{\Lambda_c^{d-2}}{16\pi G_N^{(d+1)}} \left[K^{\mu\nu} - K g^{\mu\nu} - (d-1) g^{\mu\nu} - \frac{1}{d-2}\left({}^{g}R^{\mu\nu} - \frac{1}{2}{}^{g}R g^{\mu\nu} \right) \right]. \tag{3.23}$$

The second term comes from the Hawking term

$$S_{\mathrm{body}} = \frac{1}{16\pi G_N^{(d+1)}} \int \mathrm{d}^d x \sqrt{-g}(2K + \ldots)\,. \tag{3.24}$$

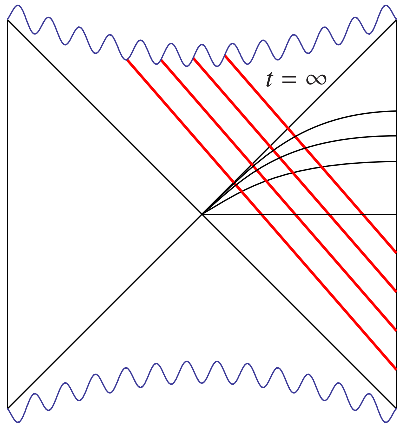
The last terms come from counter-terms from the boundary metric. For a planar AdS_{d+1} black hole one obtains the stress tensor for an ideal fluid:

$$T^{\mu\nu} = \frac{(\pi T)^d}{16\pi G_N^{(d+1)}} (\eta^{\mu\nu} + d u^\mu u^\nu)\,, \tag{3.25}$$

where $u^a = (\partial/\partial t)^a$ is the time-like Killing field. The planar black hole can be extended to a d-parameter family of solutions. One just has to boost the solution in the spatial directions of $R^{d-1,1}$, which from the gravity point of view are just coordinate transformations.

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{r^2\left(1 - \frac{r_+^d}{r^d}\right)} + r^2 \left[-\left(1 - \frac{r_+^d}{r^d}\right) u_\mu u_\nu + P_{\mu\nu} \right] \mathrm{d}x^\mu\, \mathrm{d}x^\nu\,, \quad P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu\,. \tag{3.26}$$

By promoting $r_+ = r_+(x)$, $u_\mu = u_\mu(x)$, and $P_{\mu\nu} = P_{\mu\nu}(x)$ and plugging into the Einstein equations. However, they are not solved, unless u and r_+ are numbers. However, we expect that this approximates a single domain of a locally equilibrium fluid. This is the case, if the derivatives $\partial_\mu r_+$, $\partial_\mu u^\nu$ are sufficiently small. The question remains how to put together all solutions of the form, namely how to patch together pieces of black holes to make a new solution. That is a problem, since gravity is intrinsically non-linear. We want to develop a perturbation technique which allows us to find an approximate solution in some perturbative expansion. The perturbation theory for black holes will work as long as $\varepsilon = \partial u^\mu / T \ll 1$ and $\partial \log(T)/T \ll 1$. ε will be the parameter of the perturbative expansion. A problem of perturbation theory is to choose a appropriate starting point. The planar black hole written in Schwarzschild coordinates $(t, r, \mathbf{x})$ is a bad choice for perturbation theory. The reason is that these coordinates are ill-behaved on the horizon. Therefore, we will switch to a different set of coordinates.



We will use ingoing Eddington-Finkelstein coordinates. The planar black hole without boosting is then given by:

$$\mathrm{d}s^2 = 2\,\mathrm{d}v\,\mathrm{d}r - r^2\left(1 - \frac{r_+^d}{r^d}\right)\mathrm{d}r^2 + r^2\,\mathrm{d}x^2\,. \tag{3.27}$$

The Killing field $(\partial/\partial t)^a$ is just vanishing on the future horizon. The boosted solution will look as follows:

$$\mathrm{d}s^2 = -2u_\mu\,\mathrm{d}x^\mu\,\mathrm{d}r - r^2 f\left(\frac{r}{r_+}\right)u_\mu u_\nu\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu + r^2 P_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\,, \quad f(r) = 1 - \frac{1}{r^d}\,. \tag{3.28}$$

Call $G^{(0)}$ the metric (3.28) with $r_+(x)$, $u^\mu(x)$, whereas $x = \{v, \mathbf{x}\}$. $G^{(0)}$ has an ε-expansion itself. Consider the full metric G_{AB} to be

$$G_{\mathrm{AB}} = \sum_{k=0}^{\infty} \varepsilon^k G_{\mathrm{AB}}^{(k)}\,. \tag{3.29}$$

In particular, $G_{\mathrm{AB}}^{(1)}$ is a correction term needed such that $G_{\mathrm{AB}}^{(0)} + G_{\mathrm{AB}}^{(1)}$ solves $R_{\mathrm{AB}} + dG_{\mathrm{AB}} = 0$ at order ϵ. We will be solving Einstein's equations in a derivative expansion in coordinates x, which parameterize the boundary, but **exactly** in r. Suppose we have solved for G_{AB} to $\mathcal{O}(\varepsilon^k)$. To get a solution at $\mathcal{O}(\varepsilon^{k+1})$, we just have to solve a set of equations

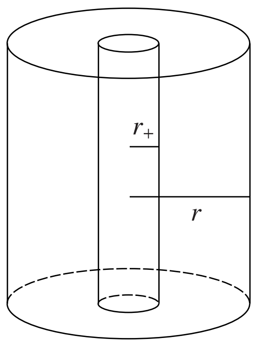
$$\mathscr{H}(G_{\mathrm{AB}}^{(k+1)}(r_+^{(0)}, u_\mu^{(0)})) = S_{k+1}\,. \tag{3.30}$$

$\mathscr{H}$ must be an ordinary differential operator involving r. Einstein's equations have become linear ordinary differential equations in one variable, but with complicated sources. If r_+ and u^μ are constants one has a large symmetry group consisting of v- and x-translations. These symmetries can be used to simplify the equations further.

AdS/CFT has a universal sub-sector, which is pure gravity in AdS.

$$S_{\mathrm{bulk}} = \frac{1}{16\pi G_N^{\mathrm{d}t'}}\int \mathrm{d}x^{\mathrm{d}t'}\,\sqrt{-G}(R - 2\Lambda)\,. \tag{3.31}$$

G_{AB} is the bulk metric whose boundary contains $g_{\mu\nu}$ (fluid dynamics background) in its conformal class. The equations of motion are $R_{\mathrm{AB}} + dG_{\mathrm{AB}} = 0$. A simple solution is AdS_{d+1}, the vacuum of dual CFT.



Another interesting **solution** is the Schwarzschuld-AdS black hole:

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\,\mathrm{d}\Omega_{d-1}^2\,, \tag{3.32}$$

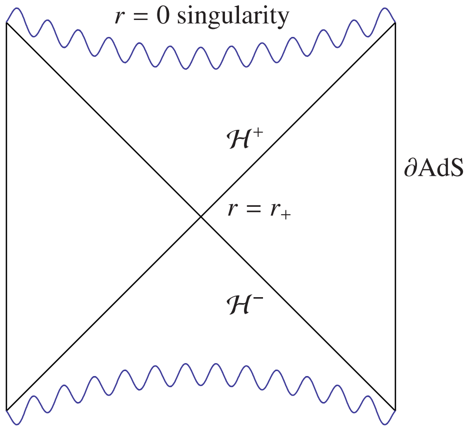
whereas $\mathrm{d}\Omega_{d-1}^2$ is the round metric on S^{d-1} and

$$f(r) = r^2 + 1 - \frac{r_+^{d-2}}{r^{d-2}}(1 + r_+^2)\,. \tag{3.33}$$

$r = r_+$ is the location of the black hole horizon. For $r_+ \geq 1$ this solution gives the thermal density matrix of field theory at $T = T_H$ with Hawking temperature T_H.

$$T_H = \frac{f'(r_+)}{4\pi}\,. \tag{3.34}$$

The global SAdS is dual to thermal CFT on $\mathbb{R} \times S^{d-1}$. Penrose diagram:



To do fluid dynamics, we require the curvature scales to slowly vary: $r_+/l_{d+1} \gg 1$. The temperature gradients shall be small compared to the local temperature. We could instead start with AdS_{d+1} with $R^{d-1,1}$ boundary and introduce boundary curvature as a part of the hydrodynamic gradient expansion. $r_+ \gg 1$ leads to the planar AdS_{d-1} black hole.

$$\mathrm{d}s^2 = -r^2\left(1 - \frac{r_+^d}{r^d}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2\left(1 - \frac{r_+^d}{r^d}\right)} + r^2\,\mathrm{d}\mathbf{x}_{d-1}^2\,, \tag{3.35}$$

where $r^2\,\mathrm{d}\mathbf{x}_{d-1}^2$ is the Euclidian line element square on $\mathbb{R}^{d-1}$. The horizon is non-compact and the spatial geometry is R^{d-1}. The planar AdS black hole is dual to thermal CFT on $R^{d-1,1}$ with $T = dr_+/(4\pi)$. Using the planar black hole as global equation solution we want to systematically derive hydrodynamics. The bulk gravity requires boundary terms in order for the classical variation problem to be well-posed: $S = S_{\text{bulk}} + S_{\text{body}}(g_{\mu\nu})$. $S_{\text{body}}(g_{\mu\nu})$ is a functional of the boundary metric.
There are $(d+1)(d+2)/2$ components of Einstein's equations.

i.) Constraint equations:

These are basically obtained by dotting the Einstein's equations with the unit normal vector n^A of the AdS boundary.

$$\mathcal{E}^{\text{AB}} = R^{\text{AB}} + dG^{\text{AB}}\,, \quad \mathcal{E}_M = \mathcal{E}_{MN}n^N\,. \tag{3.36}$$

The constraint equations will not play a role in determining $G_{\text{AB}}^{(k+1)}$. They constrain the allowed form of $r_+(x)$ and $u^\mu(x)$. These functions have to satisfy the hydrodynamical conservation

$$\mathcal{E}_\mu = \nabla_\nu (T_\mu{}^\nu)^{(k)} = 0\,. \tag{3.37}$$

$(T_\mu{}^\nu)^{(k)}$ is the stress tensor, evaluated at the k-th order of derivatives.

ii.) Dynamical equations:

These determine $G_{\text{AB}}^{(k+1)}$.

To solve the system explicitly, it is useful to use symmetries. For constant r_+ and u^μ we have an SO$(d-1)$ spatial rotation symmetry (can be seen in a local inertial frame). Decompose $G^{(k)}_{\text{AB}}$ into SO$(d-1)$ irreducible representations:

scalars	$G^{(k+1)}_{vv}$ $G^{(k+1)}_{vr}$ $G^{(k+1)}_{rr}$ $\sum_{k=1}^{d-1} G^{(k+1)}_{ii}$
vectors	$G^{(k+1)}_{iv}$ $G^{(k+1)}_{ir}$
tensors (symmetric and traceless)	$G^{(k+1)}_{ij}$

By a certain gauge choice we set $G^{(k)}_{rr} = G^{(k)}_{ir} = 0$. The dynamical equations become a decoupled set of linear ordinary differential equations. In the vector sector the dynamical equation is just

$$\mathscr{H}_{d-1}\mathcal{O} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}\mathcal{O}\right) . \tag{3.38}$$

So we have to solve

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r^{d-1}}\frac{\mathrm{d}}{\mathrm{d}r}G^{(k+1)}_{iv}\right) = S_{k+1}(r,x) . \tag{3.39}$$

In the tensor sector one has similarly

$$\mathscr{H}_{\frac{d(d+1)}{2}}\mathcal{O} = \frac{\mathrm{d}}{\mathrm{d}r}\left[r^{d+1}f(r)\frac{\mathrm{d}\mathcal{O}}{\mathrm{d}r}\right] , \quad f(r) = 1 - \frac{1}{r^d} . \tag{3.40}$$

The boundary conditions are:

- solution should be normalizable (fall off fast enough as $r \mapsto \infty$),
- regular at $r = r_+$.

$$G_{\text{AB}}\,\mathrm{d}x^A\,\mathrm{d}x^B = -2S(r,x)u_\mu(x)\,\mathrm{d}x^\mu\,\mathrm{d}r + \chi_{\mu\nu}(r,x)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu , \tag{3.41}$$

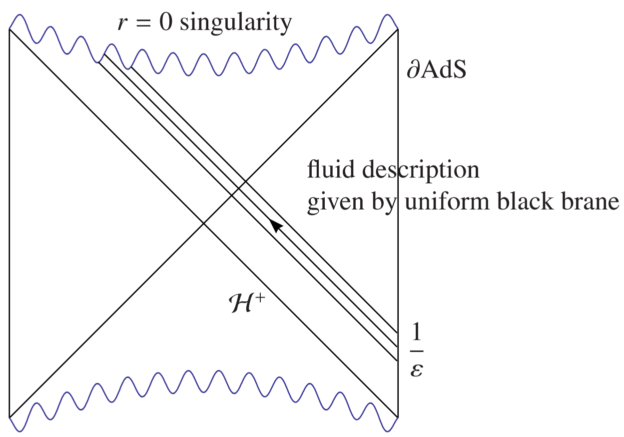
with

$$S(r,x) = 1 + \varepsilon(0) + \varepsilon^2(\bullet) + \ldots , \quad \chi_{\mu\nu}(r,x) = P_{\mu\nu} - f\left(\frac{r}{r_+}\right)u_\mu u_\nu + \#\sigma_{\mu\nu} + \ldots . \tag{3.42}$$

From G_{AB} we can extract $T_{\mu\nu}$ and find

$$T^{\mu\nu} = \alpha T^d(du^\mu u^\nu + \eta^{\mu\nu}) - 2\eta\sigma^{\mu\nu} , \quad \alpha = \frac{\pi^d}{16\pi G_N^{(d+1)}} , \quad \eta = \frac{(\pi T)^{d-1}}{16 G_N^{(d+1)}} . \tag{3.43}$$

G_{AB} delivers gravity solutions dual to fluid flow.
The question arises: What is the geometry dual to fluids? The geometries are dynamical inhomogeneous black hole spacetimes with a **regular** event horizon.



x^μ = const. are radially ingoing null geodesics. The event horizon can be determined explicitly (in fact algebraically) from G_{AB}. The event horizon is the boundary of the past/of the future null infinity. Assuming the fluid flow settles down at late times in the hydrodynamic limit (gradient expansion) we can perturbatively determine the horizon location. Let us assume that the event horizon is a dimension 1 null hypersurface

$$r - r_H = S_H = 0 . \tag{3.44}$$

$$r_H(x) = r_+(x) + \sum_{k=0}^{\infty} \varepsilon^k r_{(k)}(x) . \tag{3.45}$$

Find $r_{(k)}(x)$ using the fact that the surface is null:

$$G^{(\mathrm{AB})}(\partial_A S_H)(\partial_B S_H) = 0 . \tag{3.46}$$

As an example consider the following geometry in four dimensions:

$$\mathrm{d}s^2 = -\left(1 - \frac{2m(r)}{r}\right) \mathrm{d}v^2 + 2\,\mathrm{d}v\,\mathrm{d}r + r^2\,\mathrm{d}\Omega^2 . \tag{3.47}$$

The event horizon $r = r_H(v)$ is a null-surface. Hence, it satisfies

$$r(v) = 2m(v) + 2r(v)\frac{\mathrm{d}r(v)}{\mathrm{d}v} . \tag{3.48}$$

To make contact with hydrodynamics we assume that m is slowly varying:

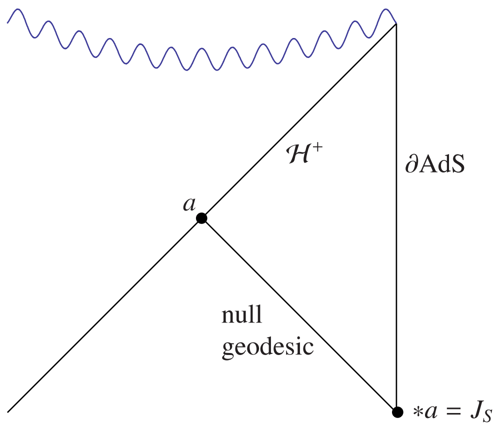
$$\dot{m}(v) \sim \mathcal{O}(\varepsilon) , \quad m\ddot{m} \sim \mathcal{O}(\varepsilon^2) , \quad \text{etc.} \tag{3.49}$$

$$\lim_{v \mapsto \infty} m(v) = m_0 . \tag{3.50}$$

Let

$$r(v) = 2m(v) + \sum_k \varepsilon^k r_{(k)}(v) , \quad r_{(1)} = 8m\dot{m} , \quad r_{(2)} = 64m\ddot{m} + 32m\dot{m}^2 . \tag{3.51}$$

The event horizon delivers and entropy current for the field theory.



a is a $d-1$ form and is pulled back along radially ingoing null-geodesics. ∂(Area of event horizon)≥ 0. The area theorem guarantees that $\nabla_\mu J^\mu_S \geq 0$. It is possible to write down this current explicitly for fluid dynamics variables:

$$J^\mu_S = Su^\mu + \frac{S}{r_+^2} u^\mu \left(A_1 \sigma_{\alpha\beta}\sigma^{\alpha\beta} + A_2 \omega_{\alpha\beta}\omega^{\alpha\beta} + A_3 R\right) + \frac{S}{r_+^2}\left(B_1 \mathscr{D}_\lambda \sigma^{\mu\lambda} + B_2 \mathscr{D}_\lambda \omega^{\mu\lambda}\right) . \tag{3.52}$$

$$B_1 + 2A_3 = 0 , \quad A_1 = \frac{2}{d^2}(d+2) - \widetilde{A}_1 , \quad A_2 = -\frac{1}{2d} , \quad B_1 = \frac{2}{d(d-2)} , \quad B_2 = \frac{1}{d-2} . \tag{3.53}$$

$$\nabla_\mu J^\mu_S \geq 0 , \quad \nabla_\mu J^\mu_S = \frac{2\eta}{T}\left(\sigma_{\mu\nu} + \frac{1}{2}\left[\frac{d}{4\pi}(1 + A_i d) - \tau_\pi\right] u^\alpha \mathscr{D}_\alpha \sigma_{\mu\nu}\right)^2 + \dots . \tag{3.54}$$