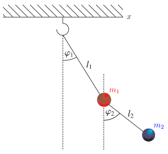
1 Introduction

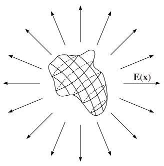
Classical mechanics



- degrees of freedom: number of independent coordinates that are needed for the complete description of a physical system
- generalized coordinates q_i (for $i = 1, \ldots, N$), where N is the number of degrees of freedom
- Lagrange function $L = T - V$ plus Euler-Lagrange equation to describe the dynamics (time evolution) of the system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0\,. \tag{1}$$

Classical field theory



- describe physical systems with infinite number of degrees of freedom
- index $i \mapsto$ spacetime variable $x = (t, \mathbf{x})$, $q_i \mapsto \phi(x)$, $\dot{q}_i \mapsto \partial_0\phi(x)$, $p_i \mapsto \pi(x) \equiv \partial\mathcal{L}(x)/\partial(\partial_0\phi(x))$
- Lagrange density $\mathcal{L}$ and Euler-Lagrange equations to describe the dynamics of the fields:

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0\,, \quad L = \int \mathrm{d}^3x\, \mathcal{L}(x)\,. \tag{2}$$

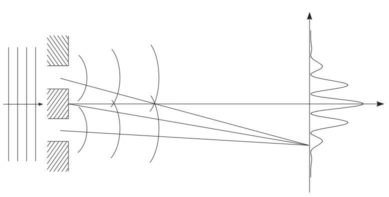
- Hamilton function H and Hamilton density $\mathcal{H}$:

$$H = \int \mathrm{d}^3x\, \mathcal{H}\,, \quad \mathcal{H} = \pi\partial_0\phi - \mathcal{L}\,. \tag{3}$$

- example: classical electrodynamics; Maxwell equations follow from the Lagrange density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu\,, \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,, \quad A^\mu = \begin{pmatrix}\phi\\ \mathbf{A}\end{pmatrix}. \tag{4}$$

Quantum field theory



- wave particle dualism: free propagation of waves and point-like interaction (double slit experiment)
- find quanta of fields: quantization of electromagnetic field leads to the photon (interpretation of the photo effect by A. Einstein)
- creation and annihilation (absorption and emission) of particles
- replace classical fields by field operators: $\phi(x) \mapsto \widehat{\phi}(x)$
- impose commutation relation for field operators:

$$[\widehat{\pi}(\mathbf{x}, t), \widehat{\phi}(\mathbf{x}', t)] = -\mathrm{i}\delta^{(3)}(\mathbf{x} - \mathbf{x}')\,, \quad [\widehat{\pi}(\mathbf{x}, t), \widehat{\pi}(\mathbf{x}', t)] = [\widehat{\phi}(\mathbf{x}, t), \widehat{\phi}(\mathbf{x}', t)] = 0\,. \tag{5}$$

2 Quantization procedure I: canonical quantization

The whole process is best explained via an example. Therefore, consider a free real scalar (spin 0) field ϕ with Lagrange density

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)\,. \tag{6}$$

From the Euler-Lagrange equation (2) follows the field equation (Klein-Gordon equation)

$$(\partial_\mu\partial^\mu + m^2)\phi(x) = 0\,. \tag{7}$$

This is a wave equation and for its solutions one can make a plane wave *Ansatz*

$$\phi_{\mathbf{k}}(x) \sim \exp(\mathrm{i}k^\mu x_\mu) = \exp(\mathrm{i}[k^0 t - \mathbf{k}\cdot\mathbf{x}])\,. \tag{8}$$

Plugging this into Eq. (7) leads to the dispersion relation $k^0 = \omega(\mathbf{k}) = \sqrt{k^2 + m^2}$. Since Eq. (7) is a linear (partial) differential equation, the **superposition principle** holds. This leads to the following solution, which is given by a Fourier decomposition:

$$\phi(x) = \int \frac{\mathrm{d}^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2\omega_{\mathbf{k}}}}\left[a(\mathbf{k})\exp(\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega_{\mathbf{k}}t)) + a^\dagger(\mathbf{k})\exp(-\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_{\mathbf{k}}t))\right]. \tag{9}$$

Eq. (9) one only describes a **free** scalar field. However, every physical system shows **interactions**!

- There can be interactions **between different** fields.
- One can also introduce interactions of the same field with itself, a so-called **self-interaction**. Since we would like to keep our example as simple as possible, we will stay with this second possibility.

The simplest self-interaction is a ϕ^3-term. This is, however, unstable, since it is not bounded from below! Better to use a quartic interaction (the Higgs field has such a self-interaction):

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}'\,, \quad \mathcal{L}' = \frac{\lambda}{4!}\phi^4\,. \tag{10}$$

Problem: Such an interacting Lagrangian $\mathcal{L}$ leads to a complicated nonlinear partial differential equation, whose exact solution is not easy. (Nevertheless, exact solutions are important and examples will come in the last few chapters of the book.) Therefore, one needs some method to deal with the problem. Another description of the system is necessary; in a whole, there exist three of them:

	operators	states	dynamics
Heisenberg picture	time-dependent	time-independent	full Hamiltonian H
Schrödinger picture	time-independent	time-dependent	full Hamiltonian H
interaction picture	time-dependent	time-dependent	free Hamiltonian H_0 (for operators)

The **interaction picture** is well-suited for our purposes, since the dynamics of operators are governed by the free Hamiltonian H_0. However, we must first transform the field operators from the Heisenberg picture to the interaction picture. This is done by the U-matrix

$$U(t, 0) = \exp(\mathrm{i}H_0 t)\exp(-\mathrm{i}Ht)\,. \tag{11}$$

The U-matrix also has an internal time-dependence, which is denoted by the second index: $U = U(t, t_0)$. The time evolution of $U(t, t_0)$ is dictated by the interaction Hamiltonian in the interaction picture H'^I:

$$\mathrm{i}\frac{\partial}{\partial t}U(t, t_0) = H'^I(t)U(t, t_0)\,. \tag{12}$$

This equation can be iteratively solved and leads to power expansion of $U(t, t_0)$ in terms of time ordered products of H'^I at different spacetime points: that corresponds to an **expansion with respect to the perturbation parameter λ**.

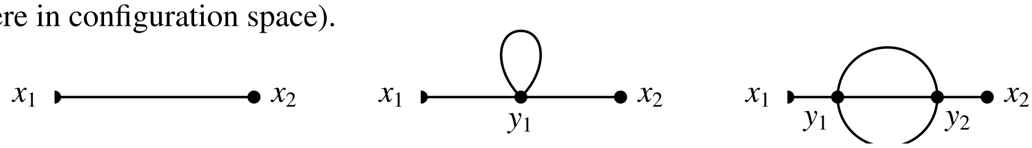
2.1 Calculation of physical quantities: Green's functions

An important example for a Green's function is the **two-point correlation function (propagator)** $\mathrm{i}\Delta(x,y) \equiv \langle 0|T(\phi(x)\phi(y))|0\rangle$ which gives the amplitude of a particle propagating from the point x to the point y. One can show that $\Delta(x,y)$ is the inverse of the differential operator which appears in the Lagrangian:

$$\int \mathrm{d}^4y\, K(x,y)\Delta(y,z) = \delta^{(4)}(x-z)\,, \quad K(x,y) = \delta^{(4)}(x-y)\left(\frac{\partial^2}{\partial t^2} - \boldsymbol{\nabla}^2 + m^2\right). \tag{13}$$

To calculate $\langle 0|T(\phi(x)\phi(y))|0\rangle$ one has to replace the fields $\phi(x)$ by the fields $\phi^I(x)$ in the interaction picture. This leads to a power series involving the interaction Hamiltonian H'^I and is a perturbation expansion with respect to the coupling constant λ.

- The j-th term in the expansion describes the interaction of two [in general n] external fields at spacetime points x_1, x_2 [x_i for $i = 1, \ldots, n$] at j internal points y_1, y_2 etc. (follows from Wick's theorem).
- Every term in this expansion can be graphically illustrated by so-called Feynman diagrams (here in configuration space).

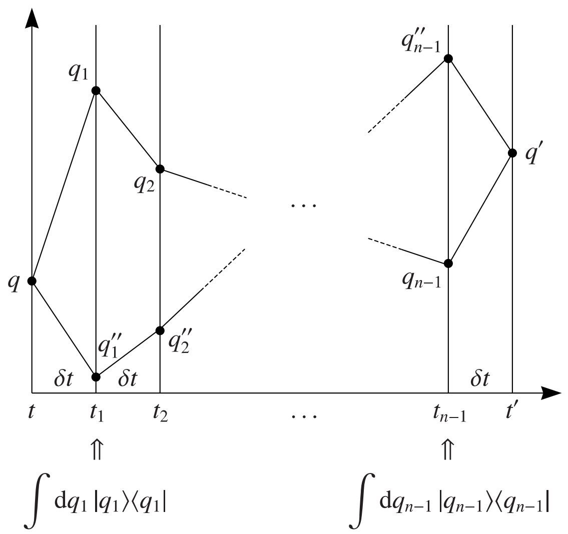


3 Quantization procedure II: path integral quantization

The starting point is the transition matrix element between the initial and final position state of a quantum mechanical system (first without field theory):

$$\langle q', t'|q, t\rangle = \langle q'|\exp(-\mathrm{i}H(t'-t))|q\rangle\,. \tag{14}$$

How can we calculate this expression?



- divide time interval into n segments with distance $\delta t = (t'-t)/n$
- include a complete set of position states $|q_i\rangle$ (i, ..., $n-1$) after each segment
- leads to matrix elements $\langle q_{n'}|\exp(-\mathrm{i}H\delta t)|q_n\rangle$, linear expansion of exponential function possible for small δt
- one expects the Hamiltonian to be of the following form:

$$H(P,Q) = \frac{P^2}{2m} + V(Q)\,. \tag{15}$$

- calculation of $\langle q'|H(P,Q)|q\rangle$ leads to additional momentum integrations

- Transformation to continuum and performance of integration over p leads to the following form of the path integral (with some prefactor N):

$$\langle q', t'|q, t\rangle = N \int \mathscr{D}q \exp(\mathrm{i}S)\,, \quad S = \int_t^{t'} \mathrm{d}\tau\, L(q(\tau), \dot{q}(\tau))\,. \tag{16}$$

- The integration is performed over function space $q(t)$, that means over all paths connecting (q, t) and (q', t') (denoted by $\mathscr{D}q$) weighted by the exponential of i times the action S.

Again we would like to calculate Green's functions

$$G(t_1, t_2) = \langle 0|T(Q^H(t_1)Q^H(t_2))|0\rangle\,, \tag{17}$$

but now in the new formalism of path integrals. Inserting complete sets of states $|q, t\rangle$, transforming the operators to the Schrödinger picture and taking the limits $t' \mapsto -\mathrm{i}\infty$, $t \mapsto \mathrm{i}\infty$ leads to the (general n-point) Green's function. The entire set of Green's functions follows from the generating functional $W[J]$:

$$G(t_1, \ldots, t_n) = \left.\frac{(-\mathrm{i})^n \delta^n W[J]}{\delta J(t_1)\ldots\delta J(t_n)}\right|_{J=0}\,, \tag{18}$$

with

$$W[J] = \lim_{\substack{t'\mapsto -\mathrm{i}\infty \\ t\mapsto \mathrm{i}\infty}} \frac{1}{\langle q', t'|q, t\rangle} \int \mathscr{D}\left(\frac{pq}{2\pi}\right) \exp\left\{\mathrm{i}\int_t^{t'} \mathrm{d}\tau\, [p\dot{q} - H(p, q) + J(\tau)q(\tau)]\right\}. \tag{19}$$

$W[J]$ corresponds to transition amplitude from a ground state at t to a ground state at t' in the presence of an external source $J(\tau)$: $W[J] = \langle 0|0\rangle_J$. The pre-factor $\langle q', t'|q, t\rangle$ is irrelevant for connected Green's functions. In field theory the approach is similar and the generating functional for the interacting Lagrangian $W[J]$ results from free functional $W_0[J]$ as follows:

$$W[J] = \left[\exp\left\{\int \mathrm{d}^4x\, \mathcal{L}'\left(\frac{\delta}{\delta J}\right)\right\}\right] W_0[J]\,, \tag{20}$$

with

$$W_0[J] = \int \mathscr{D}\phi\, \exp\left\{\int \mathrm{d}^4x\, (\mathcal{L}_0 + J\phi)\right\} = \exp\left(\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\Delta(x, y)J(y)\right). \tag{21}$$

- The full generating functional $W[J]$ follows from the action of the exponential of the interaction Lagrangian $\mathcal{L}'$ on the free part $W_0[J]$, whereas all fields in $\mathcal{L}'$ are replaced by functional derivatives with respect to the external current J.
- The expansion of the exponential function yields a perturbative expansion of $W[J]$ with respect to the coupling, which can again be graphically illustrated.

For the quantization of fermions in the path integral formalism, the currents η and also the classical fields ψ have to be anticommuting numbers (Grassmann numbers, with $\{a, b\} = ab + ba$):

$$\{\psi(x), \psi(x')\} = \{\psi(x), \overline{\psi}(x')\} = \{\overline{\psi}(x), \overline{\psi}(x')\} = 0\,, \tag{22a}$$

$$\{\eta(x), \eta(x')\} = \{\eta(x), \overline{\eta}(x')\} = \{\overline{\eta}(x), \overline{\eta}(x')\} = 0\,. \tag{22b}$$

3.1 Rules of Grassmann algebra

- Grassmann numbers anticommute: $\{\theta_i, \theta_j\} = 0$ for $i, j = 1, 2, \ldots, n$; $\{\theta, \theta\} = 0$ hence $\theta^2 = 0$.
- Every expansion of a Grassmann-valued function stops after the linear term: $p(\theta) = P_0 + \theta P_1$.
- Integration corresponds to differentiation.
- Jacobian for integral transformations is the inverse compared to that for commuting numbers.
- Gauß integral over a set of Grassmann variables (with a matrix A) yields:

$$G(A) \equiv \int \mathrm{d}\theta_n \ldots \mathrm{d}\theta_1\, \exp\left(\frac{1}{2}\sum_{i,j=1}^{n} A_{ij}\theta_i\theta_j\right) = \sqrt{\det(A)}\,. \tag{23}$$

What we want to use here, is the rule for a functional derivative:

$$\frac{\delta}{\delta J(x)} \int \mathrm{d}^4y\, G(y)J(y) = \int \mathrm{d}^4y\, G(y)\delta(x-y) = G(x)\,. \tag{24}$$

The product rule (plus Leibniz rule) of differentiating ordinary functions also holds for functional derivatives:

$$\begin{aligned}
&\frac{\delta}{\delta J(x)} \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, J(x_1)G(x_1)J(x_2)H(x_2) = \\
&\quad = \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, G(x_1)\delta(x_1-x)J(x_2)H(x_2) + \int \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, J(x_1)G(x_1)H(x_2)\delta(x_2-x) = \\
&\quad = G(x)\int \mathrm{d}^4x_2\, J(x_2)H(x_2) + H(x)\int \mathrm{d}^4x_1\, J(x_1)G(x_1)\,.
\end{aligned} \tag{25}$$

With these rules (and the chain rule) we obtain:

$$\begin{aligned}
&\int \mathrm{d}^4x \left[\frac{\delta}{\delta J(x_1)}\right]^4 W_0[J] = \int \mathrm{d}^4x \left[\frac{\delta}{\delta J(x_1)}\right]^4 \exp\left(\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\Delta(x,y)J(y)\right) = \\
&\quad = \int \mathrm{d}^4x_1 \left[\frac{\delta}{\delta J(x_1)}\right]^3 \int \mathrm{d}^4y_1\, J(y_1)\Delta(y_1,x_1) \exp\left(\frac{1}{2}\int \mathrm{d}^4x\, \mathrm{d}^4y\, J(x)\Delta(x,y)J(y)\right) = \\
&\quad = \int \mathrm{d}^4x_1 \left[\frac{\delta}{\delta J(x_1)}\right]^2 \int \mathrm{d}^4y_1\, \Delta(x_1,x_1)W_0[J] \\
&\qquad + \int \mathrm{d}^4x_1 \left[\frac{\delta}{\delta J(x_1)}\right]^2 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)J(y_2)\Delta(y_1,x_1)\Delta(y_2,x_1)W_0[J] = \\
&\quad = \int \mathrm{d}^4x_1 \frac{\delta}{\delta J(x_1)} \int \mathrm{d}y_2\, \Delta(x_1,x_1)J(y_2)\Delta(y_2,x_1)W_0[J] \\
&\qquad + 2\int \mathrm{d}^4x_1 \frac{\delta}{\delta J(x_1)} \int \mathrm{d}^4y_1\, J(y_1)\Delta(y_1,x_1)\Delta(x_1,x_1)W_0[J] + \\
&\qquad + \int \mathrm{d}^4x_1 \frac{\delta}{\delta J(x_1)} \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\, J(y_1)J(y_2)J(y_3)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(y_3,x_1)W_0[J] = \\
&\quad = \int \mathrm{d}^4x_1\, \Delta(x_1,x_1)^2 W_0[J] + \int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)J(y_2)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(x_1,x_1)W_0[J] + \\
&\qquad + 2\int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)J(y_2)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(x_1,x_1)W_0[J] + \\
&\qquad + 3\int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)J(y_2)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(x_1,x_1)W_0[J] + \\
&\qquad + \int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\, \mathrm{d}^4y_4\, J(y_1)J(y_2)J(y_3)J(y_4)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(y_3,x_1)\Delta(y_4,x_1)W_0[J] = \\
&\quad = \int \mathrm{d}^4x_1\, \Delta(x_1,x_1)^2 W_0[J] + 6\int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, J(y_1)J(y_2)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(x_1,x_1)W_0[J] \\
&\qquad + \int \mathrm{d}^4x_1 \int \mathrm{d}^4y_1\, \mathrm{d}^4y_2\, \mathrm{d}^4y_3\, \mathrm{d}^4y_4\, J(y_1)J(y_2)J(y_3)J(y_4)\Delta(y_1,x_1)\Delta(y_2,x_1)\Delta(y_3,x_1)\Delta(y_4,x_1)W_0[J]
\end{aligned} \tag{26}$$