Mitschrieb zur Vorlesung: Classical Solutions in Gauge Theories

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Mitschrieb der Vorlesung Classical Solutions in Gauge Theories
von Herrn Prof. Dr. Klinkhamer im Sommersemester 2008
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Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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Chapter 1

Introduction

1.1 Standard Model

In our visible universe it holds that $N_\mathrm{p} = N_\mathrm{e} \approx 10^{80}$. The critical density of the universe if defined by

$$\varrho_\mathrm{crit} \equiv \frac{3H_0^2}{8\pi G} \approx 10^{-29}\,\frac{\mathrm{g}}{\mathrm{cm}^3}\left(\frac{H_0}{75\,\frac{\mathrm{km}}{\mathrm{s\cdot Mpc}}}\right)\,. \tag{1.1}$$

Hence, the number of protons is given by

$$N_\mathrm{p} \approx 0,1\varrho_\mathrm{crit} \times \frac{(3000\,\mathrm{Mpc})^3}{m_\mathrm{p}} = 10^{-30} \times (10^{28})^3 \times 10^{24} = 10^{78}\,. \tag{1.2}$$

It is remarkable that all these protons and electrons, respectively, are the same. Perhaps, there exist **non**linear equations for the e, μ, τ, u, d, ... Let us now look at the properties of some particles of the standard model [1, 2]. ($\hbar = c = 1$ will be set.)

leptons	e (0,5)	μ (106)	τ (1784)
	ν_e (≈ 0)	ν_μ (≈ 0)	ν_τ (≈ 0)
quarks	u (≈ 6)	c (≈ 154)	t (≈ 174 000)
(×3 for color)	d' (≈ 10)	s' (≈ 200)	b' (≈ 5000)

The fields describe the interactions by gauge theories. We present a table about how force particles communicate with matter particles:

						force particles (integer spin, 1 or 2)					
						γ (0)	W/Z (80 GeV)	gluon (200 MeV, effective)	graviton (0)		
m	p	spin	τ	μ	e	-1	g_w	0	0,5	106	1784
a	a		ν_τ	ν_μ	ν_e	0			≈ 0	≈ 0	≈ 0
tt	r	1/2	b	s	d	-1/3	g_w	g_s	10	200	5000
er	t		t	c	u	2/3			6	154	174000

One can give gluons an effective mass à la Yukawa, because their interaction length is limited. Gauge symmetry fixes many of the coupling constants. In order to describe the Lorentz invariant, unitary and renormalizable gauge theory of the Standard model one needs the Lie algebra $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. In 1956, parity violation of the weak interaction was discovered, which is incorporated in the Standard Model by putting the left-handed particles in $\mathrm{SU}(2)_L$-doublets. However, where parity violation originated, is still a mystery. U(1) is not the gauge group of the photon, $\mathrm{U}(1)_\mathrm{em}$, but depends on the hypercharge Y. Furthermore, the Higgs mechanism is needed to describe, how particles acquire an inertial mass, so how to get a mass scale into the theory. This all-together is the Yang-Mills-Higgs theory.

Chapter 2

On fields and classical solutions

In theoretical particle physics one needs to compute probability amplitudes $\langle \text{particles out}|\text{particles in}\rangle$. The tool to calculate these amplitudes is quantum field theory. We will have a brief review of quantum field theory. First, the canonical formulism and then the so-called path integral formalism will be considered.

2.1 Canonical formalism

We start with free fields. A free scalar (spinless) particle with definite three-momentum $\mathbf{p}$ and energy $p_0 \equiv \sqrt{|\mathbf{p}|^2 + m^2}$ is given by a plane wave. (In practice one works with wave packages, but the plane wave is the very crucial object.) Temporarily use finite three-volume V with periodic boundary conditions, from which discrete momenta result. We want to use the Pauli matric $g^{\mu\nu} = \delta^{\mu\nu}$. Therefore, the covariant vector is given by $x_\mu = (\mathbf{x}, \mathrm{i}ct)$ with $\hbar = c = 1$. We consider a **Hilbert space** with vectors corresponding to physical states and take a orthonormal basis with the vacuum state $|0\rangle$, which corresponds to the state with no particles. A one particle state with one plane wave with three momentum $\mathbf{p}_1$ going through it, is denoted by $|\mathbf{p}_1\rangle$. A two particle state with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$ is denoted by $|\mathbf{p}_1, \mathbf{p}_2\rangle$, etc. All these states build up the **Fock space**. A complete set of operators (such that any operator can be written in terms of these) are the annihilation and creation operators $\widehat{a}(\mathbf{p})$ and $\widehat{a}^\dagger(\mathbf{p})$. It holds that

$$\widehat{a}(\mathbf{p})|\mathbf{p}\rangle = |0\rangle\,, \quad \widehat{a}(\mathbf{p})|0\rangle = 0\,, \quad \widehat{a}(\mathbf{p})|\mathbf{q}\rangle = 0\,, \quad \widehat{a}(\mathbf{p})|\mathbf{p},\mathbf{q}\rangle = \sqrt{2}|\mathbf{p}\rangle\,. \tag{2.1}$$

$$\widehat{a}^\dagger(\mathbf{p})|0\rangle = |\mathbf{p}\rangle\,, \quad \widehat{a}^\dagger(\mathbf{p})|\mathbf{p}\rangle = \sqrt{2}|\mathbf{p},\mathbf{p}\rangle\,. \tag{2.2}$$

These operators obey commutation relations:

$$[\widehat{a}(\mathbf{p}), \widehat{a}^\dagger(\mathbf{q})] = \delta_{\mathbf{p},\mathbf{q}}\,. \tag{2.3}$$

With these operators we can construct a Hamiltonian:

$$\widehat{H} = \sum_{\mathbf{p}} p_0 \widehat{a}^\dagger(\mathbf{p})\widehat{a}(\mathbf{p})\,. \tag{2.4}$$

The state $|\alpha\rangle$ has energy E_α:

$$\widehat{H}|\alpha\rangle = E_\alpha|\alpha\rangle\,, \tag{2.5}$$

with

$$E_\alpha = \sum_{\mathbf{p}} p_0\,. \tag{2.6}$$

Another hermitian combination is inspired by the Fourier transform

$$\widehat{\phi}_{\text{in}}(x) = \sum_{\mathbf{p}} \frac{1}{\sqrt{2p_0V}} \left(\widehat{a}_{\text{in}}(\mathbf{p}) + \widehat{a}^\dagger_{\text{in}}(\mathbf{p})\exp(-\mathrm{i}px)\right)\,. \tag{2.7}$$

with label “in” for free fields. $\widehat{\phi}_{\text{in}}(x)$ has the following properties:

i.) locality: $[\widehat{\phi}_{\text{in}}(x), \widehat{\phi}_{\text{in}}(y)] = 0$ outside the light cone ($(x-y)^2 > 0$)

ii.) They obey the Klein-Gordon equation: $(\Box - m^2)\widehat{\phi}_{\text{in}}(x) = 0$.

iii.) $\widehat{\phi}_{\text{in}}|0\rangle = \sum_{\mathbf{p}} \frac{1}{\sqrt{2p_0 V}} \exp(-\mathrm{i}px)|\mathbf{p}\rangle$ corresponds to a particle located at x.

Recall, that it holds for Fourier transformation

$$\delta(\mathbf{x}) \sim \int \mathrm{d}^3p \, \exp(\mathrm{i}px) \,. \tag{2.8}$$

Consider now a system in state $|c\rangle = \alpha|a\rangle + \beta|b\rangle$, where $|a\rangle$ and $|b\rangle$ are orthogonal. The probability to find $|a\rangle$ is $P_a = |\alpha|^2$. More generally, if the system is in state $|c\rangle$, then the probability to measure state $|a\rangle$ is given by $|\langle a|c\rangle|^2$ (the square of the overlap).
At that point we are coming to **interacting fields**. Think of an elastic scattering experiment. The physical state $|\alpha\rangle$ has two widely separated particles at $t = \pm\infty$. The complete situation is **characterized** by the particle configuration either at $t = -\infty$ **or** $t = +\infty$. But $|\alpha\rangle$ still refers to all times, because the evolution in quantum mechanics is deterministic. **Assume** that for this characterization free fields suffice (modulo bound states). Then the state $|\alpha\rangle$ can be written as $|\alpha\rangle = |\mathbf{p}, \mathbf{q}\rangle_{\text{in}}$ or $|\alpha\rangle = |\mathbf{p}', \mathbf{q}'\rangle_{\text{out}}$. There are two conceptionally different sets of orthonormal basis vectors $|\text{in}\rangle$ and $|\text{out}\rangle$. Since they are in the same Hilbert space, they must be related by a unitary operator $\widehat{S}$ in the way $|\text{in}\rangle = \widehat{S}|\text{out}\rangle$. All information about the physics of scattering is stored in the operator $\widehat{S}$, the so-called S-matrix. Take $|a\rangle_{\text{in}}$ and $|b\rangle_{\text{out}}$ to be different physical states. The overlap gives the probability to have particles $\{a\}$ at $t = -\infty$ and $\{b\}$ at $t = +\infty$:

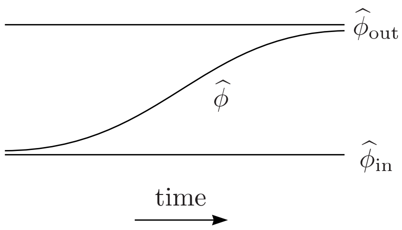
$$P = |_{\text{out}}\langle b|a\rangle_{\text{in}}|^2 = |_{\text{in}}\langle b|S|a\rangle_{\text{in}}|^2 \,. \tag{2.9}$$

Introduce an **interpolating** field $\widehat{\phi}(x)$, so that

$$\text{“}\lim_{x_0 \mapsto -\infty}\text{”}\widehat{\phi}(x) = \widehat{\phi}_{\text{in}} \,, \tag{2.10}$$

and

$$\text{“}\lim_{x_0 \mapsto +\infty}\text{”}\widehat{\phi}(x) = \widehat{\phi}_{\text{out}} \,. \tag{2.11}$$



We **postulate** that $\widehat{\phi}$ satisfies the field equation:

$$(\Box - m^2)\widehat{\phi}(x) = -j[\widehat{\phi}(x)] = -\frac{\delta}{\delta\phi}\widehat{\mathcal{H}}_I[\phi] \,. \tag{2.12}$$

For example, from

$$\mathcal{H}_I = \frac{g^2}{4!}\phi^4 \,, \tag{2.13}$$

it results that

$$j = \frac{g^2}{3!}\phi^3 \,. \tag{2.14}$$

In order to solve this nontrivial field equation, define a matrix $\widehat{U}(x)$ (evolution operator) such that $\widehat{\phi}(x) = \widehat{U}^\dagger(x)\widehat{\phi}_{\text{in}}(x)\widehat{U}(x)$ holds.

$$\lim_{x_0 \mapsto -\infty} \widehat{U}(x) = \mathbb{1} \,, \quad \lim_{x_0 \mapsto +\infty} \widehat{U} = \widehat{S} \,. \tag{2.15}$$

$\widehat{U}$ itself satisfies a differential equation, which is linear in $\widehat{U}(x)$:

$$\partial_{x_0}\widehat{U}(x_i) = \mathrm{i} \int\limits_{y_0 = x_0} \mathrm{d}^3y \, \mathcal{H}_I[\widehat{\phi}_{\text{in}}(y)]\widehat{U}(x_0) \equiv \mathrm{i}\widehat{H}_I U\widehat{U}(x_0) \,. \tag{2.16}$$

This equation can be solved **iteratively** and one can take the limit $x_0 \mapsto \infty$.

$$\widehat{S} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{\mathrm{i}^n}{n!} \int \mathrm{d}^4 y_1 \ldots \mathrm{d}^4 y_n \, T\left\{ \widehat{H}_{\mathrm{int}} \widehat{\phi}_{\mathrm{in}}(y_1) \ldots \mathcal{H}(\widehat{\phi}_{\mathrm{in}}(y_n)) \right\} , \tag{2.17}$$

where T is the time ordering operator. With $S[\widehat{\phi}_{\mathrm{in}}]$ one readily calculates all matrix elements ${}_{\mathrm{in}}\langle b|S|a\rangle_{\mathrm{in}}$. This gives a **perturbation series** ($\mathcal{H}_I \propto g^2$). The perturbation series is calculated with the help of Feynman diagrams. That is technically complicated, but, in principle, straightforward.
What we want to do instead, is to return to the field equations and look for other solutions, namely nonperturbative ones. Are there other fields that contribute to the amplitude and that do not appear as a perturbative solution.

2.2 Euclidian path integral

In general, the Green's functions need to be calculated. It is defined as follows:

$$G(x_1, \ldots, x_n) = \langle T(\widehat{\phi}(x_1) \ldots \widehat{\phi}(x_n)) \rangle . \tag{2.18}$$

One can calculate it with canonical operators. Another way to formulate it is as Euclidian path integral:

$$G(x_1, \ldots, x_n) \simeq \int \mathscr{D}\varphi \, \phi(x_1) \ldots \phi(x_n) \exp(-A_E[\phi]) . \tag{2.19}$$

One integrates over all values of the fields at each point. The product $\phi(x_1) \ldots \phi(x_n)$ is weighted by the Euclidian function

$$A_E = \int \mathrm{d}^4 \, (\mathcal{L}_0 + \mathcal{L}_I) , \tag{2.20}$$

with the free Lagrangian

$$\mathcal{L}_0 = \frac{1}{2} \sum_{m=1}^{4} \partial_m \phi \partial_m \phi + \frac{1}{2} \mu^2 \phi^2 , \tag{2.21}$$

and the interacting part

$$\mathcal{L}_I = \frac{g^2}{4!} \phi^4 . \tag{2.22}$$

One way to continue is to introduce the Euclidian generating functional:

$$W[J] = \int \mathscr{D}\varphi \, \exp\left(- \int \mathrm{d}^4 x \, (-\mathcal{L}_0 + \mathcal{L}_I + J\varphi) \right) = \int \mathscr{D}\varphi \, \exp \text{ß}left(- \int \mathrm{d}^4 x \, \mathcal{L}_I \left(\frac{\delta}{\delta j} \right) W_0[J] , \tag{2.23}$$

with

$$W_0[J] = \int \mathscr{D}\varphi \, \exp\left(- \int \mathrm{d}^4 x \, (\mathcal{L}_0 + J\phi) \right) . \tag{2.24}$$

By taking arbitrary derivatives of $W[J]$ with respect to J one obtains the Green's function. $W_0[J]$ is in fact a Gaussian integral. Recall

$$\int_{-\infty}^{+\infty} \mathrm{d}x \, \exp(-ax^2 + bx + c) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{1}{4} b \cdot \frac{1}{a} \cdot b \right) . \tag{2.25}$$

From that one can compute

$$W_0[J] = N \exp\left(\frac{1}{2} \int \mathrm{d}^4 x \int \mathrm{d}^4 y \, J(x) \Delta(x,y) J(y) \right) , \tag{2.26}$$

with propagator

$$\Delta(x,y) = \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\exp(\mathrm{i}k(x-y))}{k^2 + \mu^2} . \tag{2.27}$$

Hence, we have found Feynman's perturbation series

$$W[J] = \left[1 + \sum_{n=1}^{\infty} \frac{1}{n!}\left(-A_I\left(\frac{\delta}{\delta J}\right)\right)^n\right] W_0[J]\,. \tag{2.28}$$

$W_0[J]$ contains propagators and $(-A_I(\delta/\delta J))^n$ the vertices. A way to organize calculations is with Feynman diagrams. Return to the path integral for the Green's functions, but rescale $\phi = \widetilde{\phi}/g$:

$$G_E \propto \int \mathscr{D}\widetilde{\phi}\,\widetilde{\phi}\ldots\widetilde{\phi}\,\exp\left\{-\frac{1}{g^2}\int \mathrm{d}^4x\,\left(\frac{1}{2}(\partial_m\widetilde{\phi})^2 + \frac{\mu^2}{2}\widetilde{\phi}^2 + \frac{1}{4!}\widetilde{\phi}^4\right)\right\}\,. \tag{2.29}$$

In that way on can pull out the dependence on the coupling constant out of the integral. The exponential factor is in fact

$$\exp\left(-\frac{1}{g^2 h}A[\widetilde{g}]\right)\,. \tag{2.30}$$

Since g^2 and $\hbar$ always come together, one can either take the limit $g^2 \mapsto 0$ (weak coupling limit) or $\hbar \mapsto 0$ (semiclassical limit). The Euler-Lagrangian equations originate from $\delta\phi(x)$ of $\mathcal{L}(\phi, \partial_\mu\phi)$:

$$0 = \delta\mathcal{L} = \delta\phi\frac{\delta\mathcal{L}}{\delta\phi} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\mu\delta\phi\,, \tag{2.31}$$

By partial integration one obtains:

$$0 = \delta\phi\left(\frac{\delta\mathcal{L}}{\delta\phi} - \partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\right)\,. \tag{2.32}$$

Since $\delta\phi$ is arbitrary the expression in the round brackets has to vanish and these are just the classical field equations. For the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}\mu^2\phi^2 + \frac{g^2}{4!}\phi^4\,, \tag{2.33}$$

the field equations are for example given by:

$$-\Box\phi + \mu^2\phi + \frac{g^2}{3!}\phi^3 = 0 \Rightarrow (\Box - \mu^2)\phi = \frac{g^2}{3!}\phi^3\,. \tag{2.34}$$

Since $\phi = 0$ solves equation (2.34) it makes sense to perturbate around the "point" $\phi = 0$. Now, one can ask the question, if there are other saddle points next to $\phi = 0$, around one can perturbate. These points are not covered by the Feynman series [3] and show us new physics that are not found by perturbation theory.

Chapter 3

Low dimensional examples

3.1 General remarks

We are going to consider classical field theory with energy density $T_{00} \geq 0$ with the normalization $T_{00}|_{\text{vacuum}} = 0$. A **dissipative** solution is a solution, for which the energy flows away. Mathematically, that means

$$\lim_{t\mapsto\infty} \max_{\mathbf{x}} T_{00}(\mathbf{x}, t) = 0\,. \tag{3.1}$$

At every point in space the energy density goes to zero, if the time is sufficiently large. The Klein-Gordon equation, the Maxwell equations, etc. **only** have dissipative solutions. However, there exist some field theories, as for example the Yang-Mills-Higgs theory, with (non-trivial) **non**-singular, **non**-dissipative solutions of finite total energy, sometimes even time-independent solutions.

3.2 Kink

The solution exists in a (1+1)-dimensional $[x_0, x_1]$ theory with a single real scalar field $\varphi(x^0, x^1)$. The potential $U(\varphi)$ is non-negative and it holds that $U(\varphi_{\text{vac}}) = 0$. It is given by:

$$U = \frac{\lambda}{2}(\varphi^2 - a^2)^2\,. \tag{3.2}$$

We are looking for time-independent, non-dissipative solutions $\varphi(x_1)$ of the field equations

$$\boxed{\frac{\partial^2}{\partial x_1^2}\varphi = 2\lambda\varphi(\varphi^2 - a^2)\,.} \tag{3.3}$$

This is a non-linear ordinary differential equation. It exists the trivial solution $\varphi(x_1) = 0$, the classical vacuum of the theory. However, it turns out that there is a non-trivial solution of the form

$$\varphi(x_1) = a\tanh(\sqrt{\lambda}a x_1)\,, \tag{3.4}$$

which is the so-called kink. It is not defined for negative λ, unlike the Feynman series. So, this solution really is something new. We want to check, if (3.4) is in fact a solution of (3.3) by using the notation $x_1 \mapsto x$ and rescaling $\varphi = a\widetilde{\varphi}$, $ax = \widetilde{x}$:

$$\frac{\partial^2}{\partial\widetilde{x}}\widetilde{\varphi} = 2\lambda\widetilde{\varphi}(\widetilde{\varphi}^2 - 1)\,. \tag{3.5}$$

Recall

$$\sinh(x) \equiv \frac{\exp(x) - \exp(-x)}{2}\,, \quad \cosh(x) = \frac{\exp(x) + \exp(-x)}{2}\,, \tag{3.6}$$

$$\cosh^2(x) - \sinh^2(x) = 1\,, \quad \frac{\partial\sinh(x)}{\partial x} = \cosh(x)\,, \quad \frac{\partial\cosh(x)}{\partial x} = \sinh(x)\,, \frac{\partial\tanh(x)}{\partial x} = \frac{1}{\cosh^2(x)}\,. \tag{3.7}$$

Inserting

$$\widetilde{\varphi} = \tanh(\sqrt{\lambda}\widetilde{x})\,, \tag{3.8}$$

one obtains for the left-hand side of (3.5)

$$-2\lambda\frac{\sinh(\widetilde{x})}{\cosh^3(\widetilde{x})}\,, \tag{3.9}$$

and for the right-hand side of (3.5):

$$2\lambda\tanh\left(\frac{\sinh^2(\widetilde{x})-\cosh^2(\widetilde{x})}{\cosh^2(\widetilde{x})}\right) = -2\lambda\frac{\sinh(\widetilde{x})}{\cosh^3(\widetilde{x})} \tag{3.10}$$

so left-hand side and right-hand side are the same.

3.2.1 General analysis of the kink solution

First, we observe that the field equation (3.3) follows from variation of an energy functional $E[\varphi(x)]$ in the following way:

$$0 = \delta E = \delta\int \mathrm{d}x\,\left\{\frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + U(\varphi)\right\}\,. \tag{3.11}$$

We compare with the fictitious particle motion $q(t)$ of a particle of **unit** mass in a potential equal to $-U(q)$. Let us look at the variation of the Lagrange function $L = T - V$:

$$0 = \delta\int \mathrm{d}t\,L = \delta\int \mathrm{d}t\,\left\{\frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - (-U(q))\right\}\,, \tag{3.12}$$

which is called the Hamilton principle. From

$$\delta\int \mathrm{d}t\,L = 0\,, \tag{3.13}$$

one obtains the Newton equation

$$\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} = -V'(q) = F\,. \tag{3.14}$$

Let us make a little dictionary:

field talk	particle talk
field φ	coordinate q
position x	time t
potential $U(\varphi)$	potential $V = -U(q)$

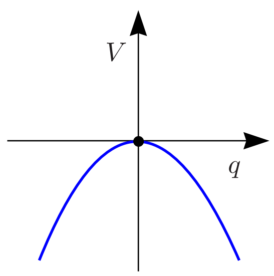
Every motion of the "particle" in the potential corresponds to a time-independent solution of the field equations, but not, in general, to a field solution of finite energy.

$$\delta E = \delta\int \mathrm{d}t\,\left(\frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - (-U(q))\right) = 0\,. \tag{3.15}$$

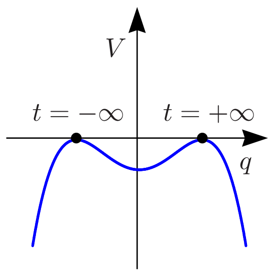
This describes, how a point particle behaves in a potential $V(q) = -U(q)$. For finite energy E we need

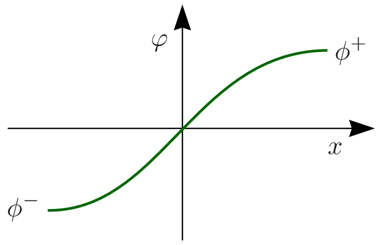
$$\lim_{x\mapsto\pm\infty}\varphi(x) = \varphi_\pm\,, \tag{3.16}$$

such that $U(\varphi_\pm) = 0$. So, the "particle" motion must start and end at the top where $V = 0$. If $U(\varphi)$ has a single zero, the ground state of the theory is unique and there are no non-trivial, time-independent solutions of finite energy. This corresponds to motions, where the particle stays forever at the maximum of the potential.



It results that $q(t) = 0$ and hence $\varphi(x) = \varphi_{\text{vac}} = \text{const}$. Now, if U has more than one zero, the theory has more than one ground state and there always exist non-trivial, time-independent solutions of finite energy. φ moves monotonically from one zero of U at $x = -\infty$ to an adjacent zero of U at $x = \infty$. Conventionally, solutions, for which φ is monotonically increasing, are called lumps and solutions, for which φ is monotonically decreasing, are called antilumps.





For any $U(\varphi)$ with two zeros one expects that behavior. A soliton can easily be found by quadrature (for any U):

$$E_{\text{part}} = T + V = \frac{1}{2}\left(\frac{\mathrm{d}q}{\mathrm{d}t}\right)^2 - U(q) = 0\,. \tag{3.17}$$

Translation into field language leads to

$$\frac{1}{2}(\partial_x\varphi)^2 = U(\varphi) \Rightarrow x - x_0 = \pm\int\limits_{\varphi_0}^{\varphi} \mathrm{d}\eta\,\frac{1}{\sqrt{2U(\eta)}}\,. \tag{3.18}$$

The field energy is given by

$$E_{\text{field}} = \int \mathrm{d}x\,\left(\frac{1}{2}(\partial_x\varphi)^2 + U(\varphi)\right) = \int \mathrm{d}x\,(\partial_x\varphi)^2\,. \tag{3.19}$$

For φ^4-theory

$$U = \frac{\lambda}{2}(\varphi^2 - a^2)^2\,,\quad a^2 = \frac{\mu^2}{\lambda}\,, \tag{3.20}$$

one obtains $\varphi = a\tanh(\mu x)$ for the form of the lump. From the energy of the lump (the lumps of φ^4-theory are frequently called “kinks” in the literature)

$$E_{\text{kink}} = \frac{4}{3}\frac{\mu^3}{\lambda}\,, \tag{3.21}$$

one realizes that this configuration cannot be obtained by perturbation theory around $\lambda = 0$.

3.2.2 Kink stability: by topology

For **finite energy** condition it holds that $\varphi(+\infty) - \varphi(-\infty) = 2na$, where a is the fundamental scale from the potential U. $n = 0$ corresponds to the vacuum and $n = 1$ corresponds to the kink. The case $n = -1$ is called antikink. The above condition can also be written in the form

$$\int\limits_{-\infty}^{+\infty} \partial_x\varphi\,\mathrm{d}x = 2na\,. \tag{3.22}$$

We can now define a **current** (with μ, ν=0, 1)

$$j_\mu(x) \equiv \varepsilon_{\mu\nu}\partial^\nu\varphi\,, \tag{3.23}$$

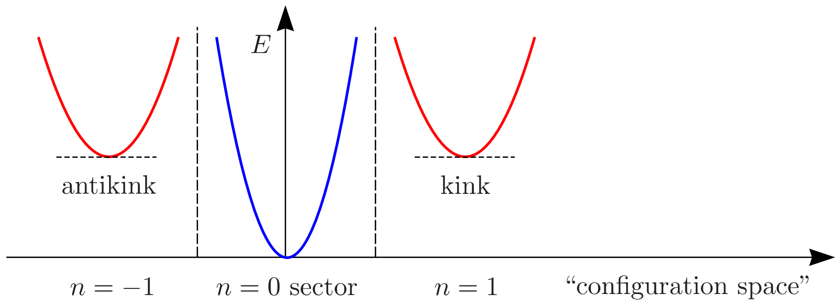
for which it holds that

$$\partial^\mu(\varepsilon_{\mu\nu}\partial^\nu\varphi) = \varepsilon_{\mu\nu}\partial^\mu\partial^\nu\varphi = 0\,, \tag{3.24}$$

whereas $\varepsilon_{\mu\nu}$ is the antisymmetric Levi-Civita symbol in two dimensions. Hence, the current is conserved and so there must exist a conserved topological charge:

$$Q \equiv \int\limits_{-\infty}^{+\infty} j_0\,\mathrm{d}x = \int\limits_{-\infty}^{+\infty} \partial_x\varphi\,\mathrm{d}x = 2na\,. \tag{3.25}$$

The kink ($n = 1$) is stable, because it is the lowest energy configuration in the $n = 1$ sector. There are infinitely high energy barriers between the sectors.



The barriers are infinitely high, because the space in between is infinite. If one obtained, for example, solutions on a circle, this would not be the case. Notice, that the current j^μ does not follow from the invariance of $\mathcal{L}$ under any symmetry transformation. It is, therefore, not a Noether current.

3.2.3 Footnote on Noether theorem

We use $\mathcal{L}(\varphi_i(x), \partial_\mu(\varphi_i(x)))$. The field equations are given by

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi_i} = 0\,. \tag{3.26}$$

Consider a continuous symmetry transformation

$$\varphi_i \mapsto \varphi_i + \overline{\delta\varphi_i}\,, \quad \overline{\delta\varphi_i} = \varepsilon T_{ij}\varphi_j\,. \tag{3.27}$$

The variation is given by

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi_i}\overline{\partial\varphi_i} + \frac{\partial\mathcal{L}}{\delta(\partial_\mu\phi_i)}(\partial_\mu\overline{\delta\varphi_i}) \overset{(3.26)}{=} \left(\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\right)\overline{\delta\varphi_i} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}(\partial_\mu\overline{\delta\varphi_i}) =$$
$$= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}\overline{\delta\varphi_i}\right) = \varepsilon\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}T_{ij}\varphi_j\right)\,, \tag{3.28}$$

Hence, to each symmetry generator of a continuous symmetry corresponds a conserved current:

$$J_i^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi_i)}T_{ij}\varphi_j\,. \tag{3.29}$$

This Noether current only applies to a solution of the field equations.

3.2.4 Kink stability: directly

Let us look at the kink in a more mathematical manner. Be S_∞ the points at spatial infinity and $M_V = \{\varphi|U(\varphi) = 0\}$. Finite energy condition implies the mapping $S_\infty \mapsto M_V$:

$$\lim_{x\mapsto S_\infty} \varphi(x) \in M_V\,. \tag{3.30}$$

Topologically distinct mapping often can be labeled by an integer n. This is the basis of all topological solitons. We have to study the field equation

$$\Box\varphi + U'(\varphi) = 0\,, \tag{3.31}$$

where the prime denotes differentiation with respect to φ. Insert $\varphi(x,t) = \varphi_k(x) + \delta(x,t)$, where φ_k is the static, non-dissipative kink solution and δ is a small perturbation. This leads to

$$\Box\delta + U''(\varphi_k)\delta = 0\,. \tag{3.32}$$

We want to retain only terms of first order in the perturbation: $U(\varphi_k + \delta) = U(\varphi_k) + U'(\varphi_k)\delta$. Therefore, we express a general small perturbation as a superposition of normal modes, where a_k are arbitrary complex coefficients

$$\delta(x,t) = \mathrm{Re}\left\{\sum_n a_n\psi_n(x)\exp(\mathrm{i}\omega_n t)\right\}\,, \tag{3.33}$$

and obtain the equation

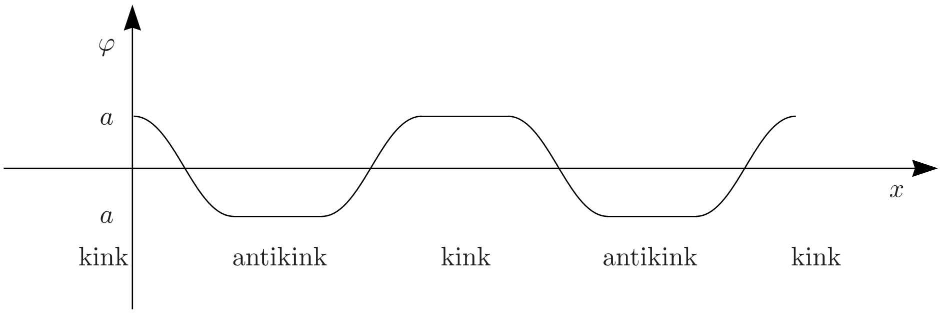
$$-\frac{\mathrm{d}^2\psi_n}{\mathrm{d}x^2} + U''(\varphi_k)\psi_n = \omega_n^2\psi_n\,, \tag{3.34}$$

which mathematically corresponds to a one-dimensional Schrödinger equation with a potential $U''(\varphi_k)$. We are now interested in the spectrum: our solution is stable with respect to small perturbations, if and only if none of the energy eigenvalues of the Schrödinger equation is negative. Spatial translation invariance tells us about a zero eigenvalue, namely if $\varphi_k(x)$ is a solution, then also

$$\varphi_k(x+a) = \varphi_k(x) + a\frac{\mathrm{d}\varphi_k}{\mathrm{d}x} + \mathcal{O}(a^2)\,. \tag{3.35}$$

$$\psi_0(x) = \frac{\mathrm{d}\varphi_k}{\mathrm{d}x}\,, \tag{3.36}$$

corresponds to $\omega_0 = 0$. The kink ϕ_k is monotonic and hence, ψ_0 has **no** node. From quantum mechanics we know that fpr a one-dimensional Schrödinger equation with arbitrary potential the eigenfunction with no nodes is the eigenfunction of lowest energy ψ_0. It holds that $\omega_n^2 \geq 0$, so everything is stable (modulo zero modes). So, the Kink solution really is particle like: It has finite energy, has a localized energy density and a Lorentz transformation gives a moving soliton. However, multi-kink solutions are restricted.



So, kink and antikink would correspond to the **same** particle, but a very **unusual** particle.

3.3 Sine Gordon soliton

It is again a (1+1)-dimensional scalar theory with the potential

$$U = \frac{\alpha}{\beta^2}(1-\cos(\beta\varphi)) = \frac{1}{2}\alpha\varphi^2 + \frac{1}{4!}\alpha\beta^2\varphi^4 + \mathcal{O}(\varphi^6)\,, \tag{3.37}$$

whereas in the last term the cosine was expanded in a power series. The field equation is given by

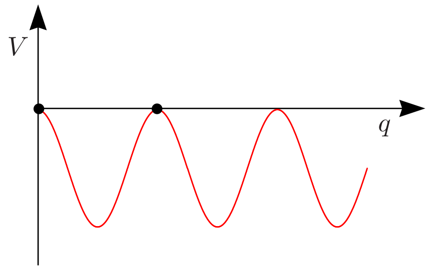
$$\partial_x^2\varphi = \frac{\alpha}{\beta}\sin(\beta\varphi)\,. \tag{3.38}$$

Introducing the quantities $\beta\varphi = \widetilde{\varphi}$ and $\sqrt{\alpha}x = \widetilde{x}$ results in

$$\partial_{\widetilde{x}}^2\widetilde{\varphi} = \sin\widetilde{\varphi}\,, \tag{3.39}$$

with its solution

$$\widetilde{\varphi}_s = 4\arctan(\exp(\widetilde{x}))\,. \tag{3.40}$$



Hence, the soliton is given by

$$\varphi_s = \frac{4}{\beta} \arctan(\exp(\sqrt{\alpha} x)) \,, \tag{3.41}$$

and its energy is

$$E_S = \frac{8\sqrt{\alpha}}{\beta^2} \sim \frac{m_{\text{boson}}}{\text{coupling}} \,. \tag{3.42}$$

One can also do the quadrature with $1/2(\partial_x \varphi)^2 = U = 1 - \cos\varphi$:

$$x = \int_{\pi}^{\varphi} \mathrm{d}\eta \, \frac{1}{\sqrt{2(1-\cos(\eta))}} = \int_{\pi}^{\varphi} \mathrm{d}\eta \, \frac{1}{2\sin\left(\frac{\eta}{2}\right)} = \ln\left\{\tan\left(\frac{\varphi}{4}\right)\right\} \,, \tag{3.43}$$

by using $\cos(2\eta) = \cos^2\eta - \sin^2\eta$.
The full quantum theory is known. Coleman found out that the Sine Gordon theory is equivalent to the massive Thirring model [4, 5]. The Thirring model is a (1+1)-dimensional Dirac theory with a four fermion interaction. Be $\psi(x_0, x_1)$ a Dirac field. With the definitions $\gamma_0 = \sigma_1$, $\gamma_1 = \mathrm{i}\sigma_2$ and "γ_5" $= -\sigma_3$ the Lagrangian of the massive Thirring model is given by

$$\mathcal{L}_{\text{mTh}} = \overline{\psi} \mathrm{i}\gamma_\mu \partial^\mu \psi - m' \overline{\psi}\psi - \frac{1}{2} g j_\mu j^\mu \,, \quad j^\mu \equiv \overline{\psi}\gamma^\mu \psi \,, \tag{3.44}$$

where m' is the bare (and not the physical) mass. The equivalence of Sine Gordon theory and massive Thirring theory is realizable by identifying

$$\frac{\beta^2}{4\pi} = \frac{\pi}{\pi + g} \,, \tag{3.45}$$

and the quantum operators

$$S_\pm =: \exp(\pm \mathrm{i}\beta\varphi) : \,, \quad \sigma_\pm = Z\overline{\psi}(\mathbb{1} \pm \gamma_5)\psi \,, \tag{3.46}$$

where Z is a cut-off-dependent constant. If $\beta^2 = 4\pi$ it follows $g = 0$ and the Thirring model is a free massive Dirac field theory. All Green functions made out of $S_\pm$ are the same as the Green functions made out of the $\sigma_\pm$. Furthermore, the topological current is related to the Noether current:

$$j^\mu = -\frac{\beta}{2\pi} \varepsilon^{\mu\nu} \partial_\nu \varphi \,, \quad j^\mu = \overline{\psi}\gamma^\mu\psi \,. \tag{3.47}$$

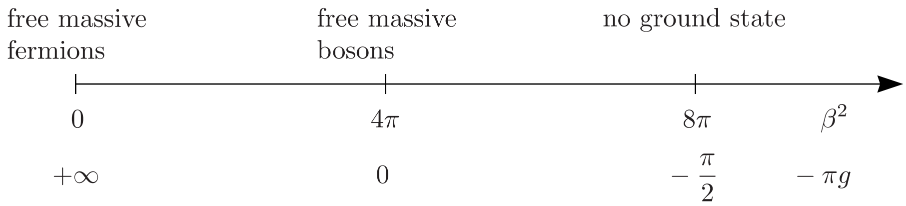
For small β it holds that

$$\langle \beta\varphi(x) \rangle \sim 4 \arctan(x - a) \equiv f(x - a) \,. \tag{3.48}$$

The current for an S-$\overline{\text{S}}$ pair (soliton-antisoliton pair) is

$$\langle j_0 \rangle = \frac{1}{2\pi} \frac{\mathrm{d}}{\mathrm{d}x} [f(x-a) - f(x+a)] \,. \tag{3.49}$$

Integration gives $+1$ and -1 as charge (fermion number) Mandelstam proved that the fermion field appears as a coherent state of a Bose field. Let us have a look at the phase diagram of the theory:



The striking feature is that a theory, which is "obviously" a theory of fermions is equivalent to a theory, which is obviously a theory of bosons.

	massive Thirring	Sine Gordon
fermion	fundamental	composite
boson	composite	fundamental

3.4 Catalogue of classical solutions

model	spacetime	topological class	name	application
U(1) Yang-Mills-Higgs (⊂ SM)	2	$\pi_1(\mathrm{U}_1)$	vortex	superconductivity cosmology
nonlinear sigma (perhaps from SM)	3	$\pi_3(\mathrm{SU}(2))$	skyrmions	nuclear physics
GUTs (⊃ SM)	3	$\pi_2(\mathrm{G/H})$	magnetic monopole	cosmology
SU(3) Yang-Mills (⊂ SM)	4	$\pi_3(\mathrm{SU}(3))$	BPST instanton	particle physics, QCD

For example, nucleons can be described as skyrmions. A skyrmion (for different sigma-models) behaves like a fermion. For $N \mapsto \infty$ one obtains the nonlinear sigma-model out of the standard model. Hence, our world is a perturbation around $N = \infty$. The above solutions are based on direct topology and they give stable solutions. Solutions based on indirect topology are unstable, as the following ones:

model	spacetime	topological class	name	application
SU(2) Yang-Mills-Higgs (⊂ SM)	3	$\pi_3(\mathrm{SU}(2))$	sphaleron S	ΔB at high T
SU(2) Yang-Mills-Higgs (⊂ SM)	3	$\pi_4(\mathrm{SU}(2))$	sphaleron S^*	particle physics

S^* controls scattering a high energies, but it cannot be discovered by LHC, because it is a 20 TeV effect.

Chapter 4

Derrick and scaling

This is ready for solitons in higher-dimensional scalar field theories, but it is a no-go theorem.
Consider

1.) a $(D{+}1)$-dimensional spacetime,

2.) a set of scalar fields $\varphi_1, \ldots, \varphi_N$ with flavour index,

3.) and the Lagrangian $\mathcal{L} = \sum_{a=1}^{N} \left\{ \frac{1}{2} \left[(\partial_t \varphi_a)^2 - (\boldsymbol{\nabla}_D \varphi_a)^2 \right] - U(\varphi_a) \right\}$, with $a = 1, \ldots, N$ and $U(\{\phi_a\}) \geq 0$ and $\min U(\{\phi_a\}) = 0$.

Derrick's theorem says that the only time-dependent classical solutions of finite energy E for $D \geq 2$ are the ground states $\overline{\varphi}_a \equiv \overline{\varphi}_a \equiv \text{const.}$ with $U(\{\overline{\varphi}_a\}) = 0$ [6].
So, either look for time-dependent, but non-dissipative solutions in scalar theories with $D > 1$ or introduce higher-spin fields (e.g. gauge bosons).

4.1 Proof

First of all we construct the Hamilton density, which is the T^{00} component of the energy momentum tensor:

$$\mathcal{H} = T^{00} = \sum_{a=1}^{N} \frac{1}{2} \left[\Pi_a^2 + (\boldsymbol{\nabla}_D \varphi_a)^2 \right] + U(\{\varphi_a\}) \,. \tag{4.1}$$

with $\Pi_a = \partial_t \varphi_a$. Let $\overline{\varphi}_a(\mathbf{x})$ be a time-independent solution of

$$\boldsymbol{\nabla}_D^2 \varphi_a - \frac{\partial U}{\partial \varphi_a} = 0 \,, \tag{4.2}$$

namely the N independent equations of motion. Consider the total energy

$$E(1) = T_1 + V_1 \,, \quad T_1 = \int \mathrm{d}^D x \sum_a \frac{1}{2} (\boldsymbol{\nabla}_D \overline{\varphi}^a)^2 \,, \quad V_1 = \int \mathrm{d}^D x \, U(\overline{\varphi}_a) \,. \tag{4.3}$$

We deform the configuration we started with by rescaling the space coordinate:

$$\varphi_a^\lambda = \overline{\varphi}_a(\lambda \mathbf{x}) \,, \tag{4.4}$$

with $\lambda \geq 0$. We look at the variation of the energy $E(\lambda) = T_\lambda + V_\lambda$ as a function of the variational parameter λ:

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E(\lambda) = (2 - D) T_\lambda - D V_\lambda \,. \tag{4.5}$$

For $\lambda = 1$ the configuration $\{\overline{\varphi}_a\}$ was applied to be a solution:

$$0 = \left. \frac{\mathrm{d}E}{\mathrm{d}\lambda} \right|_{\lambda=1} = (2 - D) T_1 - D V_1 \,. \tag{4.6}$$

For $D \geq 2$ it follows that $T_1 = V_1 = 0$. It holds that $\overline{\varphi}_a \equiv$ const. such that $U(\{\varphi_a\}) \equiv 0$. For $D = 2$ it holds that $V_1 = 0$. Hence, $\overline{\varphi}_a$ is constant such that $U(\{\varphi_a\}) \equiv 0$. Alternatively, look at

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} E(\lambda) = (2 - D)\lambda^{1-D} T_1 - D\lambda^{D-1} V_1 \stackrel{!}{=} 0\,. \tag{4.7}$$

We obtain out of that the solution with respect of λ:

$$\lambda^2 = \frac{DV_1}{(2 - D)T_1}\,. \tag{4.8}$$

For $V_1 > 1$ and $T_1 > 0$ one can look at dimension constraints that are encoded in this equation. $D > 2$ implies $\lambda \neq 0$ and this is a contradiction, because we suppose that λ is real. For $D = 2$ we also get a contradiction, because the rescaling will be infinite.

Remarks

1.) By imposing that the kinetic term in the Lagrangian contains higher derivatives one would get higher powers of lambda in the energy functional that could stabilize it. That does not guarantee the existence of a solution, but does even not exclude it. But then, one has to introduce an additional mass scale in the Lagrangian.

2.) One would have to introduce certain ideas, how gauge fields behave under rescaling. But this shall not be considered in this lecture.

Chapter 5

Vortex solution

5.1 Abelian Higgs model in (2+1) dimensions

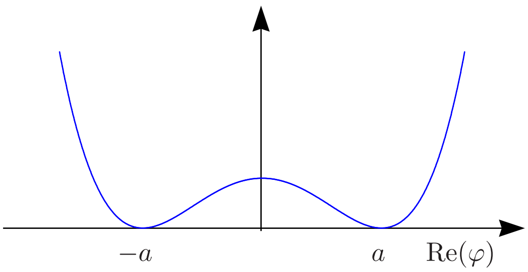
We consider a slice through (3+1) dimensions and use this system to study in a sophisticated way the basic structure of topological gauge theory solutions. This can be interesting in different solid state problems as for example superconductivity.

- We consider a complex scalar field $\varphi = \varphi_1 + \mathrm{i}\varphi_2$, where φ_1, $\varphi_2 \in \mathbb{R}$ and
- a U(1) gauge field A_μ with $\mu = 0, 1, 2$ and $(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1)$.

The gauge invariant action S is given by

$$S = \int \mathrm{d}^3x\, \mathcal{L} = \int \mathrm{d}^3x \left[\frac{1}{2}\overline{D_\mu\varphi}D^\mu\varphi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\overline{\varphi}\varphi)\right], \quad V(\overline{\varphi}\varphi) = \frac{\lambda}{8}(\overline{\varphi}\varphi - a^2)^2\,, \tag{5.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.



The gauge transformation is

$$\varphi \mapsto \exp(\mathrm{i}\alpha(x))\varphi\,, \quad A_\mu \mapsto A_\mu + \frac{\mathrm{i}}{e}\partial_\mu\alpha(x)\,. \tag{5.2}$$

One can check gauge invariance after defining the covariant derivative:

$$D_\mu\varphi = \partial_\mu\varphi - \mathrm{i}eA_\mu\varphi\,. \tag{5.3}$$

$V(\overline{\varphi}\varphi)$ is invariant under gauge transformations. So is the $F_{\mu\nu}F^{\mu\nu}$ term, because it is antisymmetric in μ, ν and the derivatives behave symmetric. The only nontrivial contribution comes from the derivative term

$$D_\mu\varphi \mapsto \exp(\mathrm{i}\alpha(x))D_\mu\varphi\,. \tag{5.4}$$

The conformal invariance is violated because of the explicit mass scale a. (For physicists the conformal symmetry group is just the rescaling group.) Furthermore it holds that $[\phi] = 1/2$, $[A_\mu] = 1/2$, $[e] = 1/2$ and $[\lambda] = 1$. So, even the λ part itself violates conformal symmetry, but the existence of a makes it explicit.
We now do a rescaling of fields to make the fields dimensionless:

$$\varphi = \widetilde{\varphi}a\,, \quad aex = \widetilde{x}\,, \quad A_\mu(x) = \widetilde{A}_\mu(\widetilde{x})\,, \quad e\widetilde{A} = \overline{A}\,, \quad \frac{\lambda}{e^2} = \overline{\lambda}\,. \tag{5.5}$$

So, one obtains:

$$S=\frac{a}{e}\int \mathrm{d}^3x\,\left[\frac{1}{2}|\widetilde{D}_\mu\widetilde{\phi}|^2-\frac{1}{4}\widetilde{F}_{\mu\nu}^2-\frac{\overline{\lambda}}{8}\left(|\widetilde{\varphi}|^2-1\right)\right]\,,\quad \widetilde{D}_\mu\widetilde{\varphi}=(\widetilde{\partial}_\mu-\mathrm{i}\widetilde{A}_\mu)\widetilde{\varphi}\,. \tag{5.6}$$

At the classical level the prefactor a/e is not important, because it cancels out in the equations of motion. But quantum mechanically it matters, because it will change Feynman's path-integral weight $\exp(\mathrm{i}/\hbar S)$.
From now on we will use the old notation even for the dimensionless fields (without tilde). Impose gauge $A_0=0$. The equations of motion associated with A_0 must be imposed as a constraint:

$$\partial_i\dot{A}_i+\varepsilon_{ab}\dot{\varphi}_a\varphi_b=0\,. \tag{5.7}$$

ε_{ij} is the totally antisymmetric tensor in two dimensions. $a=1,\,2,\,\dots$ is the label counting of $\varphi=\varphi_1+\mathrm{i}\varphi_2$.

$$L=\int \mathrm{d}^2x\,\mathcal{L}=T-V\,,\quad T=\frac{1}{2}\int \mathrm{d}^2x\,\left[\dot{\varphi}_a\dot{\varphi}_a+\dot{A}_i\dot{A}_i\right]\,, \tag{5.8}$$

$$V=\frac{1}{2}\int \mathrm{d}^2x\,\left[|\boldsymbol{\nabla}\varphi|^2+F_{12}^2+\frac{\lambda}{4}(|\varphi|^2-1)^2\right]\,. \tag{5.9}$$

For a configuration of finite $E=T+V$ we need to impose boundary conditions at $|\mathbf{x}|\mapsto\infty$:

1.) $|\varphi|\mapsto 1$: $\arg(\varphi(\vartheta))=\arg(\varphi(\vartheta+2\pi))$, because the field can be written as $\varphi=|\varphi|\exp(\mathrm{i}\arg(\vartheta))$ and it should be single-valued.

2.) $\boldsymbol{\nabla}\varphi\mapsto 0$

The group manifold of U(1) is an S_1. The according map would then be $S^1_\infty\to \mathrm{U}(1)=S^1_{\mathrm{gauge}}$. The real axis has to be chopped in parts with length 2π. One classifies the solutions by the topology imposed by the behavior of the fields at $|\mathbf{x}|\mapsto\infty$.

i.) Hence, one introduces in this way topologically different sections (winding number n).

ii.) Observe the crucial role of the gauge field. Namely, without A_μ only $n=0$ is possible.
Go to polar coordinates ϱ, ϑ and z. In three space dimensions it can be found:

$$\boldsymbol{\nabla}\psi=\partial_\varrho\psi\widehat{e}_\varrho+\frac{1}{\varrho}\partial_\vartheta\psi\widehat{e}_\vartheta+\partial_z\psi\widehat{z}\,, \tag{5.10}$$

$$\boldsymbol{\nabla}\cdot\mathbf{A}=\frac{1}{\varrho}\partial_\varrho(\varrho A_\varrho)+\frac{1}{\varrho}\partial_\vartheta A_\vartheta+\partial_z A_z\,, \tag{5.11}$$

$$\boldsymbol{\nabla}\times\mathbf{A}=\frac{1}{\varrho}(\partial_\vartheta A_z-\partial_z A_\vartheta)\widehat{e}_\varrho+(\partial_z A_\varrho-\partial_\varrho A_z)\widehat{e}_\vartheta+\frac{1}{\varrho}(\partial_\varrho(\varrho A_\vartheta+\partial_\vartheta A_\varrho)\widehat{e}_z\,. \tag{5.12}$$

The gauge field A_μ plays a crucial role in making the energy finite at nontrivial winding number $n\neq 0$. Consider the kinetic energy for φ without A_μ:

$$\int\limits_0^\infty \mathrm{d}\varrho\,\varrho\int\limits_0^{2\pi}\mathrm{d}\vartheta\,\left|\frac{1}{\varrho}\partial_\vartheta\varphi\right|^2\,. \tag{5.13}$$

Introducing A_μ the kinetic term is:

$$\int\limits_0^\infty \mathrm{d}\varrho\,\varrho\int\limits_0^{2\pi}\mathrm{d}\vartheta\,\left|\frac{1}{\varrho}\partial_\vartheta\varphi-A_\vartheta\varphi\right|^2\,. \tag{5.14}$$

At spatial infinity it holds that $\varphi\mapsto\exp(\mathrm{i}n\vartheta)$ and

$$\lim_{\varrho\mapsto\infty}\varrho A_\vartheta=n\in\mathbb{Z}\,. \tag{5.15}$$

The kinetic term for the gauge field also gives a finite contribution:

$$\int\limits_0^\infty \mathrm{d}\varrho\,\varrho\int\limits_0^{2\pi}\mathrm{d}\vartheta\,\frac{1}{\varrho^4}<\infty\,. \tag{5.16}$$

Without gauge field the energy would only be finite for trivial winding number.

iii.) Introduce a quantized magnetic flux:
By Stokes' theorem we know that

$$\int_S (\nabla \times \mathbf{A})\, \mathrm{d}\mathbf{S} = \int_{\partial S} \mathbf{A}\, \mathrm{d}l\,. \tag{5.17}$$

Hence, one obtains:

$$\varphi = \int \mathrm{d}^3x\, B = \int \mathrm{d}^3x\, F_{12} = \lim_{\varrho \mapsto \infty} \int_0^{2\pi} \mathrm{d}\vartheta\, \varrho A_\vartheta = 2\pi n\,. \tag{5.18}$$

Considering static fields $\dot{A} = \dot{\phi} = 0$ one recognizes that Gauss' law is satisfied. With $T = 0$ is follows that $E = V$. Firstly check the scaling:

$$E \sim \int \mathrm{d}^2x\, [F_{12}^2 + |D\phi|^2 + U(\phi)]\,. \tag{5.19}$$

Let R be the size of the solution $E \sim R^{-2} + R^0 + R^2$. Hence, a solution is not excluded by scaling. The field equations follow from variation of the action. One obtains

$$D_m D_m \varphi = \frac{\lambda}{2} \varphi(|\varphi|^2 - 1)\,, \tag{5.20}$$

and a generalized Maxwell equation:

$$\partial_m F_{mn} = -\frac{\mathrm{i}}{2}(\varphi \overline{D_n \varphi} - \varphi D_n \varphi)\,. \tag{5.21}$$

These partial differential equations are highly nontrivial. For $\lambda = 1$ there is a miracle that occurs. Bogomolny [7] reduces these second order field equations to first order ones. Take $n > 0$ (for $n < 0$ replace $\oplus \leftrightarrow \ominus$). Rewrite

$$E = \frac{1}{2} \int \mathrm{d}^2x\, \Big[\{(D_1 \oplus \mathrm{i}D_2)\varphi\} \overline{\{(D_1 \oplus \mathrm{i}D_2)\varphi\}} \oplus F_{12} + \Big\{ F_{12} \oplus \frac{1}{2}(|\varphi|^2 - 1) \Big\}^2 \oplus \mathrm{i} \{\partial_2(\overline{\varphi} D_1 \varphi) - \partial_1(\overline{\varphi} D_2 \varphi)\} \Big] \geq \pi |n|\,. \tag{5.22}$$

The last term vanishes by the boundary conditions. Furthermore, it holds that $E = \pi|n|$ if and only if

$$(D_1 \oplus \mathrm{i}D_2)\varphi = 0\,, \tag{5.23}$$

and

$$F_{12} \oplus \frac{1}{2}(|\varphi|^2 - 1) = 0\,. \tag{5.24}$$

These are the so-called Bogomolny equations. These minimize the energy functional and therefore they automatically solve the field equations. Nonobvious but true is that **all** solutions of the field equations satisfy the Bogomolny equations. (The last statement does not hold for Yang-Mills theory.)
For rotationally invariant solutions of the Bogomolny equations we make the following Ansatz [8]:

$$\varphi(\phi, \theta) = h(\phi) \exp(\mathrm{i}n\theta)\,, \quad A_\varrho = 0\,, \quad A_\theta = \frac{n}{\varrho} a(\varrho)\,. \tag{5.25}$$

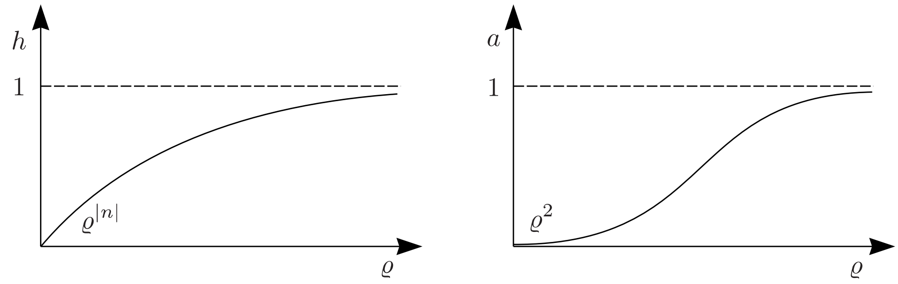
Hence, one obtains the equations

$$\varrho \partial_\varrho h - n(1 - a)h = 0\,, \quad \frac{2n}{\varrho} \partial_\varrho a + n^2 - 1 = 0\,. \tag{5.26}$$

The boundary conditions are given by

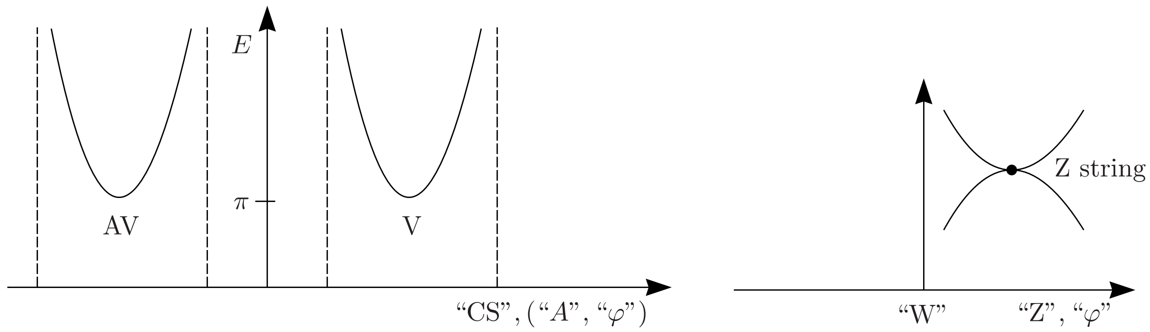
$$h(0) = a(0) = 0\,, \quad h(\infty) = a(\infty) = 1\,. \tag{5.27}$$

By solving these equations numerically one gets:



For $n = \pm 1$ there exist vortex and antivortex solutions of Abrikosov (condensed matter) and Nielsen-Odesen [hep-ph]. For $|n| > 1$: coincident vortex solution. With $\lambda < 1$ they are stable, but with $\lambda > 1$ they are unstable. For $\lambda = 1$ there exists a $2n$-parameter family of n-vortex solutions. The " massive photon" would correspond to the Z^0 of the electroweak standard model.

Abelian Higgs model	electroweak standard model
massive vector	Z
massive scalar	Higgs scalar
	+ all other fields



Chapter 6

Homotopy groups

We take a heuristic approach starting with the simplest homotopy group π_1.

6.1 Definition

a.) $\pi_1(X)$ is the first (pointed) homotopy group, whose elements are equivalence classes of maps of the unit circle S_1 into the topological space X.

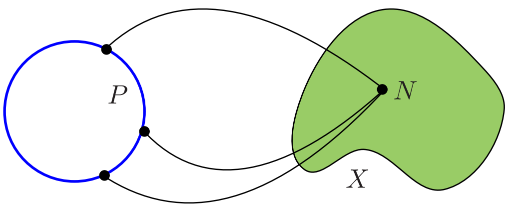
b.) S_1 has a fixed point P and orientation and the topological space X has a fixed point N and and orientation for all maps $P \mapsto N$.

c.) Two maps are equivalent or homotopic, if they can be continuously deformed into each other. Each equivalence class is an element of $\pi_1(X)$ and the elements form a group.

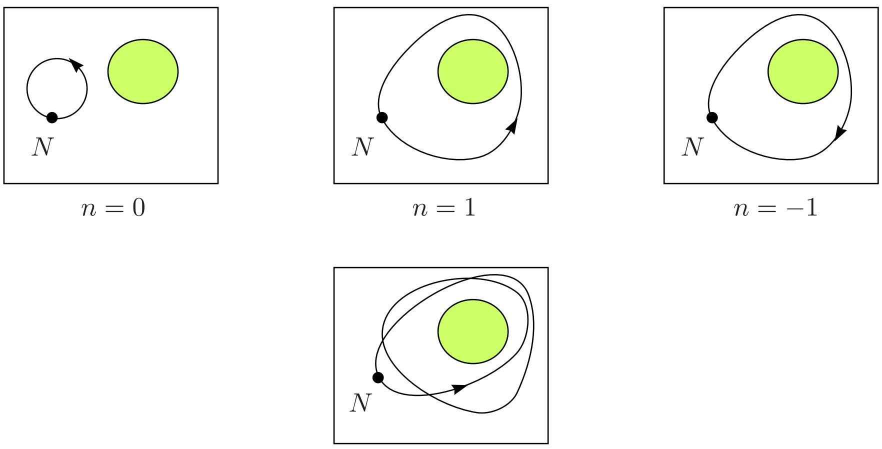
Two maps $X_1(\theta)$ and $X_2(\theta)$ $(\theta \in S_1)$ and map

$$F(\tau, \theta) \equiv \tau X_1(\theta) + (1-\tau) X_2(\theta)\,, \tag{6.1}$$

is **continuous**, then X_1 and X_2 are in the same equivalence class. The identity element of $\pi_1(X)$: whole S_1 is mapped into the single point N



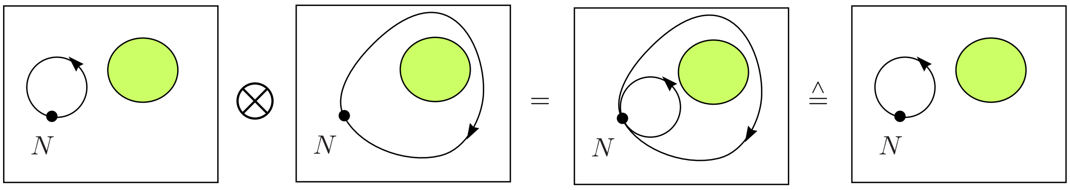
Consider a further example: $X = \overline{X} =$ "$\mathbb{R}^2 -$ hole$''$. Elements of $\pi_1(\overline{X})$:



$n = 0$ $\quad$ $n = 1$ $\quad$ $n = -1$ $\quad$ $n = 2$

$$\pi_1(\overline{X}) = \mathbb{Z}\,. \tag{6.2}$$

So, the homotopy group tells us something about the structure of the topological space.
The group multiplication $\otimes$ of $\pi_1(X)$ is, that you do one map after the other.



(6.3)

$$-1 + 0 = -1\,. \tag{6.4}$$

The inverse element follows by change of orientation:

$$[-1]^{-1} = [1]\,. \tag{6.5}$$

$\pi_1(\overline{X})$ is isomorphic to $\mathbb{Z}$. Furthermore, it holds that $\pi_1(\mathbb{R}^2) = 0$. Hence, the homotopy groups are different and therefore the topological structure of the two spaces. $\mathbb{R}^2$ is simply connected, but not $\mathbb{R}^2 - D^2$.
A further example is $\pi_1(S_1) = \mathbb{Z}$. The equivalence classes are $[\exp(\mathrm{i}n\varphi)]$ with $\varphi \in (0, 2\pi]$ as the S^1 parameter. Furthermore, we know $\pi_1(S_1) \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$. Note also that $[\exp(\mathrm{i}\varphi)]$ is the generator. All the elements follow by multiplying the generator arbitrarily often with itself. Furthermore, it holds that

$$[\exp(\mathrm{i}\varphi)] = \left[\exp\left(\frac{\mathrm{i}\varphi^2}{2\pi}\right)\right] = \left[\exp(\mathrm{i}\sqrt{2\pi\varphi})\right]\,, \tag{6.6}$$

so these maps all correspond to the generator of the homotopy group.
We now want to consider a further example, namely $\pi_1(S_2) = 1$. ("You cannot lasso a baseball.") This does for example not hold for $\pi_1(T_2) = 1$. Furthermore, it holds that $\pi_1(\mathrm{SU}(2)) = 1$ and $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2 = \{1, -1\}$. SU(2) is isomorphic to S^3, because one can write elements in the form

$$\overline{g} = x_0 \mathbb{1}_2 + x_q \mathrm{i}\sigma^q\,, \quad x_0^2 + x_q^2 = 1. \tag{6.7}$$

Another possibility to write up elements in SU(2) is

$$\overline{g} = \cos\left(\frac{|\theta|}{2}\right) + \sin\left(\frac{|\theta|}{2}\right)\widehat{\theta}\mathrm{i}\boldsymbol{\sigma}\,, \quad |\widehat{\theta}|^2 = 1\,, \quad \theta \equiv |\theta| \in [0, 2\pi]\,, \tag{6.8}$$

with the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{6.9}$$

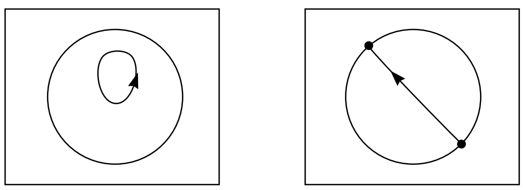
It holds that $\pi_1(\mathrm{SO}_3) = \mathbb{Z}_2 = \{1, -1\}$. The SO(3) is a three-ball with antipodal points on the surface identified. An element can be written as

$$g = \mathbb{1}_3 + \widehat{\theta} \cdot \mathbf{X} \sin\theta + |\widehat{\theta} \cdot \mathbf{X}|^2 (1 - \cos\theta)\,, \quad \theta \equiv |\theta| \in [0, \pi]\,, \tag{6.10}$$

with

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\,, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,, \quad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{6.11}$$

$\pi_1(\mathrm{SO}(3)) = \Big\{$



$\Big\} = \mathbb{Z}_2\,.$ (6.12)

6.2 Examples

For $n \leq m$ it holds that $\pi_n(S_m) = \delta_{nm}\mathbb{Z}$. However, it holds that $\pi_3(S_2) = \mathbb{Z}$ (from Hopf map) and furthermore $\pi_4(S_3) = \mathbb{Z}_2$ (from the Freudental suspension of Hopf). For X a non-Abelian group manifold, some examples are $\pi_1(\mathrm{U}_n) = \mathbb{Z}$ for $n \geq 1$, $\pi_1(\mathrm{SU}(n)) = 1$ for $n \geq 2$, $\pi_2(G) = 1$ for any Lie group G, $\pi_3(G) = \mathbb{Z}$ for any **simple** Lie group G. Recall that a **simple** Lie group has no nontrivial invariant subgroup. Recall that an invariant subgroup $N \subseteq G$ is such that $\forall\, t \in N$ and $\forall\, g \in G$ it holds that $gtg^{-1} \in N$. Let us look at further two concrete examples:

1.) The generator of $\pi_3(\mathrm{SU}(2)) = \mathbb{Z}$:

$$S_3^\infty : \widehat{x}_\mu = (x_1, x_2, x_3, x_4)\,, \quad x_\mu^2 = 1\,. \tag{6.13}$$

Then, the generator of $\pi_3(\mathrm{SU}(2))$ is given by:

$$g(\widehat{x}) = x_4\mathbb{1} + x_1\mathrm{i}\sigma_1 + x_2\mathrm{i}\sigma_2 + x_3\mathrm{i}\sigma_3 \in \mathrm{SU}(2)\,. \tag{6.14}$$

2.) Generator of $\pi_3(S_2) = \mathbb{Z}$:

Define a complex spinor

$$Z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\,, \quad z_i \in \mathbb{C}\,, \quad |z_1|^2 + |z_2|^2 = 1\,. \tag{6.15}$$

Then, the **generator** corresponds to the map $z \in S_3 \mapsto \widehat{n} \in S_2$: $n^a = Z^\dagger \sigma^a Z$ with $a = 1, 2, 3$. Note $Z \mapsto \exp(\mathrm{i}\varphi)Z$, then $\widehat{n}$ is invariant. The inverse map $S^2 \mapsto S^3$ maps points into circles. So, this is, in fact, the famous Hopf fibering of the S_3.

1.) $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$: $\mapsto D = 2$ vortex solution V

2.) $\pi_2(G/\mathrm{U}(1)) \supseteq \mathbb{Z} \mapsto D = 3$ magnetic monopole solution M

3.) $\pi_3(\mathrm{SU}(2)) = \mathbb{Z} \mapsto D = 4$ instanton solution I (indirectly related to $D = 3$ sphaleron solution S)

4.) $\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2$ indirectly related to new types of instantons I^* and sphalerons S^*

6.3 Definition of $\pi_n(X)$

a.) $\pi_n(X)$ is the n-th pointed homotopy group, whose elements are equivalence classes of maps of the unit sphere S_n into the topological space X.

b.) S_n has a fixed point P and orientation. So has X. This holds for all maps $P \mapsto N$.

c.) Two maps are equivalent or homotopic, if they can be continuously deformed into each other. Each equivalence class is an element of $\pi_n(X)$ and the elements form a group.

Chapter 7

Magnetic monopole

7.1 Dirac monopole

A magnetostatic analogue of the Coulomb field is given by

$$\mathbf{B} = g\frac{\widehat{\mathbf{r}}}{\mathbf{r}^2} = -g\boldsymbol{\nabla}\left(\frac{1}{|\mathbf{r}|}\right) . \tag{7.1}$$

Recall, that

$$\triangle\left(\frac{1}{|\mathbf{r}|}\right) = -4\pi\delta^{(3)}(\mathbf{r}) , \tag{7.2}$$

then

$$\boldsymbol{\nabla}\cdot\mathbf{B} = 4\pi g\delta^{(3)}(\mathbf{r}) . \tag{7.3}$$

So, the above $\mathbf{B}$ field is the field of a point-like magnetic charge. Also, the flux over a large sphere is given by

$$\Phi = 4\pi r^2 B = 4\pi g . \tag{7.4}$$

We use Gaussian units, such that $\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho$ holds. Let us make two remarks:

1.) The existence of magnetic monopoles is **not at all mandatory**.

2.) At the origin the field is singular, so that

$$E_{\text{ma}} = \frac{1}{4\pi}\int \mathrm{d}^3x\,\mathbf{B}^2 = \infty , \tag{7.5}$$

but this also holds for a electric Coulomb field.

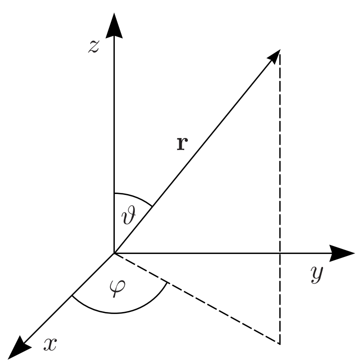
These two remarks hold for Abelian monopoles. For **non**-Abelian magnetic monopoles in a Yang-Mills-Higgs theory, the **opposite** holds. Still, it is worth to study the nonphysical Dirac monopole as it will describe the outside region of the physical non-Abelian monopole.
A vector potential A_i must be singular somewhere, because otherwise the following would hold:

$$\boldsymbol{\nabla}\cdot\mathbf{B} = \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0 . \tag{7.6}$$

One choice is the following (with cartesian coordinates $r^2 = x^2 + y^2 + z^2$):

$$A_x = g\frac{-y}{r(r+z)} , \quad A_y = g\frac{x}{r(r+z)} , \quad A_z = 0 . \tag{7.7}$$

In spherical coordinates



$$x = r\sin\vartheta\cos\varphi\,, \quad y = r\sin\vartheta\sin\varphi\,, \quad z = r\cos\vartheta\,, \tag{7.8}$$

$$\mathrm{d}x = \sin\vartheta\cos\varphi\,\mathrm{d}r + r\cos\vartheta\cos\varphi\,\mathrm{d}\vartheta - r\sin\vartheta\sin\varphi\,\mathrm{d}\varphi\,, \tag{7.9}$$

$$\mathrm{d}y = \sin\vartheta\sin\varphi\,\mathrm{d}r + r\cos\vartheta\sin\varphi\,\mathrm{d}\vartheta + r\sin\vartheta\cos\varphi\,\mathrm{d}\varphi\,, \tag{7.10}$$

$$\mathrm{d}z = \cos\vartheta\,\mathrm{d}r - r\sin\vartheta\,\mathrm{d}\varphi\,. \tag{7.11}$$

Then the 1-form of the potential can be written as

$$A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_z\,\mathrm{d}z \equiv A_r\,\mathrm{d}r + A_\vartheta\,r\mathrm{d}\vartheta + A_\varphi\,r\sin\vartheta\mathrm{d}\varphi\,, \tag{7.12}$$

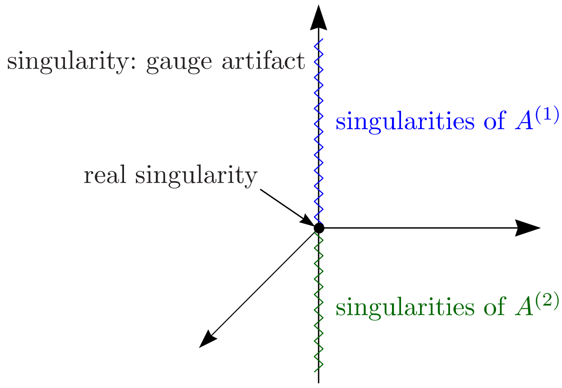
with

$$A_r^{(1)} = A_\vartheta^{(1)} = 0\,, \quad A_\varphi^{(1)} = \frac{g}{2}\frac{1-\cos\vartheta}{\sin\vartheta}\,. \tag{7.13}$$

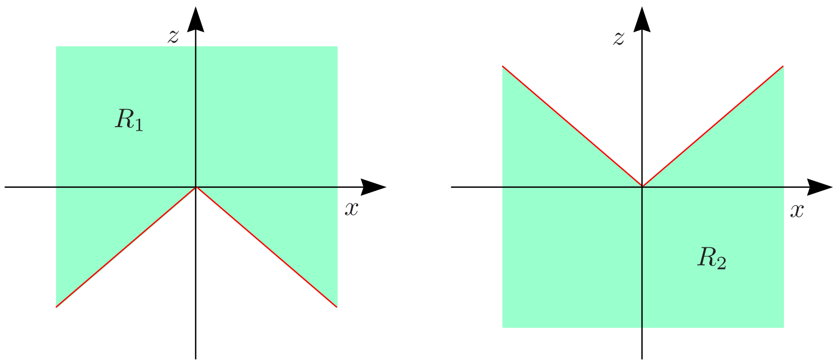
An alternative choice could be

$$A_r^{(2)} = A_\vartheta^{(2)} = 0\,, \quad A_\varphi^{(2)} = -\frac{g}{2}\frac{1+\cos\vartheta}{\sin\vartheta}\,. \tag{7.14}$$

The singularity structure of $A^{(1)}$ and $A^{(2)}$, respectively, is given by:



Both potentials are used in the so-called Wu-Yang construction [9], where two overlapping regions are considered, with in each of them **nonsingular** gauge fields [10]. The R_1-region is characterized by $0 \le \vartheta \le \pi/2 + \delta$ and $\pi/2 - \delta < \vartheta \le \pi$.



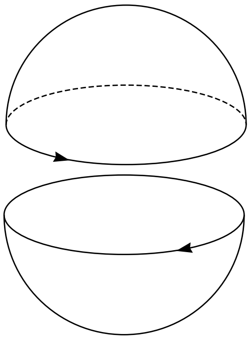
In R_1 we take

$$A_\varphi^{(1)} = \frac{g}{2}\frac{1-\cos\vartheta}{\sin\vartheta}\,, \tag{7.15}$$

and in R_2

$$A_\varphi^{(2)} = -\frac{g}{2}\frac{1+\cos\vartheta}{\sin\vartheta}\,. \tag{7.16}$$

First, let us calculate the flux Φ over the sphere, where we consider the northern hemisphere and the southern hemisphere.



Firstly, the flux is given by

$$\Phi = \int\limits_{\mathrm{NH}} \mathbf{B}\cdot \mathrm{d}\mathbf{S} + \int\limits_{\mathrm{SH}} \mathbf{B}\cdot \mathrm{d}\mathbf{S} = \oint\limits_{\partial(\mathrm{NH})} \mathbf{A}^{(1)}\,\mathrm{d}\mathbf{l} + \oint\limits_{\partial(\mathrm{SH})} \mathbf{A}^{(2)}\,\mathrm{d}\mathbf{l} =$$
$$= \int\limits_0^{2\pi} r\mathrm{d}\varphi \left[A_\varphi^{(1)} - A_\varphi^{(2)}\right] = \int\limits_0^{2\pi} r\mathrm{d}\varphi \left(2\frac{g}{r}\right) = 4\pi g\,, \tag{7.17}$$

where Stokes' theorem can be applied, because $A_\varphi^{(1)}$ is regular in the northern hemisphere and $A_\varphi^{(2)}$ is regular in the southern hemisphere. The minus sign between $A_\varphi^{(1)}$ and $A_\varphi^{(2)}$ comes from the different orientations of the northern and southern hemisphere, respectively.
Secondly, in the overlap region $R_1 \cap R_2$ the two potentials must describe the same (quantum) physics. Hence, $A^{(1)}$ and $A^{(2)}$ must be related by a gauge transformation. Indeed, take $\Omega = \exp(2\mathrm{i}ge\varphi)$ to calculate the gauge transform of $A^{(1)}$, where φ is the azimuthal angle of the spherical coordinates (with $\boldsymbol{\nabla}_\varphi = 1/(r\sin\vartheta)\partial/\partial\varphi$):

$$A_\varphi'^{(1)} = A_\varphi^{(1)} - \frac{\mathrm{i}}{e}\Omega\boldsymbol{\nabla}_\varphi\Omega^{-1} = A_\varphi^{(1)} - \frac{2g}{r\sin\vartheta} = -\frac{g}{r}\frac{1+\cos\vartheta}{\sin\vartheta}\,. \tag{7.18}$$

Now realize that $\Omega(\varphi)$ must be a single-valued function defined at each **point** along the equator. Hence, it must hold that

$$4\pi ge = 2\pi n\,, \quad \text{with } n \in \mathbb{Z}\,, \tag{7.19}$$

or

$$\boxed{ge = \frac{n}{2}\,.} \tag{7.20}$$

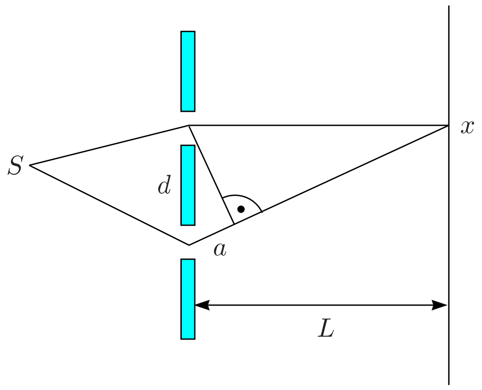
Reinstating $\hbar$ and c leads us to

$$\boxed{ge = \frac{n}{2}\hbar c\,.} \tag{7.21}$$

That is called the **Dirac quantization condition**.

- Dirac: If there exists a magnetic monopole ($g \neq 0$), then e is quantized. That means, that $\Delta(e_i - e_j)$ has a minimal step size.
- In the GUT framework (SO(10)-GUT) there exists both a quantized e and there exist magnetic monopoles (à la 't Hooft Polyakov).

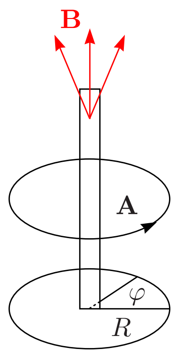
Consider an infinitely long and thin solenoid. This looks like a monopole and now there **is** a non-singular gauge potential ($\boldsymbol{\nabla}\cdot\mathbf{B} = 0$). However, the solenoid may be detectible by an Aharanov-Bohm experiment.
Reminder of a double-slit experiment:



Interference pattern with maxima for $\delta = 2\pi n$ and minima with $\delta = (2n+1)\pi$. The phase is given by

$$\delta = 2\pi \frac{a}{\lambda} = 2\pi \frac{x}{L}\frac{d}{\lambda}\,. \tag{7.22}$$

Introduce a solenoid:



$$B_z = \frac{1}{\varrho}\left(\frac{\partial(\varrho A_\varphi)}{\partial \varrho} - \frac{\partial A_\varrho}{\partial \varphi}\right)\,. \tag{7.23}$$

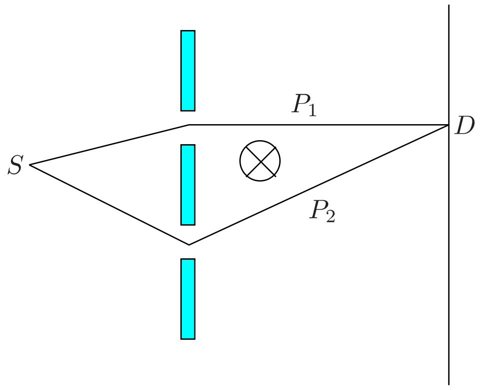
- $\varrho < R$:

$$A_\varrho = A_z = 0\,, \quad A_\varphi = \frac{B}{2}\varphi \Rightarrow B_z = B\,. \tag{7.24}$$

- $\varrho > R$:

$$A_\varphi = \frac{BR^2}{2\varrho} \Rightarrow B_z = 0\,. \tag{7.25}$$

$$\Phi_{\text{sol}} = \int \varrho\,\mathrm{d}\varphi\, A_\varphi = B\pi R^2\,. \tag{7.26}$$



A free electron can be described by the wave function

$$\psi = |\psi| \exp\left(\mathrm{i}\frac{\mathbf{p}\cdot\mathbf{r}}{\hbar}\right)\,, \tag{7.27}$$

and with electromagnetic field replace $\mathbf{p}$ by $\mathbf{p} - e\mathbf{A}$. The phase difference is given by the net path integral

$$\Delta\delta = \frac{e}{\hbar} \oint_{P_2 - P_1} \mathbf{A} \cdot \mathrm{d}\mathbf{l} = \frac{e}{\hbar} \int_{A, \partial A = P_2 - P_1} \boldsymbol{\nabla} \times \mathbf{A} \cdot \mathrm{d}\mathbf{S} = \frac{e}{\hbar} \Phi_{\mathrm{sol}} \,. \tag{7.28}$$

For $\Phi = 4\pi g$ there is **no** shift in the interference pattern, if and only if

$$\frac{4\pi g e}{\hbar} = 2n\pi \,, \tag{7.29}$$

That means that the solenoid is **invisible** and only the monopole part of the field matters. Another observation by Wu & Yang is the following: Think of **embedding** into a Yang-Mills theory (U(1) $\subset$ SU(2)). Be $W = W^a \sigma^a/(2\mathrm{i})$ a SU(2)-valued field.

$$W_r = W_\vartheta = 0 \,, \quad W_\varphi = \left(-\frac{g}{2}\right) \left(\frac{1 - \cos\vartheta}{\sin\vartheta}\right) \frac{\sigma_3}{2\mathrm{i}} \,. \tag{7.30}$$

Make a non-Abelian gauge transformation of the following form:

$$W'_\mu = \Omega \left(W + \frac{1}{\varrho} \partial_\mu \right) \Omega^{-1} \,, \quad \Omega = \begin{pmatrix} \cos(\vartheta/2) & -\exp(-\mathrm{i}\varphi)\sin(\vartheta/2) \\ \exp(\mathrm{i}\varphi)\sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \in \mathrm{SU}(2) \,. \tag{7.31}$$

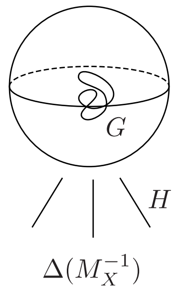
To get in Cartesian coordinates and for $g = 1/e$ (note, this is double the minimal Dirac charge) the result is

$$e W'^a_m = \varepsilon_{amn} \frac{x_n}{r^2} \,. \tag{7.32}$$

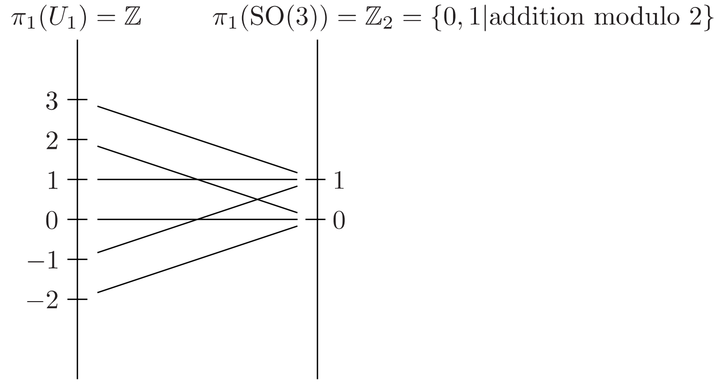
m is a spin index and a an isospin index. This is **nonsingular outside the core**. Hence, the Dirac string singularity has gone. There is something special in non-Abelian theories and therefore, one should consider such theories.

7.2 't Hooft-Polyakov magnetic monopole

For this section one can read [11, 12]. From the Wu-Yang construction, we see that for the unbroken gauge group H, the monopole is characterized by the non-trivial element of $\pi_1(H)$. Now, suppose there is a larger broken gauge group $G \supset H$. Then, there is a natural homomorphism $\pi_1(H) \mapsto \pi_1(G)$. Basic idea: If one is precisely in the **kernel** of $\pi_1(H) \mapsto \pi_1(G)$, then the singular H field can be unwrapped by exiting G fields.



The "knot" in the H fields can be untied by using G fields, which are more numerous. If so, $E_{\mathrm{mag,mon}}$ may be **finite**. For SO(3) Yang-Mills-Higgs theory, we find also the double minimal Dirac charge (U(1) = SO(2) $\subset$ SO(3)).



7.3 Georgi-Glashow model

The underlying gauge theory is SO(3) and a triplet of Higgs is introduced. The Lagrange density is given by

$$\mathcal{L} = -\frac{1}{4}G^a_{\mu\nu}G^{a,\mu\nu} + \frac{1}{2}(D_\mu\phi^a)^2 - U(\phi)\,, \quad G^a_{\mu\nu} \equiv \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + e\varepsilon_{abc}W^b_\mu W^c_\nu\,. \tag{7.33}$$

The covariant derivative is

$$D_\mu\phi_a \equiv \partial_\mu\phi_a + e\varepsilon_{abc}W^b_\mu\phi^c\,, \tag{7.34}$$

and the potential is given by

$$U \equiv \frac{\lambda}{4}(\phi^2 - v^2)\,. \tag{7.35}$$

Spontaneous symmetry breaking through

$$\langle\phi_a\rangle = \begin{pmatrix}0\\0\\v\end{pmatrix}\,, v \neq 0\,, \tag{7.36}$$

such that $G = \mathrm{SO}(3) \mapsto H = \mathrm{SO}(2)$. For all $h \in H$ muss $h\langle\phi\rangle = \langle\phi\rangle$ it must hold that $h\langle\phi\rangle = \langle\phi\rangle$. So, there is only one generator that leaves the expectation value invariant.

$$X_3\langle\phi\rangle = \begin{pmatrix}0 & 1 & 0\\-1 & 0 & 0\\0 & 0 & 0\end{pmatrix}\cdot\begin{pmatrix}0\\0\\v\end{pmatrix} = 0\,. \tag{7.37}$$

Hence, the unbroken gauge group is SU(2). There are massive gauge bosons $\mathrm{W}^\pm$ with $m_\mathrm{W} = ev$, a massless gauge boson W^3 ("photon") with $m_\gamma = 0$. Furthermore, there exists a massive scalar H with $m_\mathrm{H} = \sqrt{2\lambda}v$. But, there is no Z_0, hence, the theory is not directly important. The field equations are given by

$$D^\mu G^a_{\mu\nu} = e\varepsilon_{abc}\left[G^b_{\mu\nu}W^{c,\mu} - (D_\mu\phi^b)\phi^c\right]\,, \tag{7.38}$$

$$D^\mu D_\mu\phi^a = e\varepsilon_{abc}\left[(D_\mu\phi^b)W^{c,\mu} + \lambda\phi^a(v^2 - \phi^2)\right]\,. \tag{7.39}$$

't Hooft and Polyakov made the following ansatz (units $M_\mathrm{W} = ev = 1$):

$$W^a_0 = 0\,, \quad eW^a_m = \varepsilon_{amn}\frac{x_n}{r^2}(1 - h(r))\,, \quad \phi^a = \frac{x^a}{r}\widetilde{g}(r)\,, \tag{7.40}$$

with $r^2 = x^2 + y^2 + z^2$. The boundary conditions are given by

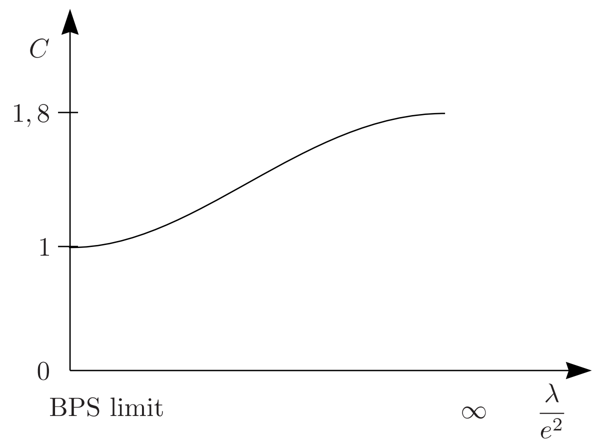
$$h(0) = 1\,, \quad h(\infty) = 0\,, \quad \widetilde{g}(0) = 0\,, \quad \widetilde{g}(\infty) = 1\,. \tag{7.41}$$

Inserting this in the field equations ($3\times3+3 = 12$ (reduction: $3\times3+1 = 10$) partial differential equations) reduce under the Ansatz to two ordinary differential equations. With $g \equiv r\widetilde{g}$ one obtains:

$$r^2g'' = g\left[2h^2 + \frac{\lambda}{e^2}(g^2 - r^2)\right]\,, \quad r^2h'' = h(h^2 - 1 + g^2)\,. \tag{7.42}$$

From Wu & Yang, we know that this is a magnetic monopole with charge $g = 1/e$. The above ordinary differential equations can be solves numerically. From the energy functional it follows that

$$E_{\mathrm{mag,mon}} = \frac{4\pi}{e^2}M_W C\left(\frac{\lambda}{e^2}\right)\,. \tag{7.43}$$



The analytic solution in the BPS limit is given by

$$\widetilde{g}_{\text{BPS}} = \frac{\cosh(r)}{\sinh(r)} - \frac{1}{r}\,, \quad h_{\text{BPS}} = \frac{r}{\sinh(r)}\,. \tag{7.44}$$

Bogomolny trick (just as for vortices):

$$E = \int \mathrm{d}^3x \left[\frac{1}{4}G^a_{ij}G^a_{ij} + \frac{1}{2}D_i\varphi_a D_i\varphi_a + U(\varphi)\right] = \int \mathrm{d}^3x \left[\frac{1}{4}(G^a_{ij} - \varepsilon_{ijk}D_k\varphi_a)^2 + \frac{1}{2}\partial_n J_n + U(\varphi)\right]\,, \tag{7.45}$$

with

$$\partial_n J_n = \varepsilon_{ijk}\partial_k(G^a_{ij}\varphi_a)\,. \tag{7.46}$$

One can interpret $G^a_{ij} - \varepsilon_{ijk}D_k\varphi_a$ as the Bogomolny equations, $1/2\partial_n J_n$ is a surface term and gives the magnetic charge by integration. $U(\varphi)$ vanishes in the BPS limit. If one finds a solution to the Bogomolny equations

$$B^a_n \equiv -\frac{1}{2}\varepsilon_{nij}G^a_{ij} = \pm D_n\varphi_a\,, \tag{7.47}$$

one obtains a solution for the second order differential equations. By inserting the HP Ansatz one gets the two equations

$$h' = -\widetilde{g}h\,, \quad \widetilde{g}' = \frac{1-h^2}{r^2}\,. \tag{7.48}$$

From that $C(0) = 1$ follows. "**Define**" an electromagnetic field $F_{\mu\nu}$. (The standard Maxwell equations are $\partial_\mu F^{\mu\nu} = J^\nu$ and $\partial_\mu \widetilde{F}^{\mu\nu} = 0$.)

$$F_{\mu\nu} \equiv \widehat{\varphi}^a G^a_{\mu\nu} - \frac{1}{e}\varepsilon_{abc}\widehat{\varphi}^a D_\mu\widehat{\varphi}^b D_\nu\widehat{\varphi}^c = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e}\varepsilon_{abc}\widehat{\varphi}^a(\partial_\mu\widehat{\varphi}^b)(\partial_\nu\widehat{\varphi}^c)\,, \tag{7.49}$$

with

$$\widehat{\varphi}^a = \frac{\varphi^a}{|\varphi|}\,, \quad A_\mu \equiv \widehat{\varphi}^a W^a_\mu\,. \tag{7.50}$$

Remarks:

1.) The above definition is SO(3) gauge-invariant.

2.) For $\varphi = (0,0,1)|\varphi|$ it holds that $F_{\mu\nu} = \partial_\mu W^3_\nu - \partial_\nu W^3_\mu$. Hence, the photon-like field buried in the magnetic monopole is W^3_μ.

For the 't Hooft-Polyakov Ansatz one finds

$$F_{0i} = E_i = 0\,, \quad F_{ij} = -\varepsilon_{ijk}\frac{1}{e}\frac{r^k}{r^3}\,. \tag{7.51}$$

$1/er^k/r^3$ is a magnetic monopole of charge $g = 1/e$.

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{e}\varepsilon_{abc}\widehat{\varphi}^a(\partial_\mu\widehat{\varphi}^b)(\partial_\nu\widehat{\varphi}^c)\,. \tag{7.52}$$

$$\partial_\mu\widetilde{F}^{\mu\nu} \equiv \partial_\mu\left(\frac{1}{2}\varepsilon^{\mu\nu\varrho\sigma}F_{\varrho\sigma}\right) = K^\nu = -\frac{1}{2e}\varepsilon^{\mu\nu\varrho\sigma}\varepsilon_{abc}\partial_\nu\widehat{\varphi}^a\partial_\varrho\widehat{\varphi}^b\partial_\sigma\widehat{\varphi}^c\,. \tag{7.53}$$

This is a **topological current**, which means automatic conservation $\partial_\mu K^\mu = 0$, regardless of solving the field equations. This then means that there is a constant charge:

$$Q = \int \mathrm{d}^3x\, K^0 = -\frac{1}{2e}\int_{S^2_\infty} \varepsilon_{ijk}\varepsilon_{abc}\widehat{\varphi}^a\partial_i\widehat{\varphi}^b\partial_k\widehat{\varphi}^c = \frac{1}{e}\times\text{winding number} \overset{\text{'t HP}}{=} \frac{1}{e}\,, \tag{7.54}$$

because for the 't Hooft-Polykov Ansatz we had that $\phi|_\infty$ is really a map $S^2_\infty \mapsto S^2$ (mod 2) and this map is nontrivial (hedgehog). Generally, the group G is broken to the group H. Then, the vacuum manifold $\widehat{\varphi}$ is given by the coset space G/H. For a nontrivial solution we need $\pi_2(G/H) \neq 1$. There exists a mathematical theorem: $\pi_2(G/H)$ is isomorphic to the kernel of the natural homomorphism of $\pi_1(H) \mapsto \pi_1(G)$. The Georgi-Glashow SU(3) model is not relevant for the electroweak interaction. Are there monopoles in other theories?

1.) In the electrowaek standard model the answer is "no, but a litte bit". This is easy to see for the so-called weak mixing angle $\theta_w = 0$. One has $G = \mathrm{SU}(2)$ and $H = 1$.

$$\pi_2(G/H) = \pi_2(\mathrm{SU}(2)) = \pi_2(S^3) = 1\,, \tag{7.55}$$

$$\widehat{\phi} \in S^3 \mapsto \pi_2(S^3) = 1\,. \tag{7.56}$$

However, there exists a magnetic **dipole** (sphaleron S). The magnetic monopoles are "confined".

2.) For the GUT theories the answer if "yes, if $\mathrm{U_{em}}(1)$ is embedded in a **simple** group. Then, it holds that

$$\pi_2(G^{\mathrm{simple}}/\mathrm{U_{em}}(1)) = \pi_1(\mathrm{U_{em}}(1))\,. \tag{7.57}$$

$$G = \mathrm{SO}(10) \xrightarrow{M_\mathrm{X}} \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}_Y(1) \xrightarrow[M_\mathrm{W}]{} \mathrm{U_{em}}(1)\,. \tag{7.58}$$

The mass of the monopole is gigantic:

$$M_{\mathrm{mag,mon}} \sim \frac{M_\mathrm{X}}{\frac{g^2_{\mathrm{GUT}}}{4\pi}} \sim 10^{-6}\,\mathrm{g}\left(\frac{M_\mathrm{X}}{10^{16}\,\mathrm{GeV}}\right)\,. \tag{7.59}$$

This leads to the so-called primordial monopole disaster.

7.4 Primordial Monopole Disaster

Recap magnetic monopole: In SO(3) Yang-Mills-Higgs theory with

$$G = \mathrm{SO}(3) \xrightarrow[M_\mathrm{W}=ev]{} H = \mathrm{U}(1)\,, \tag{7.60}$$

and a Higgs triplet. There exists a classical solution of the field equations with:

1.) It is a non-perturbative effect: $E = \frac{M_\mathrm{W}}{\alpha} C\left(\frac{\lambda}{\alpha}\right)$, $\alpha \equiv \frac{e^2}{4\pi} \neq 0$

2.) It has a finite size: $R_{\mathrm{con}} = \mathcal{O}(M_\mathrm{W}^{-1})$

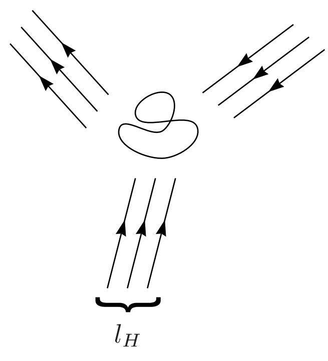
3.) It has a magnetic charge: $g = 1/e$ ($r \gg R_{\mathrm{con}}$: Coulomb like magnetic field)

4.) $\phi^\infty(\vartheta, \varphi) \in$ vacuum manifold $\equiv \{\phi | V(\phi) = 0\} = G/H$ for the existence require a nontrivial mapping, possible if $\pi_2(G/H) \neq 1$

Set now $\hbar = c = k = 1$ and $M_{\mathrm{pl}} \equiv 1/\sqrt{G_N} \sim 10^{19}\,\mathrm{GeV}$. We want to make a suggestion about how many monopoles there exist nowadays. **Expansion** of the universe:

$$\frac{1}{t^2} = \frac{NT^4}{M^2_{\mathrm{pl}}}\,, \quad N \sim 1000\,. \tag{7.61}$$

The horizon distance is $l_H = t$. There is a phase transition at $T \sim M_\mathrm{X} \sim 10^{16}\,\mathrm{GeV}$ (disorder).



A phase transition combined with causality cannot make the universe homogeneous. Over typical distances of the horizon at that time there exists misalignment and there is a change that there is a non-trivial mapping of the vacuum manifold. The result of that is a deformed monopole, where perhaps the Coulomb-type hair is a little bit changed around.
So, we wan calculate the number density of magnetic monopoles [13].

$$\frac{n_{\text{mag,mon}}}{n_\gamma} \sim \frac{l_H^{-3}}{M_{\text{X}}^3} \sim \frac{1}{t^3 M_{\text{X}}^3} \sim N^{\frac{3}{2}} \left(\frac{M_{\text{X}}}{M_{\text{pl}}}\right)^3 \approx 10^{-6} \,. \tag{7.62}$$

Nowadays, n_γ is given by the 3K Cosmic Background Radiation: $n_\gamma(\text{now}) \approx 100\,\text{cm}^{-3}$. The ratio does not change under the expansion of the universe. Let us calculate the density of magnetic monopoles:

$$\varrho_{\text{mag,mon}}(\text{now}) = m_{\text{mag,mon}} \cdot \frac{n_{\text{mag,mon}}}{n_\gamma} \cdot n_\gamma(\text{now}) \sim 10^{-6}\,\text{g} \cdot 10^{-6} \cdot 100\,\text{cm}^{-3} \approx 10^{-10}\,\frac{\text{g}}{\text{cm}^3} \,. \tag{7.63}$$

But, the observed density of the universe is given by

$$\varrho_{\text{obs}} \sim \varrho_c \approx 10^{-29}\,\frac{\text{g}}{\text{cm}^3} \,. \tag{7.64}$$

So, that is a contradiction, called the primordial monopole disaster. Inflation was introduced in order to cure that problem.

Chapter 8

Instantons

8.1 Definition

An instanton is a localized finite-action classical solution in Euclidian field theory. Here, we want to consider SU(2) Yang-Mills-Higgs theory over the manifold $M = \mathbb{R}^4$. (It is localized in Euclidian time. That is why 't Hooft gave it the name "instanton".) It holds that "∂M"$= S^3$ and for any simple Lie group: $\pi_3(G) \neq 1$. (This is remarkable and the reason, why we can focus on SU(2).) The nontrivial behavior of the gauge fields at infinity leads to the instantons.

8.2 BPST instanton

The reference to read is [14]. It holds that $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ and $G = \mathrm{SU}(2)$. We will write the gauge fields as $A_\mu \equiv A_\mu^a \sigma^a/2$. The Yang-Mills field strength tensor is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g[A_\mu, A_\nu] \equiv \frac{\mathrm{i}}{g}[\mathscr{D}_\mu, \mathscr{D}_\nu]\,, \quad \mathscr{D}_\mu \equiv \partial_\mu - \mathrm{i}gA_\mu\,. \tag{8.1}$$

Furthermore, we introduce

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma}F_{\varrho\sigma} \Rightarrow \widetilde{\widetilde{F}}_{\mu\nu} = F_{\mu\nu}\,. \tag{8.2}$$

The gauge transformation is

$$A'_\mu = \Omega\left(A_\mu + \frac{\mathrm{i}}{g}\partial_\mu\right)\Omega^{-1}\,, \quad F'_{\mu\nu} = \Omega F_{\mu\nu}\Omega^{-1}\,. \tag{8.3}$$

The Euclidian action is given by

$$S_E = \frac{1}{2}\int \mathrm{d}^4x\ (-\mathrm{Tr}(F_{\mu\nu}F_{\mu\nu})) \geq 0\,. \tag{8.4}$$

From that, the field equations follow:

$$D_\mu F_{\mu\nu} = 0\,, \tag{8.5}$$

with

$$D_\mu A_\nu = \partial_\mu A_\nu - \mathrm{i}g[A_\mu, A_\nu]\,, \tag{8.6}$$

$$D_\mu F_{\varrho\sigma} = \partial_\mu F_{\varrho\sigma} - \mathrm{i}g[A_\mu, F_{\varrho\sigma}]\,. \tag{8.7}$$

The field equations (8.5) are $4 \times 3 = 12$ second order partial differential equations for A_μ. We immediately realize the vacuum solution $A_\mu = 0$. The question is, if there are other solutions and the answer is "Yes, by construction!". For $x^2 \equiv x_1^2 + x_2^2 + x_3^2 + x_4^2 \mapsto \infty$ the finite-action condition requires $\overline{F} \mapsto 0$, which indicates the BPST solution. Hence, it must hold

$$\overline{A}_\mu \mapsto -\frac{\mathrm{i}}{g}\partial_\mu \overline{\Omega}\overline{\Omega}^{-1}\,, \tag{8.8}$$

at infinity. Take $\overline{\Omega}$ topologically **nontrivial**:

$$\overline{\Omega} = \widehat{x}_a \mathrm{i}\sigma_a + \widehat{x}_4 \mathbb{1}\,, \quad \widehat{x}_\mu \equiv \frac{x_\mu}{|x|}\,, \tag{8.9}$$

which is a mapping of $S_3 \mapsto S_3$ with unit winding number. (That means $\overline{\Omega} \not\sim \overline{\overline{\Omega}} = 1$.) We make the PBST Ansatz

$$\overline{A}_\mu = \left(-\frac{\mathrm{i}}{g}\partial_\mu \overline{\Omega}\overline{\Omega}^{-1}\right) f(x)\,. \tag{8.10}$$

The boundary conditions are $f(\infty) = 1$ and $f(0) = 0$. That is of order $\mathcal{O}(1/g)$. g is fixed by the non-Abelian structure of the theory. The first boundary conditions comes from the finite-action requirement and the second from claiming continuity.
Now, require $\overline{A}$ to solve the self-duality equations

$$F_{\mu\nu} = \widetilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\varrho\sigma} F_{\varrho\sigma}\,, \tag{8.11}$$

which are 6×3 **first** order partial differential equations. With $\overline{A}$ inserted we find that the self-duality equations reduce to a single ordinary differential equation:

$$f' = \frac{2f}{x}(1-f)\,. \tag{8.12}$$

This equation has the solution

$$\overline{f} = \frac{x^2}{x^2+\varrho^2}\,, \tag{8.13}$$

with ϱ being a constant > 0. The second order field equations are then solved **automatically** due to the non-Abelian Bianchi identity

$$D_\mu \widetilde{F}_{\mu\nu} = 0\,, \tag{8.14}$$

which follows from the Jacobi identity

$$[D_\mu, [D_\nu, D_\varrho]] + \text{cycl.} = 0\,, \tag{8.15}$$

and is a general property of the theory. The summary of our logic is as follows. By using the self-duality equations it holds that

$$D_\mu \overline{F}_{\mu\nu} = D_\mu \widetilde{\overline{F}}_{\mu\nu} = 0\,. \tag{8.16}$$

Now we are coming to some remarks:

1.) We are not working in **Minkowski space**. In Minkowski space it would hold that $\widetilde{\widetilde{F}} = -F$ (due to $\varepsilon^{0123} = 1$, $\varepsilon_{0123} = -1$). Then, self-duality is mirrored in $F = \widetilde{F}$. From that it follows that $\widetilde{F} = \widetilde{\widetilde{F}} = -F$ and therefore $F = 0$ and $S_E = 0$.

2.) There is again something like to Bogomolnyi trick behind the BPST instanton, namely "topology" and "squares".

   – All finite action Yang-Mill fields are classified by the so-called Pontyagin index.

$$q \equiv \frac{g^2}{16\pi^2}\int \mathrm{d}^4x \left\{-\mathrm{Tr}(F_{\mu\nu}\widetilde{F}_{\mu\nu})\right\}\,. \tag{8.17}$$

   q is a topological invariant an it is $\in \mathbb{Z}$. From all configurations that are reached by Feynman diagrams one obtains zero. These are topologically trivial. We want to proof that q is a topological invariant, using the techniques from differential forms. A will now be a Lie algebra valued 1-form. We use the exterior derivative

$$\mathrm{d} \equiv \mathrm{d}x^\mu \frac{\partial}{\partial x^\mu}\,, \tag{8.18}$$

with the remarkable property $d^2 = 0$. The so-called wedge product is denoted by "$\wedge$" and defined as follows:

$$F = \mathrm{d}A + A \wedge A\,. \tag{8.19}$$

It is a way to combine forms to get a higher order form. It is a skew-symmetric bilinear operation. (In this way anomalies can be well understood.) The gauge field is a Lie algebra valued 1-form:

$$A = A_\mu\,\mathrm{d}x^\mu = A^a_\mu(x) T^a\,\mathrm{d}x^\mu\,. \tag{8.20}$$

By using

$$\mathrm{d}F = 0 + \mathrm{d}A \wedge A - A \wedge \mathrm{d}A = F \wedge A - A \wedge F\,, \tag{8.21}$$

one finds the Bianchi identity

$$DF \equiv \mathrm{d}F + [A, F] = 0\,. \tag{8.22}$$

The definition of the Pontyagin charge is

$$q = \frac{1}{8\pi^2}\int \varrho\,, \tag{8.23}$$

with the 4-form

$$\varrho \equiv \mathrm{Tr}(F \wedge F)\,. \tag{8.24}$$

Under the change $A \mapsto A + \delta A$ one obtains $\delta q = 0$. For $F \mapsto F + \delta F$ we obtain

$$\delta F = \mathrm{d}\delta A + A \wedge \delta A + \delta A \wedge A\,, \tag{8.25}$$

and

$$\delta\varrho = 2\mathrm{Tr}(F \wedge \delta F) = 2\mathrm{Tr}(F \wedge \mathrm{d}\delta A) + 2\mathrm{Tr}\left\{(F \wedge A - A \wedge F) \wedge \delta A\right\}\,, \tag{8.26}$$

where we have used the cyclicity of the trace. Furthermore the second term can be rewritten by using the Bianchi identity

$$\delta\varrho = 2\mathrm{Tr}(F \wedge \mathrm{d}\delta A) + 2\mathrm{Tr}\left\{\mathrm{d}F \wedge \delta A\right\} = 2\mathrm{d}(\mathrm{Tr}(F \wedge \delta A))\,. \tag{8.27}$$

This is a total derivative. Therefore we obtain

$$\delta q = \int \delta\varrho = 0\,, \tag{8.28}$$

if there is no boundary or if δA vanishes on the boundary.
In fact, there is a so-called Chern-Simons current

$$K_\mu = -\varepsilon_{\mu\nu\kappa\lambda}\mathrm{Tr}\left\{\frac{1}{2}A_\nu \partial_\kappa A_\lambda - \frac{\mathrm{i}g}{3}A_\nu A_\kappa A_\lambda\right\}\,, \tag{8.29}$$

with

$$\partial_\mu K_\mu = -\frac{1}{4}\mathrm{Tr}(F\widetilde{F})\,. \tag{8.30}$$

It follows that

$$q = \frac{g^2}{4\pi}\int_{S^3} K_\mu\,\mathrm{d}^3 S^\mu \in \mathbb{Z}\,, \tag{8.31}$$

because it is the winding number. q is gauge-invariant, but K_μ **not**.

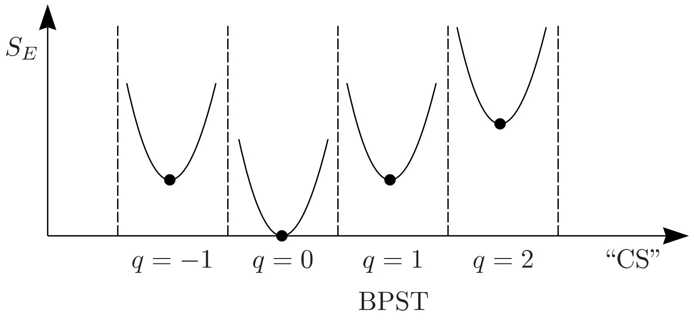
– For $q \geq 0$ we can write up the following identity:

$$\int \mathrm{d}^4x\,\frac{1}{4}\left\{-\mathrm{Tr}[(F - \widetilde{F})(F - \widetilde{F})]\right\} \geq 0\,. \tag{8.32}$$

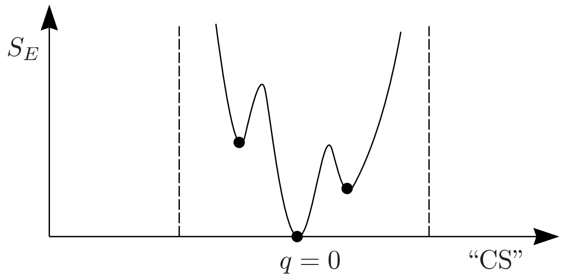
$$\int \mathrm{d}^4x\,\frac{1}{2}\left\{-\mathrm{Tr}(FF)\right\} \geq \frac{1}{2}\int \mathrm{d}^4x\,\left\{-\mathrm{Tr}(F\widetilde{F})\right\}\,, \tag{8.33}$$

where the first term is the Euclidian action and the second term is equal to $8\pi^2/g^2 q$. For $q \leq 0$ use $\mathrm{Tr}[(F + \widetilde{F})(F + \widetilde{F})]$ and obtain $S_E \geq -8\pi/g^2 q$ and therefore

$$S_E \geq \frac{8\pi^2}{g^2}|q|\,. \tag{8.34}$$



3.) If one has a self-dual solution, the field equations are automatically solved. **All** self-dual solutions are known. That goes under the name of the ADHM-construction [15], what leads to the Atiyah-Jones conjecture [16]: These ADHM solutions are all there is. This would mean, that for SU(2) Yang-Mills theory there are self-duality equations that are equivalent to the field equations. For the vacuum sector ($q = 0$) there would be only one solution: $A = 0$. However, in 1989, the Atiyah-Jones conjecture was proven false: There are infinitely many non-self-dual solutions of the field equations.

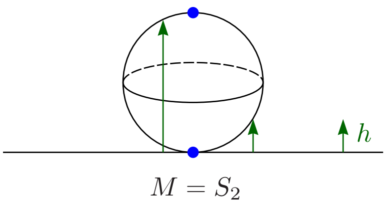


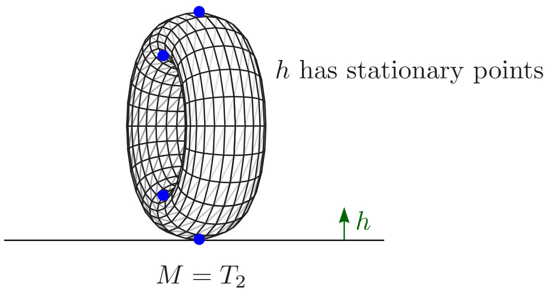
These are the "sphalerons" of $\mathbb{R}^4$ Yang-Mills theory.

Chapter 9

Sphalerons

Are there classical solutions in the electroweak standard model? The answer ist “Yes, but not those discussed up till now.” Most likely, there are “no topological solitons in the electroweak standard model (QCD)”. The topology rises in another way. There is an inspiration from mathematics, namely the Morse theory. We have a manifold M and an function h over M. Morse theory relates topology of M to stationary points of h. In the following discussion the S_2 and T_2 will be considered. The value of the function h shall be the height to the reference point. If M is **compact**, h has a maximum and a minimum.

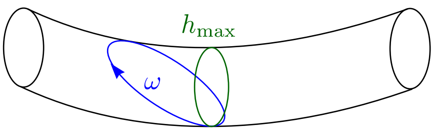




The existence of stationary points can be **proven** by the **mini-max** procedure. In physics, but in particular particle physics, for M one takes the configuration space $M \equiv \{\text{space of static finite-energy field configurations}\}$. Furthermore, choose h as the energy functional

$$E = \int \mathrm{d}^3x \, e[\text{fields}] \, . \tag{9.1}$$

The manifold M is infinite-dimensional and non-compact.



The electroweak standard model is given by

$$e = -\frac{1}{2g^2}\mathrm{Tr}(W_{mn}^2) + \frac{1}{4g'^2}B_{mn}^2 + |D_m\phi|^2 + V(\phi) + \text{fermionic terms} \, , \tag{9.2}$$

with

$$W_m \equiv W_m^a \frac{\sigma^a}{2\mathrm{i}} \, , \quad W_{mn} \equiv \partial_m W_n - \partial_n W_m + [W_m, W_n] \, , \quad B_{mn} \equiv \partial_m B_n - \partial_n B_m \, , \tag{9.3}$$

$$D_m\phi \equiv \left(\partial_m + W_m + \frac{1}{2\mathrm{i}}B_m\right)\phi \, , \quad V \equiv \lambda\left(|\phi|^2 - \frac{v^2}{2}\right) \, , \quad \Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \, . \tag{9.4}$$

The mass of the W boson is given by

$$M_\mathrm{W} = \frac{1}{2}gv \, , \tag{9.5}$$

$$M_\mathrm{H} = \sqrt{2\lambda}v \, , \tag{9.6}$$

and the weak mixing angle by $\tan(\theta_w) = g'/g$. Start with the SU(2) Yang-Mills-Higgs theory $\{W, \Phi\}$. Hence, v sets the energy scale in this theory. For the solutions we find the fermions give zero contribution. Hence, only the gauge fields are important here. For the construction of the sphaleron read [17].
We perform a mini-max procedure over a non-contractable loop in configuration space. Nontrivial mapping: U: $S_1 \times S_2 \mapsto \mathrm{SU}(2) = S_3$ (S_1: $\omega \in [0, \pi]$, S_2: $\{\vartheta, \varphi\}$, S_3: $\{S_\mu\}$). The S_1 and the S_2 can be smashed in a S_3 and it holds that $\pi_3(S_3) = \mathbb{Z}$.

$$U = U(\omega, \vartheta, \varphi) \equiv S_\mu \sigma_\mu\,, \quad \sigma_\mu \equiv (\mathbb{1}_2, \boldsymbol{\sigma})\,. \tag{9.7}$$

Concretely:

$$S_0 = \cos^2\omega + \sin^2\omega\cos\vartheta\,, \quad S_1 = \sin\omega\sin\vartheta\sin\varphi\,, \quad S_2 = \sin\omega\sin\vartheta\cos\varphi\,, \quad S_3 = \sin\omega\cos\omega(1-\cos\vartheta)\,. \tag{9.8}$$

We are coming to some remarks:

1.) $U(0/\pi, \vartheta, \varphi) = \mathbb{1}_2$ (trivial configuration)

2.) $U(\omega, 0/\pi, \varphi)$ is φ-independent

3.) covers S_3 once

The energy density is spherical symmetric:

$$E = \int \mathrm{d}^3x\, e = \frac{v}{g}\int \xi^2\,\mathrm{d}\xi\,\sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\, e(\xi)\,, \quad \xi = rgv\,. \tag{9.9}$$

Now, one can write up, what a non-contractable loop is.

- phase I^a (Higgs build-up): $\omega = \left[-\frac{\pi}{2}, 0\right]$

$$W = 0\,, \quad B = 0\,, \quad \Phi = \frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ \sin^2\omega + h\cos^2\omega \end{pmatrix}\,. \tag{9.10}$$

- phase II (gauge field build-up and removal): $\omega = [0, \pi]$

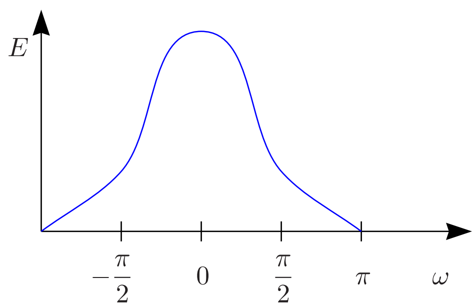
$$W_n = -f(\partial_m U)U^{-1}\,, \quad \Phi = \frac{v}{\sqrt{2}} hU \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,, \tag{9.11}$$

  with radial functions h and f with boundary conditions $f(0) = h(0) = 0$, $f(\infty) = h(\infty) = 1$.

$$e = \frac{1}{2}h'^2 + \frac{1}{4}\frac{\lambda}{g^2}(h^2-1)^2 + \sin^2\omega\frac{4}{\xi^2}f'^2 + \sin^4\omega\frac{8}{\xi^4}f^2(1-f)^2 + \sin^2\omega\frac{h^2}{\xi^2}(1-f^2)\,. \tag{9.12}$$

- phase I^b (Higgs removal): $\omega = \left[\pi, \frac{3\pi}{2}\right]$

We have a **manifest** maximum of $E(\omega)$ at $\omega = \pi/2$.



So, the **maximum** of E is at $\omega = \pi/2$, but it is still a functional of $f(r)$ and $h(r)$. Now, **minimize** over f and h (mini-max procedure). Solve the variational equations of the quantity $E(\pi/2) = E_{\pi/2}[f, h]$. These are two ordinary differential equations for f and h with the solutions $\overline{f}$ and $\overline{h}$. Now, the non-contractable loop with $\overline{f}$ and $\overline{h}$ **catches** a new stationary point: sphaleron S. The name “sphaleron” comes from Greek and means “ready to fall”.

9.1 Applications

S^* plays a role at very high-energy collisions, where perturbation theory breaks down. The relevant energy scale will be 20 TeV. At such an energy new effect will occur, which could have been measured by the SSC (Superconducting Super Collider), but not by the LHC! S is related to Baryon number violation, which has to do with the Adler-Bell-Jackiw anomaly. For $B + L$ violation instantons give a contribution

$$\Gamma \sim \exp(-2I_{\mathrm{BPST}}) = \exp\left(-\frac{16\pi^2}{g^2\hbar}\right) \sim 10^{-164}\,. \tag{9.13}$$

Sphalerons give

$$\Gamma \sim T^4 \exp\left(-\frac{E_S}{k_B T}\right)\,, \tag{9.14}$$

with $E_S \sim 10\,\mathrm{TeV}$. For large enough T the theory will be unsuppressed. Therefore, in early universe the contribution was very strong, but nowadays it is practically zero.

Bibliography

[1] S. Weinberg, "*A Model Of Leptons*," Phys. Rev. Lett. **19**, 1264 (1967).

[2] C. Bouchiat, J. Iliopoulos and P. Meyer, "*An Anomaly Free Version Of Weinberg's Model*," Phys. Lett. B **38**, 519 (1972).

[3] R. P. Feynman, "*Space-time approach to quantum electrodynamics*," Phys. Rev. **76**, 769 (1949).

[4] S. R. Coleman, "*Quantum sine-Gordon equation as the massive Thirring model*," Phys. Rev. D **11**, 2088 (1975).

[5] S. Mandelstam, "*Soliton operators for the quantized sine-Gordon equation*," Phys. Rev. D **11**, 3026 (1975).

[6] G. H. Derrick, "*Comments on nonlinear wave equations as models for elementary particles*," J. Math. Phys. **5**, 1252 (1964).

[7] E. B. Bogomolny, "*Stability Of Classical Solutions*," Sov. J. Nucl. Phys. **24**, 449 (1976) [Yad. Fiz. **24**, 861 (1976)].

[8] H. J. de Vega and F. A. Schaposnik, "*A Classical Vortex Solution Of The Abelian Higgs Model*," Phys. Rev. D **14**, 1100 (1976).

[9] T. T. Wu and C. N. Yang, "*Concept of nonintegrable phase factors and global formulation of gauge fields*," Phys. Rev. D **12**, 3845 (1975).

[10] P. A. M. Dirac, "*Quantised singularities in the electromagnetic field*," Proc. Roy. Soc. Lond. A **133**, 60 (1931).

[11] G. 't Hooft, "*Magnetic monopoles in unified gauge theories*," Nucl. Phys. B **79**, 276 (1974).

[12] A. M. Polyakov, "*Particle spectrum in quantum field theory*," JETP Lett. **20** (1974) 194 [Pisma Zh. Eksp. Teor. Fiz. **20** (1974) 430].

[13] T. W. B. Kibble, "*Topology Of Cosmic Domains And Strings*," J. Phys. A **9**, 1387 (1976).

[14] A. A. Belavin, A. M. Polyakov, A. S. Shvarts and Yu. S. Tyupkin, "*Pseudoparticle solutions of the Yang-Mills equations*," Phys. Lett. B **59**, 85 (1975).

[15] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Yu. I. Manin, "*Construction of instantons*," Phys. Lett. A **65**, 185 (1978).

[16] M. F. Atiyah and J. D. S. Jones, "*Topological Aspects Of Yang-Mills Theory*," Commun. Math. Phys. **61**, 97 (1978).

[17] F. R. Klinkhamer and N. S. Manton, "*A Saddle Point Solution In The Weinberg-Salam Theory*," Phys. Rev. D **30**, 2212 (1984).

Literature

1.) S. Coleman: " Aspects of Symmetry" (Chapters 6 - 7), CUP, 1985

2.) N. Manton, P. Sutcliffe: " Topological Solitons", CUP, 2004

3.) Ryder: " Quantum Field Theory" (Chapter 10), CUP, 1985

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