Mitschrift zur Vorlesung: Riemannian Geometry

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Vorlesung Wintersemester 2009/2010

Letzte Aktualisierung und Verbesserung: 14. Februar 2010

Mitschrieb der Vorlesung Riemannian Goemetry
von Herrn Prof. Dr. Leuzinger im Wintersemester 2009/2010
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

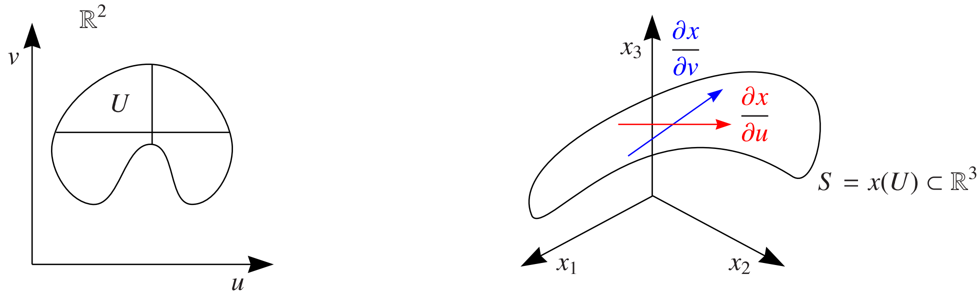
Inhaltsverzeichnis

Kapitel 1

Some History (as motivation)

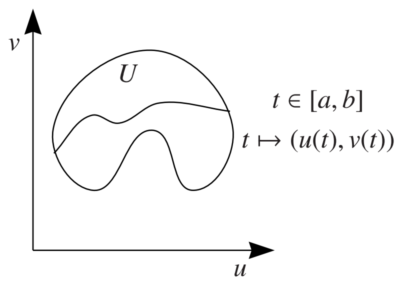
- Gauß studies in 1827: inner geometry of surfaces in $\mathbb{R}^3$

  Consider a map $x \colon U \subset \mathbb{R}^2 \mapsto \mathbb{R}^3$, which is supposed to be smooth and to have rank $= 2$. This means that the tangent vectors $\partial x/\partial u$ and $\partial x/\partial v$ should be linear independent. U be an open subset. (Think of a unit ball/disk.)



$$\frac{\partial x}{\partial u} = \left(\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u}\right), \quad \frac{\partial x}{\partial v} = \left(\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v}\right). \tag{1.1}$$

The length of a curve on S shall be computed:



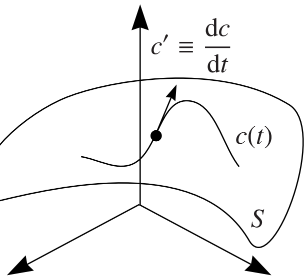
If we consider the curve $c(t) := x(u(t), v(t)) \subset S \subset \mathbb{R}^3$, the length is defined by:

$$L(c) := \int_a^b \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\| \mathrm{d}t\,. \tag{1.2}$$

With

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial t}\,, \tag{1.3}$$

one obtains:



$$\left\|\frac{\mathrm{d}c}{\mathrm{d}t}\right\|^2 \equiv c' = \langle c', c'\rangle = \left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial ju}\right\rangle (u')^2 + 2\left\langle \frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}\right\rangle u'v' + \left\langle \frac{\partial x}{\partial v}, \frac{\partial x}{\partial v}\right\rangle (v')^2 :=$$
$$:= E(u,v)(u')^2 + 2F(u,v)u'v' + G(u,v)(v')^2 \,. \tag{1.4}$$

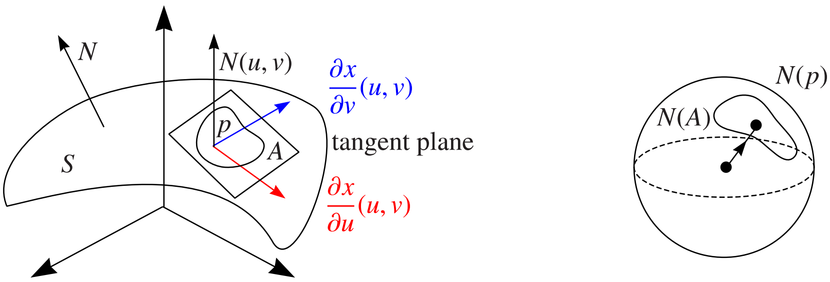
To compute the length of the curve, one needs the functions $E(u,v)$, $F(u,v)$, and $G(u,v)$:

$$L(c) = \int_a^b \sqrt{E(u')^2 + 2Fu'v' + G(v')^2}\,\mathrm{d}t\,. \tag{1.5}$$

This was the discovery made by Gauß. This can also be formulated in a different manner by introducing the first Fundamental-Form, which is a family of symmetric 2×2 matrices (for every u and v)

$$\begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix}\,, \tag{1.6}$$

of scalar products, which is positive definite. You do not need anything about the surface and that was the starting point for Riemann. He formulated a research program that is nowadays called the inner geometry of a surface. This project was formulated on the concept of **Gauß-curvature**.



Parallel-translate any unit vector to the sphere. Hence, every point $p \in S$ is mapped on a unit vector $N(p)$ on the sphere:

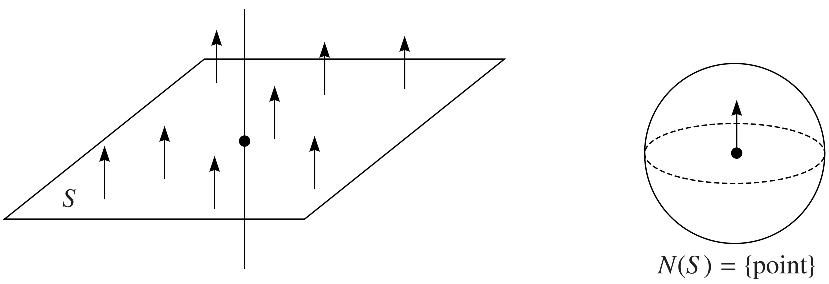
$$S \mapsto S^2(\text{unit sphere in } \mathbb{R}^3)\,, \tag{1.7}$$

with $S^2 = \{a \in \mathbb{R}^3 | \|a\| = 1\}$. Define

$$K(p) := \lim_{|A|\mapsto 0} \frac{|N(A)|}{A}\,, \tag{1.8}$$

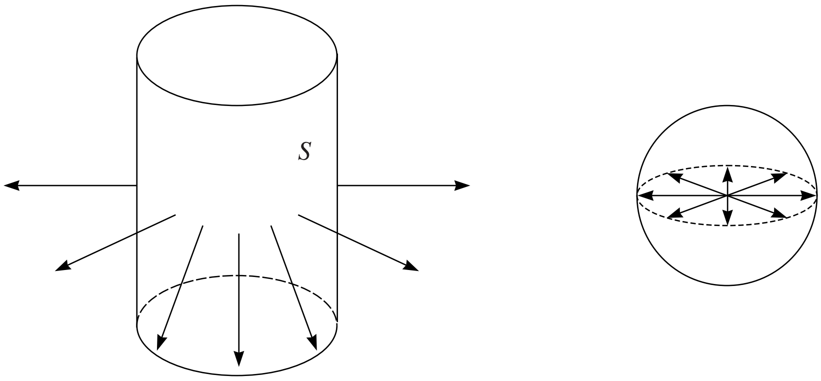
where $N(A)$ is the corresponding area on the two-sphere S^2. We are coming to some examples:

– Consider a plane:



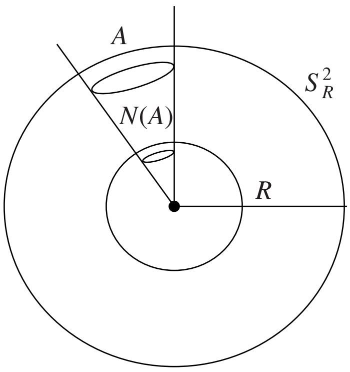
The area of a point on the two-sphere is just zero: $|N(A)| = 0$. Hence, the plane has constant Gauß curvature $K(p) = 0 \; \forall \; p$.

– Cylinder:



Hence, the map is a just a line on the two-sphere: $N(s) =$ equator. Again, $|N(A)| = 0$ and hence $K \equiv 0$. With this information about the surface one cannot distinguish, if it is a plane or a cylinder.

– Spheres of radius R:



Hence, the Gauß map is just the full S^2: $N(S_R^2) = S^2$. Since the image of A is also contracted one obtains for $R \mapsto \infty$: $K(S_R^2) \mapsto 0$. For $R \mapsto 0$ one obtains $K(S_R^2) \mapsto \infty$ and for $R = 1$ it follows that $K \equiv 1$ (since $N(A) = A$).

The Gauß curvature measures in some sense the deviation of the surface from a plane. A cylinder can be mapped isometrically (without distortion) on a plane. Gauß' discovery, which was the motivation for Riemann's programm, was the following: K is a quantity of the inner geometry of the surface, meaning that is can be computed by the scalar products of the tangent vectors spanning the tangent plane, hence the functions $E(u, v)$, $F(u, v)$, and $G(u, v)$ and their derivatives (which is the definition for „inner geometry“) („theorema egregium“).

- Riemann 1854:

  Generalize "surfaces"! The generalization is to forget about the two dimensions and about the embedding. This leads to the concept of smooth manifolds. Also generalize the first Fundamental-Form, which is called the Riemannian metric, which means that one has a family of scalar products on tangent spaces at each points.

- Poincaré $\sim$ 1900:

  Introduce topological concepts in particular to control the connection between local and global properties.

- Weyl $\sim$ 1910: Introduction of the modern concept of a **manifold**.

- Einstein $\sim$ 1915: Use Riemannian geometry as mathematical foundation of general relativity. The universe is modeled as a Riemannian manifold with field equations of the following type: $G = KT$, whereas G involves curvature and T is a mass/energy distribution.

Kapitel 2

What is a manifold?

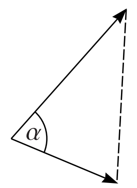
2.1 Differentiable manifolds

The basic idea is that a manifold is a space that locally looks like $\mathbb{R}^n$. First, we will repeat some concepts of linear algebra and analysis. Consider a n-dimensional **Euclidian space** $(\mathbb{R}^n, \langle , \rangle)$, where the scalar product is given by

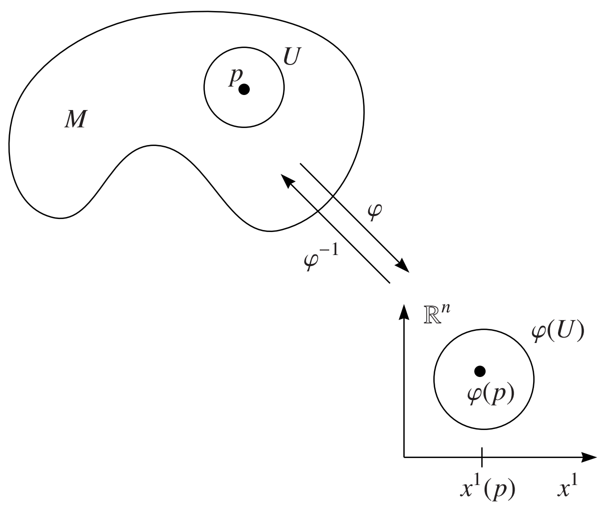
$$\langle a, b \rangle = \sum_{i=1}^{n} a_i b_i \,, \quad a = (a_1, \ldots, a_n) \,, \quad b = (b_1, \ldots, b_n) \in \mathbb{R}^n \,. \tag{2.1}$$

Define a norm by $\|a\| = \sqrt{\langle a, a \rangle}$ and the distance by $d(a, b) = \|a - b\|$. The angle between two vectors a and b is given by

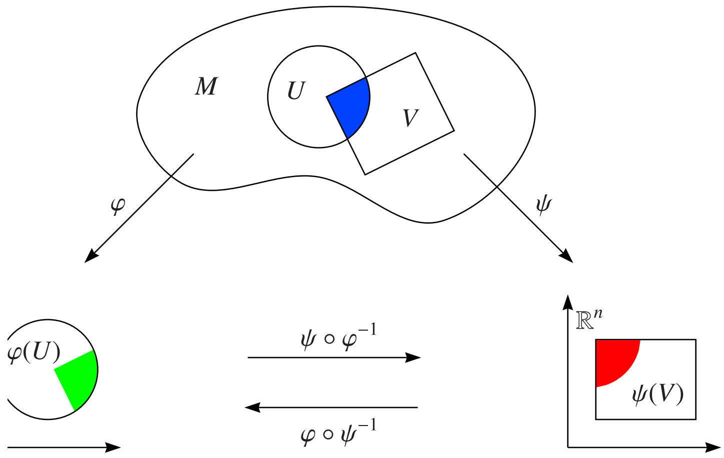
$$\cos(\alpha) = \frac{\langle a, b \rangle}{\|a\| \cdot \|b\|} \,. \tag{2.2}$$



Hence, $(\mathbb{R}^n, d)$ is a metric space. A function f: $(U \subset \mathbb{R}^n) \mapsto \mathbb{R}$ (for an open subset U) is **smooth** (or C^∞), if for every $p \in U$ all mixed partial derivatives exist and are continuous. The C^∞-functions u^i: $\mathbb{R}^n \mapsto \mathbb{R}$ ($i = 1$, …, n) with $u^i(p = (p_1, \ldots, p_n)) = p_i$ are called **coordinate functions**. A map ϕ: $(U \subset \mathbb{R}^n) \mapsto \mathbb{R}^n$ (of an open subset U) is **smooth**, if all real-valued functions $u^i \circ \phi$ are smooth for $i = 1$, …, n.
Let M be a topological space, which is Hausdorff (for example a metric space) and has a countable basis. (The Hausdorff condition is a separation condition. It means that if one chooses different points p and q on a space, there exist disjunct sets around these points.) A **coordinate system** (or **chart**) at p on M is a homeomorphism φ: $(U \subset M) \mapsto \varphi(U) \subset \mathbb{R}^n$ (with $p \in U$, whereas U is open). Homeomorphism means a mapping that is continuous and bijective such that its inverse is also continuous.



If we write $\varphi(p) = (x^1(p), \ldots, x^n(p))$ for all $p \in U$, then the functions x^i: $u^i \circ \varphi$ are called **coordinate functions** of φ. n is called the **dimension** of M. **Charts** are **compatible** in the following sense: If φ: $U \mapsto \mathbb{R}^n$ and ψ: $V \mapsto \mathbb{R}^n$ are two charts on M with $U \cap V \neq \emptyset$, then we have **coordinate changes** $\psi \circ \varphi^{-1}$: $\varphi(U \cup V) \mapsto \psi(U \cap V)$ and $\varphi \circ \psi^{-1}$: $\psi(U \cap V) \mapsto \varphi(U \cap V)$.



Hence, manifolds are modeled locally by charts.

Be M a topological space, for example M is a set equipped with a system $\mathcal{T}$ of subsets (called the **open subsets** of M), such that the following holds:

1) $M, \emptyset \in \mathcal{T}$

2) Finite intersections and units of arbitrary elements of $\mathcal{T}$ belong to $\mathcal{T}$. (The family of subsets if closed under ...)

$\mathcal{T}$ is called **topology** on M. This definition is very general and we impose additional properties: In addition M is supposed to be connected (it is not split into open subsets), Hausdorff and is should have a countable basis. A n-dimensional **differentiable atlas** on M is an collection $\mathcal{A} \subset \mathcal{T}$ of n-dimensional charts such that the following two conditions are satisfied:

$\mathcal{A}$1) Every point of M lies in at least one chart (in other words, M is "locally Euclidian" and covered by charts).

$\mathcal{A}$2) All possible coordinate changes are **smooth** (C^∞).

A chart φ of M is **compatible** with an atlas $\mathcal{A}$, if $A \cup \{\varphi\}$ is still a differentiable atlas for M. $\mathcal{A}$ is called **maximal** (**or differentiable structure for** M), if every charge compatible with $\mathcal{A}$ already belongs to $\mathcal{A}$. A differentiable atlas $\mathcal{A}$ uniquely determines a maximal atlas $\mathcal{A}^*$.

Definition

A **n-dimensional, differentiable manifold** is a topological space, which is connected, Hausdorff, countable basis and is equipped with a maximal atlas.

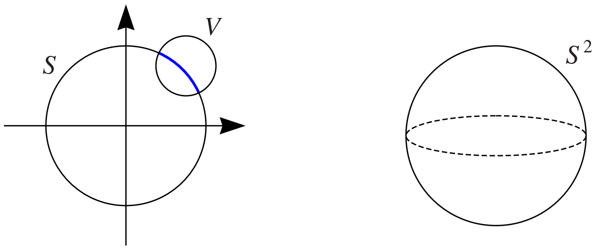
2.2 Examples

2.2.1 The n-dimensional real space

$\mathbb{R}^n$ is Hausdorff, because it is a metric space and two different points can be separated by two open balls. Furthermore, it has a countable basis, which consists of open balls with rational radius and rational center. We can start with an atlas $\mathcal{A} = \{\text{id}\}$, id: $\mathbb{R}^n \mapsto \mathbb{R}^n$. $\mathcal{A}^*$ is the maximal atlas. This is the standard smooth structure on $\mathbb{R}^n$. Remark: For $n \neq 4$ the standard differentiable structure is the only one on $\mathbb{R}^n$. But for $\mathbb{R}^4$ there exist different structures (the four dimensional Euclidian space is very special).

2.2.2 The unit sphere

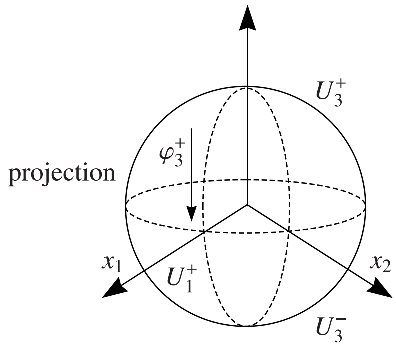
We want to consider $S^n := \{p \in \mathbb{R}^{n+1} | \|p\| = 1\}$.



They are subsets of $\mathbb{R}^{n+1}$. Since $\mathbb{R}^{n+1}$ has a topology, the subspace S^n also has a topology, the so-called subset topology induced from $\mathbb{R}^{n+1}$. This means that $U \subset S^n$ is open if and only if by definition there exists and open subset V in $\mathbb{R}^{n+1}$ such that $U = S^n \cap V$. It is an exercise to show that it is connected, Hausdorff and that is has a countable basis. (Use the **fact**: Subspaces of Hausdorff spaces with countable basis are also Hausdorff and have a countable basis.) Furthermore the sphere is a **compact** manifold.
Let us construct an atlas for S^n: For $i = 1, \ldots, n+1$ let U_i^+ (respectively U_i^-) be the open hemi-spheres

$$U_i^+ := \{p = (p_1, \ldots, p_{n+1}) \in S^n | p_i > 0\}\,, \quad U_i^- = \{p \in S^n | p_i < 0\}\,. \tag{2.3}$$

Illustration for $n = 2$:



One needs six hemispheres in this construction to cover the whole S^2. (U_3^+ and U_3^- miss, for example, the equator.) The **charts** are given by: $\varphi_i^\pm \colon U_i^\pm \mapsto \mathbb{R}^n$, which is the orthogonal projection in the direction of the i-th coordinate axis.

$$\varphi_i^\pm(p) = (u^1(p), \ldots, u^{i-1}(p), u^{i+1}(p), \ldots, u^{n+1}(p)) = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{n+1})\,. \tag{2.4}$$

This is a homeomorphism, because the map is continuous and has an inverse. We still have to show that coordinate changes are smooth. Consider $n = 2$:

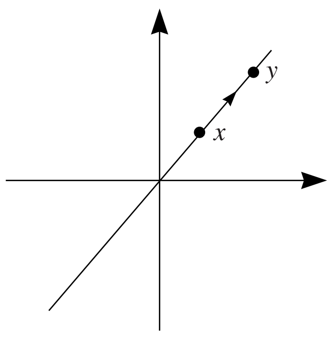
$$(u^1, u^2) \xrightarrow{(\varphi_3^+)^{-1}} (u^1, u^2, \sqrt{1 - (u^1)^2 - (u^2)^2}) \xrightarrow{\varphi_1^+} (u^2, \sqrt{1 - (u^1)^2 - (u^2)^2})\,. \tag{2.5}$$

Since $(u^1)^2 + (u^2)^2 < 1$ this map is smooth, since the square root is differentiable for $(u^1)^2 + (u^2)^2 < 1$. Take the maximal atlas: S^n is an n-dimensional smooth manifold. Remark: There exist exotic differentiable structures on spheres, for example 28 on S^7.

2.2.3 Projective space

a) n-dimensional real projective space:

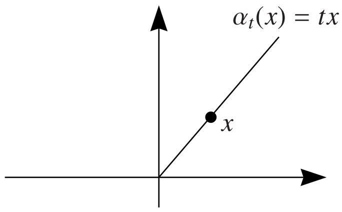
   We consider the set $X := \mathbb{R}^{n+1} \setminus \{0\}$. On X define an equivalence relation: $x \sim y \Leftrightarrow$ there exists $t \in \mathbb{R} \setminus \{0\}$, $y = tx$ (linear dependence).



The equivalence class $[x]$ of $x \in X$ is the straight line through x and y in $\mathbb{R}^{n+1} \setminus \{0\}$. $P^n\mathbb{R} := X/\sim$ (alternative definition $P^n\mathbb{R} =$ set of one-dimensional vector subspaces of $\mathbb{R}^{n+1}$) is called the **one-dimensional real projective space**. **Claim**: $P^n\mathbb{R}$ (equipped with quotient topology) is an n-dimensional differentiable manifold.

Quotient topology: Be Y a topological space with equivalence relation $\sim$. This gives rise to the set $Y := \widetilde{Y}/\sim$ (which is the set of equivalence classes). How can one make this abstract set into a topological space? To define a topology on Y use the projection map π: $\widetilde{Y} \mapsto Y$, $y \mapsto [y]$. Define a quotient topology such that this map becomes a continuous map. Quotient topology: $U \subset Y = \widetilde{Y}/\sim$ is open by definition if and only if $\pi^{-1}(U)$ is open in $\widetilde{Y}$. We want to prove the claim:

1) Countable basis: By lemma 1 (handouts) it suffices to show that π: $X \mapsto X/\sim = P^n\mathbb{R}$ is an open map, for example $\pi(U)$ is open for U open. To show that, consider the maps α_t: $X \mapsto X$, $x \mapsto tx$ $(t \in \mathbb{R})$.



For $t \neq 0$, α_t is a homeomorphism with $(\alpha_t)^{-1} = \alpha_{1/t}$. Assume that $U \subset X$ is open, then $\pi^{-1}(\pi(U)) = \cup_{t\neq 0}\alpha_t(U)$. Since $\alpha_t(U)$ is a homeomorphism, this is open. Hence (by definition of quotient topology) $\pi(U)$ is open.

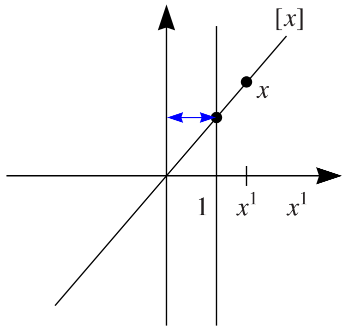
2) $P^n\mathbb{R}$ is Hausdorff: By lemma 2 (handouts) it suffices to show that $R := \{(x, y) \in X \times X | x \sim y\}$ is a closed subset (this means that the complement is open). To see that we consider the function f: $X \times X \mapsto \mathbb{R}$,

$$f(x, y) := f((x^1, \dots, x^{n+1}), (y^1, \dots, y^{n-1})) := \sum_{i\neq j}(x^i y^j - x^j y^i)^2 \,. \tag{2.6}$$

f is continuous and $f(x, y) = 0$, if and only if $x^i y^j = x^j y^i$ for all i, j and $i \neq j$. This is equivalent to $x^i/y^i = x^j/y^j =: t$, hence $x = ty$. This implies that $R = f^{-1}(\{0\})$, which is closed since $\{0\}$ is closed.

What remains, is to construct an atlas. The charts are defined as follows. Set $\widetilde{U}_i := \{x \in X | x^i \neq 0\}$ (for $i = 1, \dots, n + 1$) and $U_i := \pi(\widetilde{U}_i) \subset P^n\mathbb{R}$. Further set φ_i: $U_i \mapsto \mathbb{R}^n$,

$$\varphi_i([x]) := \left(\frac{x^1}{x^i}, \frac{x^2}{x^i}, \dots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \dots, \frac{x^{n+1}}{x^i}\right) \,. \tag{2.7}$$



One has to check the following properties:

- φ_i is independent of the choice of the representative of $[x]$.
- φ_i is injective: From $\varphi_i([x]) = \varphi_i([y])$ it follows that $y^j/y^i = x^j/x^i \Leftrightarrow y^j = (y^i/x^i)x^j$ and hence $y = tx$, $[y] = [x]$.
  φ_i: $U_i \mapsto \mathbb{R}^n$ is onto: For $a = (a_1, \dots, a_n) \in \mathbb{R}^n$ set $\widetilde{a} := (a_1, \dots, a_{i-1}, 1, a_i, a_n) \in \mathbb{R}^{n+1}$. Hence, $\varphi([\widetilde{a}]) = a$. As a result of that, φ_i is bijective and continuous. Since

$$\varphi_i^{-1}(z^1, \dots, z^n) = \pi(z^1, \dots, z^{i-1}, 1, z^i, \dots, z^n) \,, \tag{2.8}$$

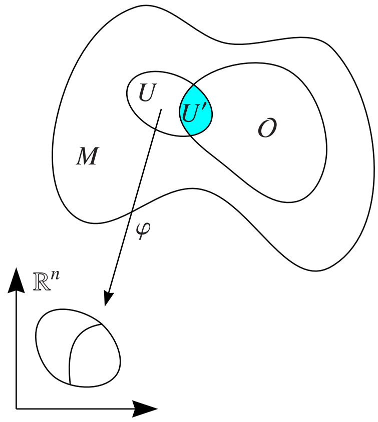
  we see that φ_i^{-1} is also continuous

– Coordinate changes $\varphi_i \circ \varphi_j^{-1}$ are C^∞ (exercise).

b) In a similar way define the complex projective space $P^n\mathbb{C} := \mathbb{C}^{n+1} \setminus \{0\}/\sim$. $\mathbb{C}$ is a two-dimensional real manifold (topologically $\mathbb{R}^2$). Hence, $P^n\mathbb{C}$ is a $2n$-dimensional manifold.

2.2.4 Construction of new manifolds

Let us continue with some indications how to construct new manifolds out of given ones. Let M be an n-dimensional differentiable manifold with complete atlas $\mathcal{A}$. For an open subset $\mathcal{O} \subset M$ set $\mathcal{A}' := \{(U', \varphi')|U' := U \cap \mathcal{O}, \varphi' := \varphi|_{U'}, (U, \varphi) \in \mathcal{A}\}$.



Then $\mathcal{A}'$ is an atlas for $\mathcal{O}$ and $\mathcal{O}$ is an n-dimensional smooth manifold (called an **open submanifold** of M).

Example

Let $M := \mathbb{R}^{n^2}$. Take the subset $\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n}|\det(A) \neq 0\}$, which is an open subset of $\mathbb{R}^n$. (Since the complement if a sequence of matrices with $\det(A) = 0$, hence $\{A \in \mathbb{R}^{n\times n}|\det(A) = 0\}$, which is closed. Remark: If f: $X \mapsto Y$ and g: $X \mapsto Y$ are continuous maps, then $\{x \in X|f(x) = g(x)\}$ is closed. This follows from definition of continuity. The general linear group $\mathrm{GL}(n, \mathbb{R})$ is a n^2-dimensional smooth manifold, which is not connected.

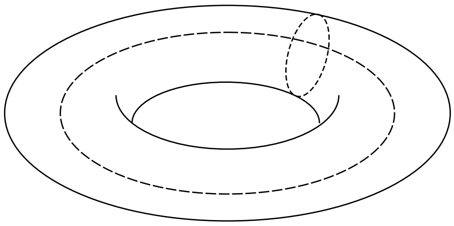
2.2.5 Construction of product manifolds

We have just considered, how manifolds can be constructed from subsets of manifolds. However, one can also build higher sets from manifolds. Let M be an m-dimensional smooth manifold and N be an n-dimensional smooth manifold. Then, $M \times N$ with product topology and charts $\varphi \times \psi$: $U \times V \mapsto \varphi(U) \times \psi(V) \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$ (where (U, φ) and (V, ψ), respectively, are charts of M and N, respectively) is an $(m + n)$-dimensional smooth manifold.

Examples

- $\mathbb{R}^k = \mathbb{R} \times \ldots \times \mathbb{R}$ (with k factors)
- $T^n := S^1 \times \ldots \times S^1$ (with n factors of $S^1 = \{x \in \mathbb{R}^2|\|x\| = 1\}$) is the n-dimensional torus.

$$T^2 = \bigcirc \times \bigcirc = \{(a, b) \in \mathbb{R}^2 \times \mathbb{R}^2|a \in S^1, b \in S^1\}\,. \tag{2.9}$$



The torus is a compact space, since S^1 is bounded and closed.

2.2.6 Lie groups and examples for manifolds

A **Lie group** is a group G, which is also a smooth manifold such that the map $G \times G \mapsto G$, $(g_1 g_2) \mapsto g_1 g_2^{-1}$ is differentiable (see later).

Examples

- $\mathrm{GL}(n, \mathbb{R})$ is a Lie group. $\mathrm{GL}(1, \mathbb{R}) = \mathbb{R}^* = \mathbb{R} \setminus \{0\}$ is not connected.
- $(\mathbb{Z}, +)$ (discrete group) is a 0-dimensional manifold.
- $\mathrm{SO}(2) := \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \middle\| 0 \in [0, 2\pi) \right\} \simeq S^1$ is a one-dimensional Lie group.

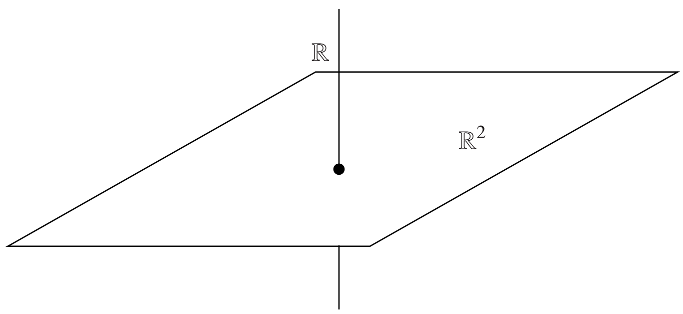
  This is a compact Lie group, since S^1 is compact (whereas $\mathrm{GL}(1, \mathbb{R})$ is a non-compact Lie group).
- $\mathrm{SU}(2) := \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \middle\| \alpha, \beta \in \mathbb{C}, \alpha\overline{\alpha} + \beta\overline{\beta} = 1 \right\}$

  Write $\alpha = x_1 + \mathrm{i}x_2$ and $\beta = x_3 + \mathrm{i}x_4$ ($\overline{\alpha} = x_1 - \mathrm{i}x_2$, $\overline{\beta} = x_3 - \mathrm{i}x_4$) and hence $\alpha\overline{\alpha} + \beta\overline{\beta} = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. The appearing matrices are $\in \mathbb{C}^2 \simeq \mathbb{R}^4$. Topologically, this set of matrices is the three-dimensional smooth manifold S^3.
- $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) | \det(A) = 1\}$. (From $\det(AB) = \det(A) \cdot \det(B)$ and $\det(A^{-1}) = (\det A)^{-1}$ it follows that $\mathrm{SL}(n, \mathbb{R})$ is a subgroup. Anyway, it is closed, since a continuous function of the set is equal to one.) The special linear group is a Lie group of $\dim = n^2 - 1$.

Fact: A closed subgroup of a Lie group is again a Lie group (see summer semester 2010). For example, this is the case for $\mathrm{SO}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) | AA^\intercal = E\}$. ($A$, $B \in \mathrm{SO}(n)$ implies $(AB)(AB)^\intercal = ABB^\intercal A = E$ and furthermore $A^{-1}(A^{-1})^\intercal = E$, hence it is a subgroup.)

Some remarks

1) Example of a "space" which is not a manifold:



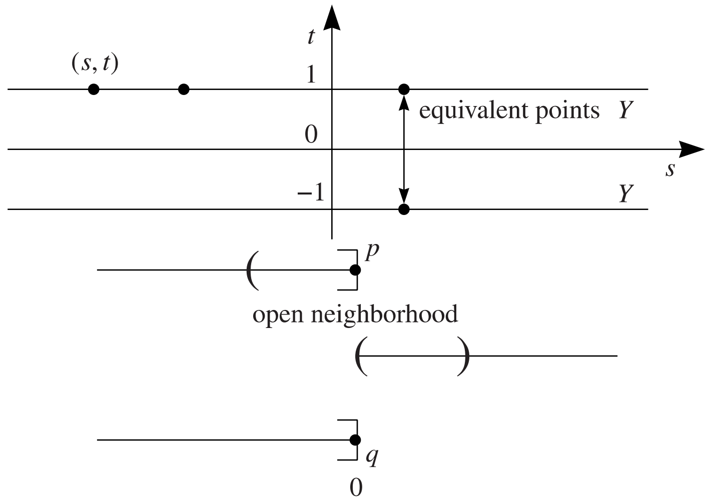
2) If instead of **smooth** coordinate transformations $\varphi \circ \psi^{-1}$ is just a homeomorphism, M is called a **topological manifold**. If alle **coordinate transformations** are real analytic C^∞, then M is called **analytic manifold**.

3) If the charts are of the form $\varphi\colon U \mapsto \mathbb{C}^n$ (or $\varphi\colon U \mapsto B$, where B is a Banachspace), then M is called a complex manifold (respectively a Banach-manifold).

4) The property "locally Euclidian" (every neighborhood of a point looks like a piece of $\mathbb{R}^n$) does not imply that the space is Hausdorff. An example for that is the "double line":

$$Y := \{(s, t) \in \mathbb{R}^2 | t = +1 \text{ or } t = -1\}, \tag{2.10}$$

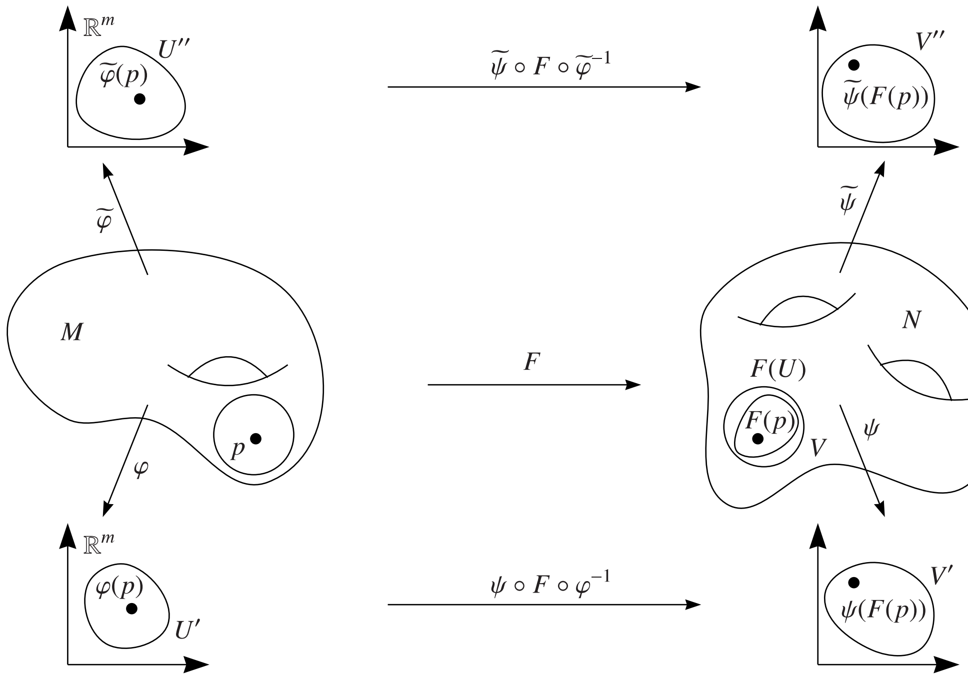
equipped with the subspace topology induced from $\mathbb{R}^2$. Set $X := Y/\sim$, where $(x, t) \sim (s', t')$ by definition if and only if $s = s'$ and $t = t'$, if $s \leq 0$ and $s = s'$, if $s > 0$ equipped with quotient topology $\pi\colon Y \mapsto X$, $(s, t) \mapsto [(s, t)]$.



The points $p = [(0, 1)]$ and $q = [(0, -1)]$ cannot be separated, hence X is not Hausdorff. But X is locally homeomorphic to $\mathbb{R}$.

2.3 Differentiable maps

A map $F\colon M \mapsto N$ (with m dimensional manifold M and n-dimensional manifold N) between smooth manifolds is called **differentiable** (or **smooth**) at $p \in M$, if for one (and hence every (*)) chart $\varphi\colon U \mapsto U' \subset \mathbb{R}^m$ (with $U \subset M$) at p and $\psi\colon V \mapsto V' \subset \mathbb{R}^n$ (with $V \subset N$) at $F(p)$ with $F(U) \subset V$ the representation of F in local coordinates $\psi \circ F \circ \psi^{-1}\colon U' \mapsto V'$ is C^∞.



$F\colon M \mapsto N$ is **differentiable or smooth**, if F is differentiable at every $p \in M$. A set of smooth maps is denoted by $C^\infty(M, N)$. To (*): The "test of smoothness" with respect to **one** chart suffices: If $\widetilde{\varphi}$ and $\widetilde{\psi}$ are other charts at p and $F(p)$ respectively, then we have:

$$\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1} \overset{\text{Trick}}{=} \widetilde{\psi} \circ (\psi^{-1} \circ \psi) \circ F \circ (\varphi^{-1} \circ \varphi) \circ \widetilde{\varphi}^{-1} = (\widetilde{\psi} \circ \psi^{-1}) \circ (\psi \circ F \circ \varphi^{-1}) \circ (\varphi \circ \widetilde{\varphi}^{-1}) . \tag{2.11}$$

Both $\widetilde{\psi} \circ \psi^{-1}$ and $\varphi \circ \widetilde{\varphi}^{-1}$ are both coordinate transformations and in C^∞. If the map $\psi \circ F \circ \varphi^{-1}$ is differentiable, so is $\widetilde{\psi} \circ F \circ \widetilde{\varphi}^{-1}$, because one just composes them with differentiable maps.

2.3.1 Special cases

- $n = 1$: $f\colon M \mapsto \mathbb{R}$ is a **differentiable function**.

- $m = 1$: c: $\mathbb{R} \mapsto N$ is a **differentiable curve**.

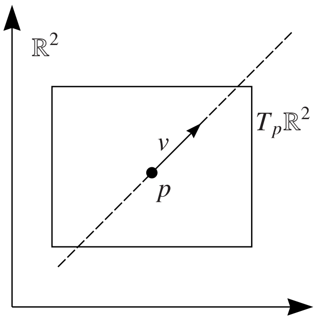
Notation: $C^\infty(M, \mathbb{R}) =: C^\infty(M)$ are all smooth functions on M. Actually, $C^\infty(M)$ is an $\mathbb{R}$-algebra: $(f \circ g)(p) = f(p) \cdot g(p)$. A differentiable map F: $M \mapsto N$ is a **diffeomorphism**, if f is bijective and F^{-1} is also C^∞. In particular, a diffeomorphism is a homeomorphism.

Remarks

- Examples for diffeomorphisms are the identity, charts, and coordinate transformations. (Why?)
- The set of diffeomorphisms $M \circlearrowright F$ is an infinite-dimensional group $\mathrm{Diff}(M)$ under composition of maps.
- **Warning**: A differentiable homeomorphism is not always a diffeomorphism! Example: $M = (\mathbb{R}, \mathrm{id})$ with the map F: $\mathbb{R} \mapsto \mathbb{R}$, $x \mapsto x^3$. The inverse map F^{-1}: $\mathbb{R} \mapsto \mathbb{R}$, $x \mapsto \sqrt[3]{x}$ is continuous, but not C^∞ at 0.

2.4 Tangent vectors

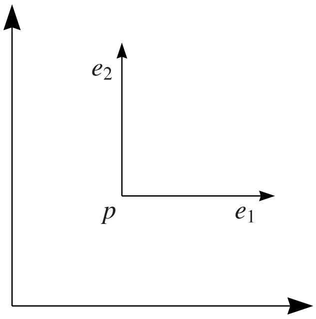
Repetition: For $p \in \mathbb{R}^n$ consider the tangent space $T_p\mathbb{R}^n = \{p\} \times \mathbb{R}^n$ at p.



For $f \in C^\infty(\mathbb{R}^n)$, $v \in T_p\mathbb{R}^n$ set

$$(\partial_v f)(p) := \lim_{t \mapsto} \frac{f(p + tv) - f(p)}{t} \,. \tag{2.12}$$

This is called the directional derivative of f at p in direction v. It measures, how the function f defined on $\mathbb{R}$ changes on the line through p in the direction of v. For $v = e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ the i-th standard basis vector we have the partial derivative $\partial_{e_i} = \partial/\partial u^i$.



Moreover, for f, $g \in C^\infty(\mathbb{R}^n)$ with α, $\beta \in \mathbb{R}$ one has

$$\partial_v(\alpha f + \beta g) = \alpha \partial_v f + \beta \partial_v g \,, \tag{2.13}$$

hence the partial derivative is a linear operator. Furthermore, the product rule holds:

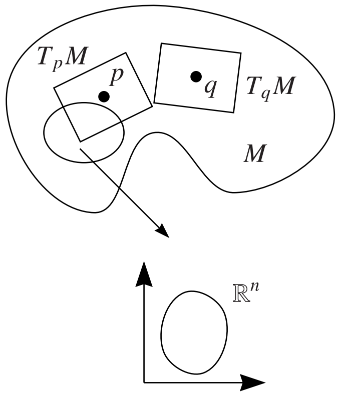
$$\partial_v(fg) = f(p)\partial_v g + \partial_v f g(p) \,. \tag{2.14}$$

Let M be a n-dimensional differentiable manifold and $p \in M$. A tangent **vector of M at p** is a map v: $C^\infty(M) \mapsto \mathbb{R}$ such that for all f and $g \in C^\infty(M)$ and α, $\beta \in \mathbb{R}$ one has:

T1) $v(\alpha f + \beta g) = \alpha v(f) + \beta v(g)$ ($\mathbb{R}$-linearity)

T2) $v(fg) = v(f)g(p) + f(p)v(g)$ (Leibniz rule)

Denote by T_pM the set of all tangent vectors of M at p. Then T_pM is an $\mathbb{R}$-vectorspace: $(v+w)(f) := v(f)+w(f)$ and $(\alpha v)(f) := \alpha v(f)$. This is called the **tangent space of M**.



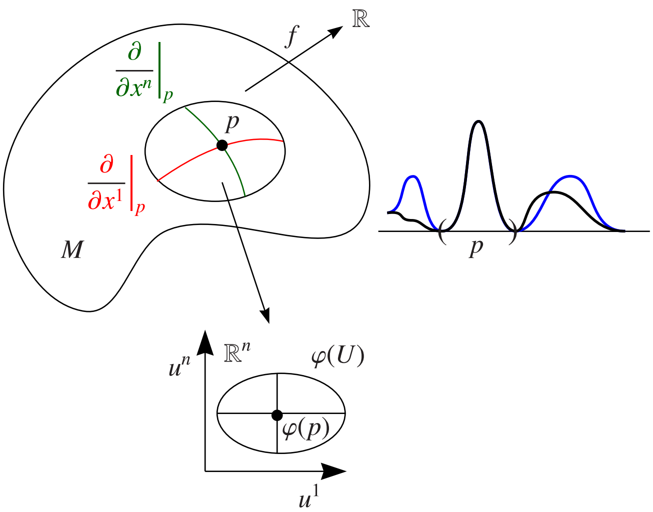
Special tangent vectors are constructed as follows: Consider a chart (U, φ) of M at p. With $x^i := u^i \circ \varphi$ we can write $\varphi = (x^1, \dots, x^n)$ ($u^i \colon \mathbb{R}^n \mapsto \mathbb{R}$, $(a_1, \dots, a_n) \mapsto a_i$). For a test function $f \in C^\infty(M)$ and $i = 1, 2, \dots, n$ we set

$$\frac{\partial f}{\partial x^i}(p) := \frac{\partial (f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) \,, \tag{2.15}$$

which is the i-th partial derivative in local coordinates. A direct computation shows that

$$\left.\frac{\partial}{\partial x^i}\right|_p : C^\infty(M) \mapsto \mathbb{R}, \ \left.\frac{\partial}{\partial x^i}\right|_p (f) := \frac{\partial f}{\partial x^i}(p) \,, \tag{2.16}$$

is a tangent vector at p.



Lemma 1 ("tangent vectors are local objects")

Let $v \in T_pM$.

1) If f, $g \in C^\infty(M)$ are equal on a neighborhood of p, then $v(f) = v(g)$.

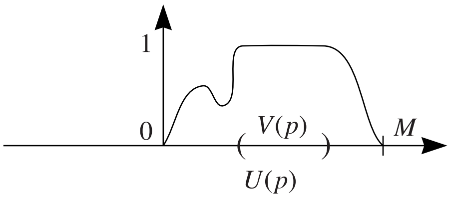
2) If $h \in C^\infty(M)$ is constant in a neighborhood of p, then $v(h) = 0$.

In order to prove this lemma one needs the existence of bump functions, which means $g \in C^\infty(M)$ such that (let $U(p)$ be a neighborhood of p)

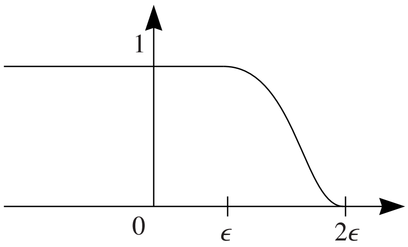
i) support $g := \overline{\{q \in M | g(q) \neq 0\}} \subset U(p)$ (fixed neighborhood of p)

ii) $0 \leq g \leq 1$ on M

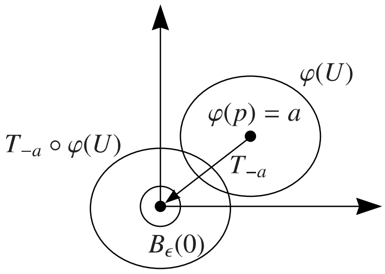
iii) $g = 1$ in a neighborhood $V(p) \subset U(p)$



Now to the construction of such a bump function: For any $\epsilon > 0$ pick a smooth function $h_\epsilon\colon \mathbb{R} \mapsto \mathbb{R}$, whose graph has the following form:



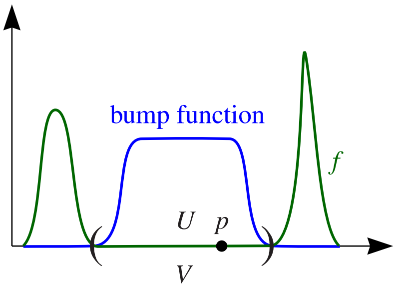
Let $p \in M$ and $\varphi\colon U \mapsto \mathbb{R}^n$ a chart at p such that $\varphi(p) = 0$ (if $\varphi(p) = a \neq 0$ then compose with translation $\widetilde{\varphi} := T_{-a} \circ \varphi$).



Then there is $\epsilon > 0$ such that $B_{\sqrt{3\epsilon}}(0) = \{b \in \mathbb{R}^n | \|b\|^2 < \sqrt{3\epsilon}\} \subset \varphi(U)$. Take the radius square function $r\colon U \mapsto \mathbb{R}$, $q \mapsto r(q) := \sum_{i=1}^n (x^i(q))^2 = \|\varphi(q)\|^2$ (with $\varphi(q) = (x^1(q), \ldots, x^n(q))$). Then $g := h_\epsilon \circ r$ has the required properties. (We take the radius square, because that is a smoooth function, whereas the radius function is not because of the square root.)

Proof of Lemma 1

1) v is linear, hence it suffices to show: If $f = 0$ in a neighborhood of p then $v(f) = 0$. (From $f = g$ it follows $f - g = 0$ and $v(f) = v(g)$ is equivalent to $v(f - g) = 0$.) To that end we consider a bump function g on U.



We have $f \cdot g \equiv 0$ on M. We have $v(0) = v(0 + 0) \overset{\text{T1}}{=} v(0) + v(0)$, which implies $v(0) = 0$. Hence, $0 = v(0) = v(f \cdot g) \overset{\text{T2}}{=} v(f) \cdot g(p) + f(p) \cdot v(g) = v(f)$.

2) By (1) we can assume that $h = \text{const.} = c$ on M. But then $v(h) = v(c \cdot 1) \overset{\text{T1}}{=} cv(1)$. It remains to show that $v(1) = 0$. As $v(1) = v(1 \cdot 1) \overset{\text{T1}}{=} v(1) \cdot 1 + 1 \cdot v(1) = 2v(1)$ and therefore we have again $v(1) = 0$. □

Remark

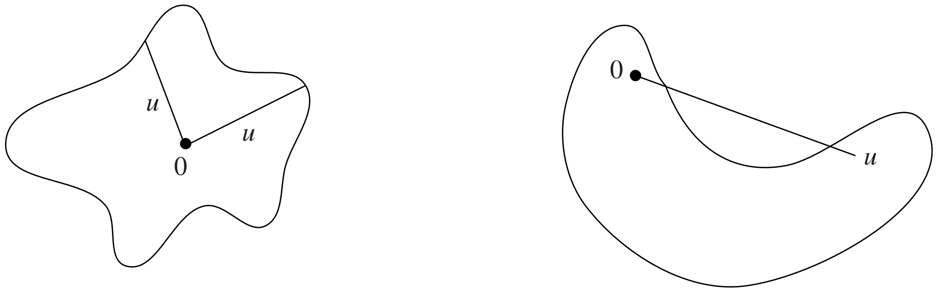
Lemma 1 shows that tangent vectors are local objects: If U is a neighborhood of p then $T_pU = T_pM$ (U is an open submanifold.)

Theorem 1 (Basis for T_pM)

Let M be an n-dimensional smooth manifold, $p \in M$. If $\varphi = (x^1, \dots, x^n)$ is a chart at p. Then the tangent vectors $\partial/\partial x^i|_p$ ($i = 1, \dots, n$) form a basis of T_pM and for any $v \in T_pM$ one has $v = \sum_{i=1}^n v(x^i)\partial/\partial x^i|_p$. In particular $\dim T_pM = \dim M$.

Lemma 2

Let U be an open neighborhood of $0 \in \mathbb{R}^n$. If U is starlike with respect to 0 (which means for $u \in U$ the segment ray $\overline{0u} \subset U$) and $g \in C^\infty(U)$ then we can write



$$g(u) = g(0) + \sum_{j=1}^n u^j g_j(u)\,, \quad g_j \in C^\infty(U)\,, \quad u = (u^1, \dots, u^n)\,. \tag{2.17}$$

Proof

We want to use the Taylor-integral formula

$$g(u) - g(0) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} g(tu)\,\mathrm{d}t = \sum_{j=1}^n u^j \int_0^1 \frac{\partial g}{\partial u^j}(tu)\,\mathrm{d}t =: \sum_{j=1}^n u^j g_j(u)\,. \qquad \square \tag{2.18}$$

Proof of Theorem 1

a) $\partial/\partial x^i|_p \in T_pM$ (computation!) for $i = 1, \dots, n$. For the coordinate functions $x^k := u^k \circ \varphi$ holds

$$\left.\frac{\partial}{\partial x^i}\right|_p (x^k) \overset{\text{def}}{=} \frac{\partial}{\partial u^i}(x^k \circ \varphi^{-1})(\varphi(p)) = \frac{\partial}{\partial u^i} u^k(\varphi(p)) = \delta_{ik}\,. \tag{2.19}$$

b) linear independence:
Assume $\sum_{i=1}^n \lambda_i \partial/\partial x^i|_p = 0$ ($\lambda_i \in \mathbb{R}$). Then for $k = 1, \dots, n$ we have

$$0 = 0(x^k) = \sum_{i=1}^n \lambda_i \left.\frac{\partial}{\partial x^i}\right|_p (x^k) \overset{(a)}{=} \lambda_k\,. \tag{2.20}$$

c) $\{\partial/\partial x^i|_p | i = 1, \dots, n\}$ generate T_pM.

We can assume $\varphi(p) = 0$ (*). Let $v \in T_pM = T_pU$ (where $\varphi(U)$ is starlike with respect to 0) and set $a_k := v(x^k) \in \mathbb{R}$. Let

$$w := v - \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x^i}\right|_p\,. \tag{2.21}$$

Then we get vor every k that

$$w(x^k) = v(x^k) - \sum_{i=1}^n a_i \left.\frac{\partial}{\partial x^i}\right|_p (x^k) = v(x^k) - a_k = 0\,. \tag{2.22}$$

We want to show $w = 0$, hence $w(f) = 0$ for all $f \in C^\infty(U)$. So pick an $f \in C^\infty(U)$. We have $g := f \circ \varphi^{-1} \in C^\infty(\varphi(U))$ and Lemma 2 can be applied to g.

$$w(f) = w(f \circ \varphi^{-1} \circ \varphi) = w(g \circ \varphi) \overset{\text{L2}}{=} w\left(g(0) + \sum_{j=1}^{n} \underbrace{(u^j \circ \varphi)}_{=x^j}(g_j \circ \varphi)\right) \overset{\text{T1,T2}}{=}$$

$$= 0 + \sum_{j=1}^{n} [w(x^j)(g_j \circ \varphi)(p) + x^j(p) w(g_j \circ \varphi)] = 0\,, \tag{2.23}$$

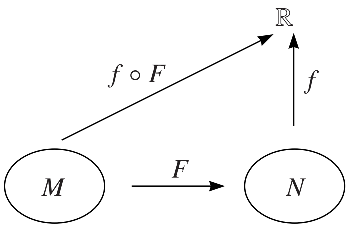
whereas the first term is zero by Eq. (2.22) and $x^j(p) = 0$ because of (*). □

2.5 Differentials (or tangent maps) of smooth maps

Let F: $M \mapsto N$ be a smooth map between smooth manifolds. The local representation of such a map in the neighborhood of points is given by $\eta \circ F \circ \varphi^{-1}$; it is C^∞.

The goal/wish is to approximate F at each point $p \in M$ by a linear map. Let $p \in M$. The **differential or tangent map** of of F in p is the linear map defined as follows:

$$\mathrm{d}F_p : T_pM \mapsto T_{F(p)}N,\ (\mathrm{d}F_p(v))(f) := v(f \circ F)\,\forall\, f \in C^\infty(N)\,. \tag{2.24}$$



We need to show that $\mathrm{d}F_p(v) \in T_{F(p)}N$.

T1) For α, $\beta \in \mathbb{R}$ and f, $g \in C^\infty(N)$:

$$\begin{aligned}\mathrm{d}F_p(v)(\alpha f + \beta g) &= v((\alpha f + \beta g) \circ F) = v(\alpha f \circ F + \beta g \circ F) \overset{\text{T1 for } v}{=} \\ &= \alpha v(f \circ F) + \beta v(g \circ F) = \alpha \mathrm{d}F_p(f) + \beta \mathrm{d}F_p(g)\,.\end{aligned} \tag{2.25}$$

T2) Show:

$$\begin{aligned}\mathrm{d}F_p(v)(f \cdot g) &= v((f \cdot g) \circ F) \overset{\text{T2 for } v}{=} v(f \circ F)(g \circ F)(p) + (f \circ F)(p) \cdot v(g \circ F) = \\ &= \mathrm{d}F_p(f) g(F(p)) + f(F(p)) + \mathrm{d}F_p(g)\,.\end{aligned} \tag{2.26}$$

Remarks

- If $F = \mathrm{Id}$: $M \mapsto M$, $p \mapsto \mathrm{Id}(p) = p$ we have $\mathrm{d}(\mathrm{Id})_p(v) = v$ for all $v \in T_pM$ and $p \in M$.
- Another notation in books for $\mathrm{d}F_p$ ist $F_*|_p$.

Lemma 2 (computation of dF_p)

Let F: $M^m \mapsto N^n$ be a differentiable map. If $\xi = (x^1, \ldots, x^m)$ is a chart at $p \in M$ and $\eta = (y^1, \ldots, y^n)$ is a chart at $F(p) \in N$ then we have for $1 \le j \le m$:

$$\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_{i=1}^{n} \frac{\partial(y^j \circ F)}{\partial x^j}(p) \left.\frac{\partial}{\partial y^i}\right|_{F(p)}\,. \tag{2.27}$$

Proof

We set

$$w := \mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) \in T_{F(p)}N\,. \tag{2.28}$$

By theorem 1 ("basis-theorem") we can write

$$w = \sum_{j=1}^{n} w(y^i)\left.\frac{\partial}{\partial y^i}\right|_{F(p)}\,. \tag{2.29}$$

By definition of the tangent vectors

$$w(y^i) = \left(\mathrm{d}F_p\left(\left.\frac{\partial}{\partial x^j}\right|_p\right)\right)(y^i) = \left.\frac{\partial}{\partial x^j}\right|_p (y^i \circ F) = \frac{\partial(y^i \circ F)}{\partial x^j}(p) = \frac{\partial(y^i \circ F \circ \xi^{-1})}{\partial u^j}(\xi(p))\,. \tag{2.30}$$

The $n \times m$-matrix

$$\left(\frac{\partial(y^i \circ F)}{\partial x^j}(p)\right)\,, \tag{2.31}$$

is called **Jacobian matrix** of F at p with respect to ξ and η.

Lemma 3 (chain rule)

If $F\colon M \mapsto N$ and $G\colon N \mapsto L$ are differentiable maps, then one has at every $p \in M$ that

$$\mathrm{d}(G \circ F)_p = \mathrm{d}G_{F(p)} \circ \mathrm{d}F_p\,. \tag{2.32}$$

Proof

Pick $v \in T_pM$ and $g \in C^\infty(L)$. Then a tangent vector of L (applied to the test function g) is given by

$$\mathrm{d}(G \circ F)_p(v)(g) \overset{\substack{\text{definition}\\ \text{of differential}}}{=} v(g \circ G \circ F) = \mathrm{d}F_p(v)(g \circ G) = \mathrm{d}G_{F(p)}(\mathrm{d}F_p(v))(g)\,. \tag{2.33}$$

Remarks

- If $F\colon M \mapsto N$ is a diffeomorphism we get

$$\mathrm{d}(F \circ F^{-1})_{F(p)} = \mathrm{d}(\mathrm{id}_N)_{F(p)} = \mathrm{id}|T_{F(p)N}\,. \tag{2.34}$$

- By lemma 3 one obtains:

$$\mathrm{d}F_p \circ \mathrm{d}F^{-1}_{F(p)}\,, \quad \mathrm{d}F_p : T_pM \mapsto T_{F(p)}N\,, \tag{2.35}$$

  is a vectorspace isomorphism. In particular

$$\dim(M) = \dim(T_pM) = \dim(T_{F(p)}N) = \dim(N)\,. \tag{2.36}$$

Theorem 2 (inverse function theorem for manifolds)

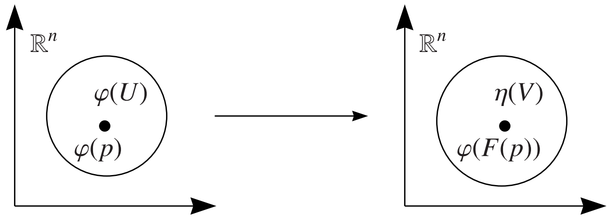
If $F\colon M \mapsto N$ is a differentiable map such that for some $p \in M$, $\mathrm{d}F_p\colon T_pM \mapsto T_{F(p)}N$ is a vectorspace-isomorphism, then there exists a neighborhood V of p in M such that $F|_V\colon V \mapsto F(V)$ is a diffeomorphism ($V \subset M$ and $F(V) \subset N$).

Proof

Choose charts φ at p and η at $F(p)$. Then

$$\mathrm{d}(\eta \circ F \circ \varphi^{-1})_p = \mathrm{d}\eta_{F(p)} \circ \mathrm{d}F_p \circ \mathrm{d}\varphi^{-1}_{\varphi(p)} : T_{\varphi(p)}\mathbb{R}^m \mapsto T_{\eta(F(p))}\mathbb{R}^n\,, \tag{2.37}$$

is a vectorspace-isomorphism (in part $m = n$), since $\mathrm{d}\eta_{F(p)}$ and $\mathrm{d}\varphi^{-1}_{\varphi(p)}$ are vectorspace-isomorphisms (since η is a diffeomorphism φ).



By Analysis II $\eta \circ F \circ \varphi^{-1}$ is a diffeomorphism from a neighborhood of $\varphi(p)$ to one of $\eta(F(p))$. Since φ and η are (as charts) local diffeomorphisms, F is a local diffeomorphism.

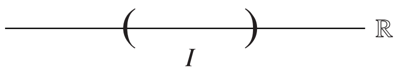
Notation

A differentiable map $F\colon M \mapsto N$ such that $\mathrm{d}F_p$ is a linear isomorphism for **all** $p \in M$ is called a local diffeomorphism. (A **diffeomorphism** is a local diffeomorphism, which is bijective.)

2.6 Tangent vectors of curves

A smooth curve in a differentiable manifold M is a differentiable map $c\colon I \mapsto M$ (where $I \subset \mathbb{R}$ is an **open interval**). We set $u := u^1$ (first coordinate function). Note that we use the chart $\varphi = \mathrm{id}|_{\mathbb{R}}$ i.e. $x^1 = u^1 \circ \varphi = u^1 = u$. In any point $t \in I$ we have

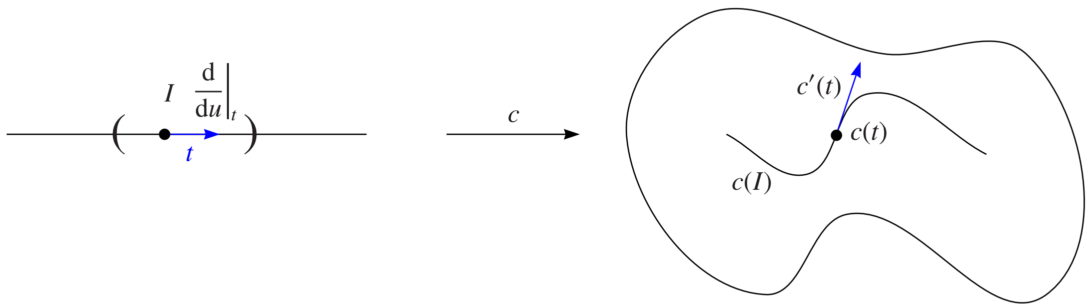
$$\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t := \left.\frac{\partial}{\partial u}\right|_t \in T_t I = T_t \mathbb{R}\,. \tag{2.38}$$



We define the tangent vector of the curve c in $c(t) \in M$ as follows:

$$c'(t) := \mathrm{d}c_t \left(\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t \right) \in T_{c(t)} M \,\forall\, t \in I\,, \tag{2.39}$$

where $\mathrm{d}c_t$ is the differential of the smooth map c.



2.6.1 Properties

Tangent vectors operate on test functions $f \in C^\infty(M)$:

1) $c'(t)(f) = \left[\mathrm{d}c_t \left(\left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t \right)\right](f) = \left.\frac{\mathrm{d}}{\mathrm{d}u}\right|_t (f \circ c) = \frac{\mathrm{d}(f\circ c)}{\mathrm{d}u}(t)$

2) In particular, if c is a smooth curve with $c(0) = p$, then $v := c'(0) \in T_p M$. With this we can write:

$$v(f) = c'(0)(f) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)(0) =: \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} (f \circ c)(t)\,. \tag{2.40}$$

This is a useful formula! **Remark:** We see later: For any $v \in T_p M$ there is a curve c such that $c(0) = p$ and $c'(0) = v$.

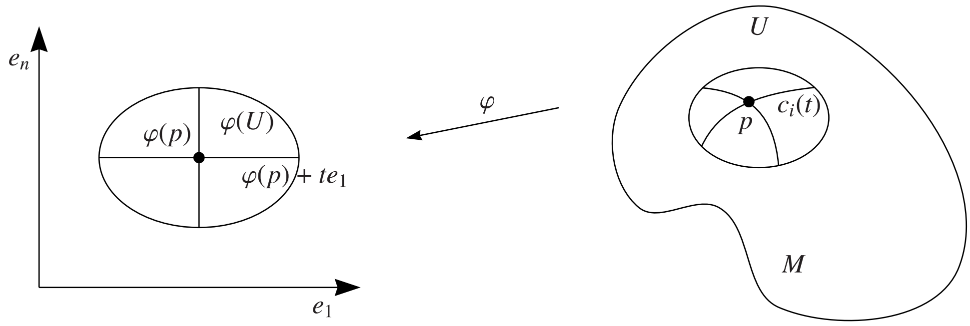
3) If c: $I \mapsto M$ is a smooth curve in M and F: $M \mapsto N$ a differentiable map, then $F \circ c$: $I \mapsto N$ is a smooth curve in N and we have

$$\mathrm{d}F_{c(t)}(c'(t)) = (F \circ c)'(t)\,. \tag{2.41}$$

This follows from the definition:

$$\mathrm{d}F_{c(t)}(c'(t)) = \mathrm{d}F_{c(t)}\left(\mathrm{d}c_t\left(\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_t\right)\right) \overset{\text{chain rule}}{=} \mathrm{d}(F \circ c)_t\left(\frac{\mathrm{d}}{\mathrm{d}u}\bigg|_t\right) \overset{\text{definition}}{=} (F \circ c)'(t)\,. \tag{2.42}$$

4) If (φ, U) is a chart at $p \in M$ and $c_i(t) := \varphi^{-1}(\varphi(p) + te_i)$ for $i = 1, \ldots, n$ and $t \in (-\varepsilon, \varepsilon)$ be the "i-th coordinate line" at p with respect to φ.



Then we have

$$c_i'(0) = \frac{\partial}{\partial x^i}\bigg|_p\,, \quad i = 1, \ldots, n\,. \tag{2.43}$$

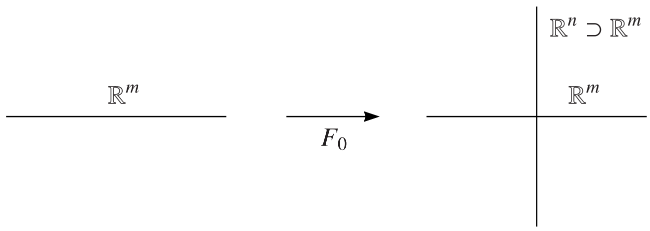
Hence, partial derivatives can be interpreted as tangent vectors to these special curves. To prove this pick a test function $f \in C^\infty(M)$. Then

$$\begin{aligned} c_i'(0)(f) &\overset{(1)}{=} \frac{\mathrm{d}}{\mathrm{d}t}(f \circ c_i)(0) = \frac{\mathrm{d}}{\mathrm{d}t}(f \circ \varphi^{-1}(\varphi(p) + te_i))(0) \overset{\text{definition of partial derivative}}{=} \\ &= \frac{\partial(f \circ \varphi^{-1})}{\partial u^i}(\varphi(p)) = \frac{\partial}{\partial x^i}\bigg|_p (f)\,. \end{aligned} \tag{2.44}$$

2.7 Special smooth maps and submanifolds

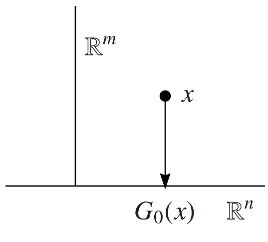
A C^∞-map F: $M^m \mapsto N^n$ is called **immersion** if $\mathrm{d}F_p$: $T_pM \mapsto T_{F(p)}N$ is injective for all $p \in M$ and submersion if $\mathrm{d}F_p$ is surjective (onto) for all $p \in M$.
Prototypes of such maps are the following:

1) F_0: $\mathbb{R}^m \mapsto \mathbb{R}^n$ $(n \geq m)$, $(x^1, \ldots, x^m) \mapsto (x^1, \ldots, x^m, 0, \ldots, 0)$ is an immersion.



One can show that if F: $M^m \mapsto N^n$ is an immersion and $p \in M$ there exist charts (U, φ) and (V, η) at p and $F(p)$ such that $\eta \circ F \circ \varphi^{-1} = F_0|_{\varphi(U)}$ ("locally every immersion looks like F_0").

2) G_0: $\mathbb{R}^m \mapsto \mathbb{R}^n$ $(n \leq m)$, $(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) \mapsto (x^1, \ldots, x^n)$ is a submersion.

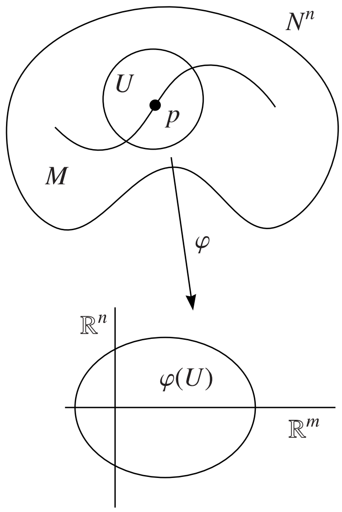


Fact: “Locally, any submersion looks like G_0.”

If $F\colon M \mapsto N$ is an injective immersion, then one can define a topology and differentiable structure on the image $F(M) \subseteq N$ such that $F(M)$ is diffeomorphic to M.

- $U \subset F(M)$ is **open** in $F(M)$, if $F^{-1}(U)$ is open in M.
- φ is a chart of $F(M)$ if and only if $\varphi \circ F$ is a chart of M.

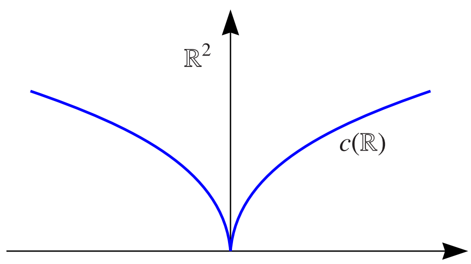
In general, the topology on $F(M)$ defined as above is **NOT** the subspace topology. $F(M) \subseteq N$ endowed with the above structures is called immersed submanifold of N.
If $F\colon M \mapsto N$ is an injective immersion and if $F(M) \subset N$ endowed with the subspace topology **is** homeomorphic to M then $F(M)$ is called **embedded submanifold** (and F is called an embedding of M into N). A topological subspace M of a differentiable manifold M is a **regular m-dimensional submanifold** if for all $p \in M$ there is a chart (U, φ) of N^n $(n \geq m)$ at p such that $\varphi(U \cap M) = \varphi(U) \cap (\mathbb{R}^m \times \{0\})$.



Using this properties one can endow M with a differentiable structure such that the inclusion $i\colon M \hookrightarrow N$ is an embedding (exercise). For more information on submanifolds see Boothby.

Examples

1) The curve $c\colon \mathbb{R} \mapsto \mathbb{R}^2$, $t \mapsto (t^3, t^2)$ is C^∞ but **not** an immersion.



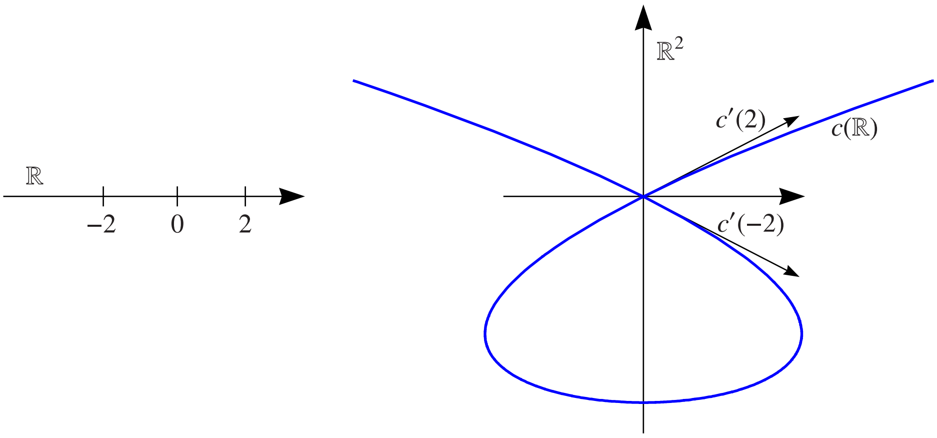
The differential of the map (the tangent vector) is not injective at every point.

$$c'(t) = (3t^2, 2t) \Rightarrow c'(0) = \mathrm{d}c_0 \left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = 0\,. \tag{2.45}$$

Hence $\mathrm{d}c_0$ is not injective. c is injective, the only problem from being an injective immersion is the origin. At every point except zero one can find a chart such that the curve is a regular submanifold of $\mathbb{R}^2$.

2) The curve $c\colon \mathbb{R} \mapsto \mathbb{R}^2$, $t \mapsto (t^3 - 4t, t^2 - 4)$

   c is not injective since $c(-2) = c(2) = (0, 0)$. But it is an immersion since the differential is injective.

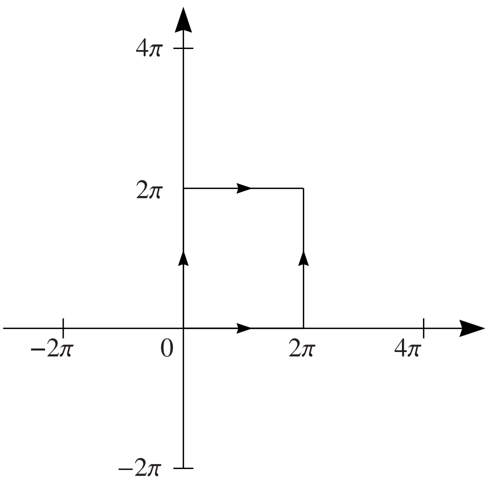


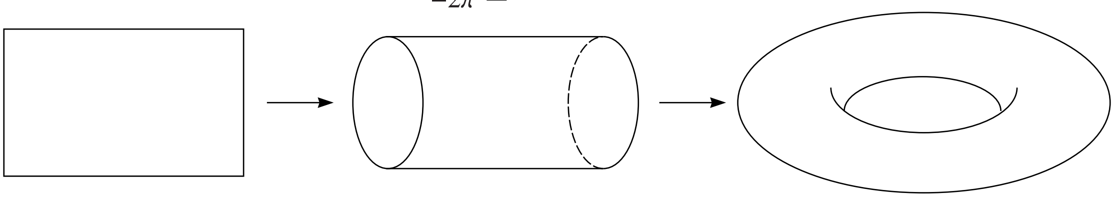
$c(\mathbb{R}) \subset \mathbb{R}^2$ with subspace topology is not homeomorphic to $\mathbb{R}$. To see this consider the sequence

$$x_n := \begin{cases} -2 + 1/n & \text{for } n \text{ even} \\ 2 + 1/n & \text{for } n \text{ odd} \end{cases}, \tag{2.46}$$

does not converge in $\mathbb{R}$ but $c(x_n)$ converges in $\mathbb{R}^2$.

3) Let us consider curves on the **torus** $T^2 := \mathbb{R}^2/\sim$ with $(x_1, y_1) \sim (x_2, y_2) \Leftrightarrow x_1 = x_2 + 2\pi k$ and $y_1 = y_2 + 2\pi l$ with k, $l \in \mathbb{Z}$.





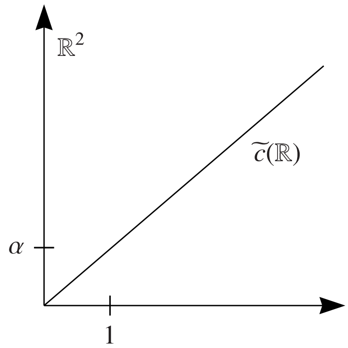
There exists a map $T^2 \underset{h}{\longrightarrow} S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C}$

$$h([(x, y)]) \mapsto (\exp(\mathrm{i}x), \exp(\mathrm{i}y)), \tag{2.47}$$

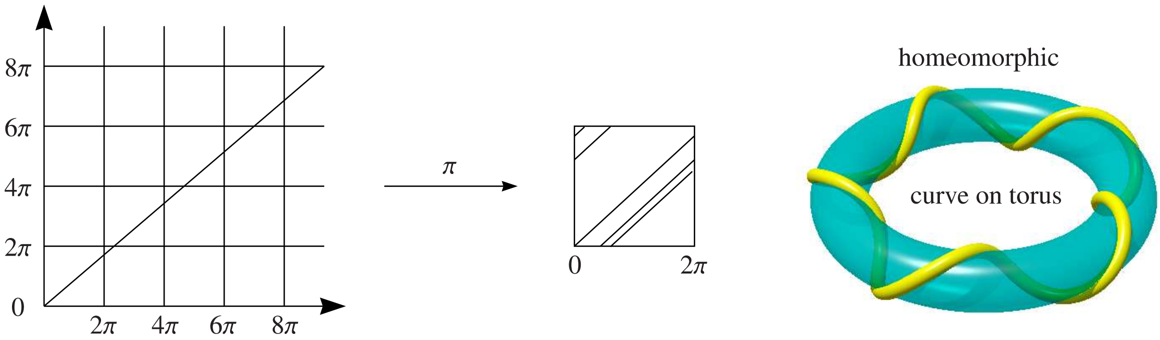
which is a homeomorphism. Now consider the curve c: $\mathbb{R} \mapsto h(T^2)$, $t \mapsto (\exp(\mathrm{i}t), \exp(\mathrm{i}\alpha t)$ with $\alpha \in \mathbb{R}$. Define the projection map:

$$\pi : \mathbb{R}^2 \mapsto T^2,\ (x, y) \mapsto [(x, y)]. \tag{2.48}$$

To construct a curve on T^2 define a curve in $\mathbb{R}^2$ and project it on T^2 by the projection map π. Hence, consider $\widetilde{c}$: $\mathbb{R} \mapsto \mathbb{R}^2$, $t \mapsto (t, \alpha t)$.

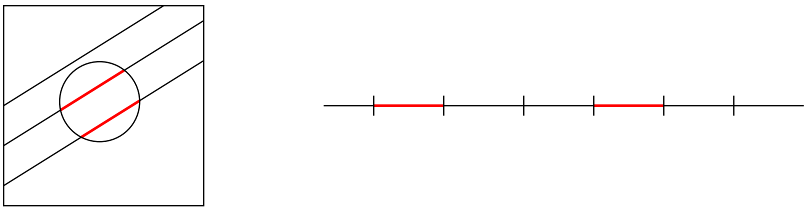


Set $c := h \circ \pi \circ \widetilde{c}$: $\mathbb{R} \mapsto S^1 \times S^1 \subset \mathbb{C}^2$. c is a differentiable map of rank 1 (i.e. a smooth curve with $c'(t) \neq 0$ $\forall\ t$).



There are now two cases:

- $\alpha \in 2\pi\mathbb{Q}$, whereas $c(\mathbb{R})$ is a closed curve (actually the embedding of a circle).
- $\alpha \notin 2\pi\mathbb{Q}$ leads to $c(\mathbb{R})$ that is close (i.e. $\overline{c(\mathbb{R})} = T^2$). This can be seen with a theorem of Kronecker (see Arnold: ordinary differential equations). c is injective and an immersion, hence $c(\mathbb{R})$ is an **immsersed submanifold** but **not** an embedding, because $c(\mathbb{R})$ with subspace topology is **not** homeomorphic to $\mathbb{R}$.



One can find a sequence that converges in the subspace topology but not on $\mathbb{R}$.

Remark (A theorem of Whitney 1936)

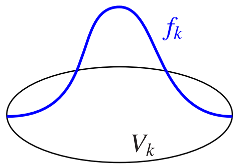
Every n-dimensional smooth manifold M can be embedded into some $\mathbb{R}^N$.

Sketch of proof for compact manifolds (for example $P^n\mathbb{R}$)

We need a tool from differential topology: **partitions of unity**. Let M be a topological space. A covering $(U_\alpha)_{\alpha \in A}$ of M (i.e. $M = \bigcup_{\alpha \in A} U_\alpha$) is **locally finite**, if every point of M has a neighborhood which intersects only **finitely** many of the U_α. A **refinement** of a covering $(U_\alpha)_{\alpha \in A}$ is a covering $(V_\beta)_{\beta \in B}$ such that every V_β is contained in some U_α. M is **paracompact** if M is Hausdorff and if every open covering has a locally finite refinement. **Fact:** M is paracompact if and only if M is locally compact with countable basis. In particular, a manifold is paracompact, since it has a countable basis. Furthermore it is locally compact, since it is locally Euclidian.

Theorem 3 (partition of unity)

Let M be a differentiable manifold (hance paracompact) and let $(U_\alpha)_{\alpha\in A} = \mathcal{A}$ be an atlas for M. There exists a locally finite refinement $(V_k)_{k\in I}$ of $\mathcal{A}$ and C^∞-function f_k: $M \mapsto \mathbb{R}$ such that firstly $f_k \geq 0$ on $\overline{V}_k$ and $f_k = 0$ on the complement $\overline{V}_k^c$.



(This is some kind of bump function.) Secondly, it holds that

$$\left(\sum_{k\in I} f_k\right)(p) = 1\,, \tag{2.49}$$

for all $p \in M$. This sum is always finite since the covering is locally finite. (For the proof see the German book Gromoll-Klingenberg-Mayer on page 275 or the English book Gallot-Hulin-Lafontaine in chapter 1.H.)
Let $(U_i, \varphi_i)_{1\leq i\leq l}$ be an atlas for M with a **finite** number of charts. Consider a partition of the unity $(f_i)_{1\leq i\leq l}$. For $1 \leq i \leq l$ set

$$\psi(p) := \begin{cases} f_i(p)\varphi_i(p) \in \mathbb{R}^n & \text{for} \quad p \in U_i \\ 0 & \text{otherwise} \end{cases} . \tag{2.50}$$

Then $\psi_i \in C^\infty$. The map ϕ: $M \mapsto \mathbb{R}^{nl+l}$, $\phi(p) := (\psi_1(p), \psi_2(p), \ldots, \psi_l(p), f_1(p), \ldots, f_l(p))$ is an immersion (can be seen by just computing the differential and showing that this is nonzero), injective (hence a bijection onto $\phi(M)$) (proof as an exercise). As M is compact, ϕ: $M \mapsto \phi(M)$ is a homeomorphism and hence an embedding (since the subspace topology is the same as that of M).

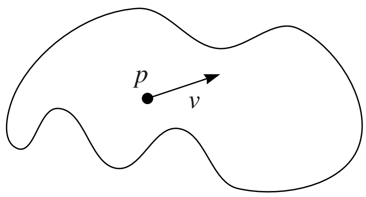
2.8 Tangent bundle and vector fields

Theorem 4

Be M a differentiable manifold and $\dim(M) = n$. The **tangent bundle of M**

$$TM := \bigcup_{p\in M} T_pM = \{(p,v)|p \in M, v \in T_pM\}\,, \tag{2.51}$$

is a $2n$-dimensional smooth manifold.



Proof

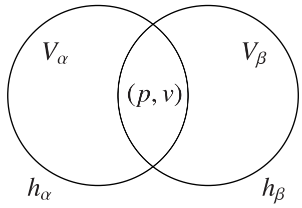
Let $(U_\alpha, \varphi_\alpha)_{\alpha\in A}$ be an atlas of M. If $\varphi_\alpha = (x_\alpha^1, x_\alpha^2, \ldots, x_\alpha^n)$ there is a basis $\{\partial/\partial x_\alpha^i|_p | i = 1, \ldots, n\}$ of T_pM. For all $p \in U_\alpha$ if $v \in T_pM$ then

$$v = \sum_{i=1}^n v(x_\alpha^i) \left.\frac{\partial}{\partial x_\alpha^i}\right|_p . \tag{2.52}$$

For each $\alpha \in A$ we get a bijection h_α: $V_\alpha := \bigcup_{p\in U_\alpha} T_pM =: TU_\alpha$ (which is a map from V_α to $\mathbb{R}^{2n}$).

$$h_\alpha((p,v)) := (x_\alpha^1(p), \ldots, x_\alpha^n(p), v(x_\alpha^1), \ldots, v(x_\alpha^n))\,. \tag{2.53}$$

The claim is that $(V_\alpha, h_\alpha)_{\alpha\in A}$ is an atlas for TM. Sketch of proof for this fact: As $\bigcup_{\alpha\in A} U_\alpha = M$ we have $\bigcup_{\alpha\in A} V_\alpha = TM$. Define a (basis of) topology in such a way that h_α are homeomorphisms. This implies that TM is Hausdorff and has a countable basis. Now to the coordinate changes: Let $(p,v) \in TU_\alpha \cap TU_\beta$ and $\varphi_\alpha(p) := (x^1(p), \ldots, x^n(p))$ and $\varphi_\beta(p) := (y^1(p), \ldots, y^n(p))$.



Furthermore

$$v = \sum_{i=1}^{n} a_i \left.\frac{\partial}{\partial x^i}\right|_p = \sum_{i=1}^{n} b_i \left.\frac{\partial}{\partial y^i}\right|_p , \quad a_i = v(x^i) , \quad b_i = v(y^i) . \tag{2.54}$$

By exercise (2.2) we have

$$\left.\frac{\partial}{\partial y^i}\right|_p = \sum_k \left(\frac{\partial x^k}{\partial y^i}\right) \left.\frac{\partial}{\partial x^k}\right|_p , \tag{2.55}$$

with the Jacobian

$$\left(\frac{\partial x^k}{\partial y^i}\right) = \mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1}) . \tag{2.56}$$

In particular for $b = (b_1, \ldots, b_n)$ and $a = (a_1, \ldots, a_n)$ it holds that $b = \mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1}) \cdot a$. This yields the formula

$$(h_\beta \circ h_\alpha^{-1})(x, a) = (y, b) = ((\varphi_\beta \circ \varphi_\alpha^{-1})(x), \mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1})(a)) . \tag{2.57}$$

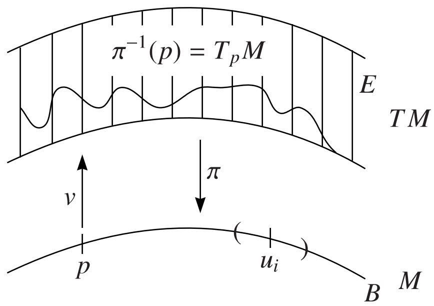
Since $(\varphi_\beta \circ \varphi_\alpha^{-1}) \in C^\infty$ and $\mathrm{d}(\varphi_\beta \circ \varphi_\alpha^{-1}) \in C^\infty$, $h_\beta \circ h_\alpha^{-1}$ is smooth.

Remark

The tangent bundle is a special case of a vector-bundle (E, B, F, π), whereas E is the so-called totale space (manifold), B the base space (manifold), F the fibre (vectorspace) and

1) π is the smooth (onto) projection map π: $E \mapsto B$, with $\pi^{-1}(p) \simeq F$.

2) $B = \bigcup_{i \in I} U_i$, $\pi^{-1}(U_i) \simeq U_i \times F$ ("locally trivial", i.e. a product)



Examples

1) $E = \mathbb{R}^m \times \mathbb{R}^k$, $B = \mathbb{R}^m$, π: $(p, q) = p$, $\pi^{-1}(p) = \mathbb{R}^k$

2) $E = TM$, $B = M$, $\pi(p, v) = p$

A **vectorfield** is a smooth map V: $M \mapsto TM$ such that $\pi \circ V = \mathrm{id}_M$ i.e. $V(p) \in T_pM$. Equivalently, V is smooth if all $f \in C^\infty(M)$. Vf is a $C^\infty(M)$ where $(Vf)(p) := V(p)(f)$ for all $p \in M$. If we define $(V + W)(p) := V(p) + W(p)$ $\forall$ w, $v \in VM$ and $(fV)(p) := f(p)V(p)$ with $f \in C^\infty(M)$. (Denote the set of all smooth vectorfields on M by VM.) The set of all vectorfields VM becomes a $C^\infty(M)$-modul.

2.8.1 Local representation of vectorfields

Let $\varphi = (x^1, \ldots, x^n)$ be a coordinate system for $U \subset M$. Then

$$\frac{\partial}{\partial x^i} : U \mapsto TU,\, p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p , \tag{2.58}$$

for $i = 1, \ldots, n$ is a vectorfield on U. Then we get from the basis theorem (theorem 1): On U every $V \in VM$ can be written in the form

$$V = \sum_{i=1}^{n} V(x^i) \frac{\partial}{\partial x^i} . \tag{2.59}$$

2.8.2 Vector fields as "differential operators"

One can treat vector fields as (geometric) vectors, but also as operators. The advantage from the treatment of operators is that in this formalism the multiplication of vector fields becomes possible.

A **derivation** of $C^\infty(M)$ is a map $\mathscr{D}$: $C^\infty(M) \mapsto C^\infty(M)$ such that the following properties hold:

1) $\mathscr{D}$ is $\mathbb{R}$-linear: $\mathscr{D}(af + bg) = a\mathscr{D}(f) + b\mathscr{D}(g)$ with $a, b \in \mathbb{R}$ and $f, g \in C^\infty(M)$
2) Leibniz rule: $\mathscr{D}(f \cdot g) = (\mathscr{D}f) \cdot g + f \cdot (\mathscr{D}g)$

These are the crucial properties for differentiation. The goal is to show that the vectorfields on M correspond to derivations of $C^\infty(M)$. The definition of tangent vectors shows that a vector field is a derivation: for $X \in VM$ it holds that $(Xf)(p) = X(p)(f)$ with $X(p) \in T_pM$. Vice versa: Let $\mathscr{D}$ be a derivation and set for $p \in M$: $v_p(f) := (\mathscr{D}f)(p)$. From the above two conditions we get that $v_p \in T_pM$ and V: $M \mapsto TM$, $p \mapsto v_p$ is a vectorfield on M. Moreover as $(Vf)(p)(= v_p(f)) = (\mathscr{D}f)(p)$ one has $Vf = \mathscr{D}f \;\forall\; f \in C^\infty(M)$. Hence, V is smooth. The advantage of this identification is that one can "multiply" vector fields: for $V, M \in VM$ define $(V \cdot M)(f) := V(W(f))$ (just by iteration). This definition makes sense since V: $C^\infty M \mapsto C^\infty M$, $f \mapsto V(f)$. The "problem" is that the product $V \cdot W$ is in general not a vector field. $\mathbb{R}$-linearity is satisfied, but not the Leibniz rule:

$$\begin{aligned}(V \cdot W)(f \cdot g) &= V(W(f \cdot g)) = V((Wf)g + f(Wg)) = \\ &= (VWf)g + (Wf)(Vg) + (Vf)(Wg) + f(VWg) .\end{aligned} \tag{2.60}$$

The trick to get rid of the blue terms which destroy the property of being a derivation is to consider the so-called Lie bracket (commutator) which is defined by

$$[V, W] := V \cdot W - W \cdot V . \tag{2.61}$$

It holds that

$$-(W \cdot V)(f \cdot g) = -(WVf)g - (Vf)(Wg) - (Wf)(Vg) - f(WVg) , \tag{2.62}$$

and therefore the object $[V, W]$ is again a vector field.
There is a general algebraic structure where such a commutator occurs:

Definition

The $\mathbb{R}$-Lie algebra is an $\mathbb{R}$-vector space L with a composition (**Lie bracket**) $[\bullet, \bullet]$: $L \times L \mapsto L$ such that $[\bullet, \bullet]$ is

1) bilinear

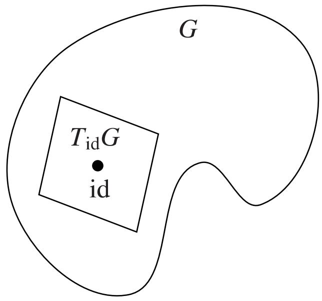
$$[ax + by, z] = a[x, z] + b[y, z] , \quad [x, ay + bz] = a[x, y] + b[x, z] , \tag{2.63}$$

2) the skew-symmetry holds $[x, y] = -[y, x]$
3) and the Jacobi-identity (cyclic permutation):

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 . \tag{2.64}$$

Remark

If G is a Lie group then $T_{\mathrm{id}}G$ is a Lie algebra. (One can see a lot of structure of the nonlinear object in the corresponding linear object.)



Example

The identity element of the Lie group

$$\mathrm{SO}(3) = \{A \in \mathbb{R}^{3\times 3} | A^{\intercal}A = E\}\,, \tag{2.65}$$

is given by

$$T_E\mathrm{SO}(3) = \{A \in \mathbb{R}^{3\times 3} | A^{\intercal} = -A\}\,. \tag{2.66}$$

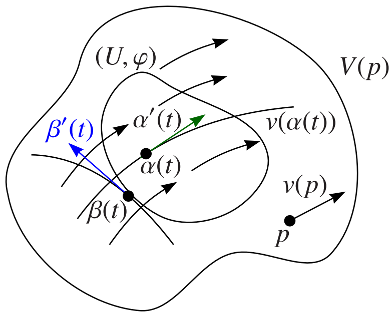
Matrices $\in T_E\mathrm{SO}(3)$ form a Lie algebra with the Lie bracket $[A,B] := AB - BA$ for A, $B \in T_E\mathrm{SO}(3)$ (matrix multiplication).

Lemma 4

$(VM, [\bullet,\bullet])$ is a $\mathbb{R}$-Lie algebra. Proof: Exercise!

2.9 Vector fields and ordinary differential equations

Let $V \in VM$ be a vector field on M. An integral curve of V is a differentiable curve α: $I \mapsto M$, $t \mapsto \alpha(t)$ such that $\alpha'(t) = V(\alpha(t))$ (*) for all $t \in I$.

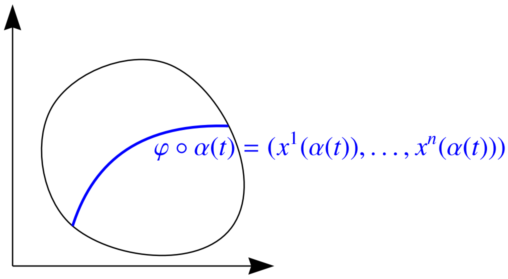


We would like to interpret $(*)$ in local coordinates: Let $\varphi = (x^1,\ldots,x^n)$ around $\alpha(t)$. Then we have

$$\alpha'(t) = \sum_{i=1}^{n} \frac{\mathrm{d}(x^i \circ \alpha(t))}{\mathrm{d}t}\, \frac{\partial}{\partial x^i}\bigg|_{\alpha(t)}\,. \tag{2.67}$$

and

$$V(\alpha(t)) = \sum_{i=1}^{n} V(x^i \circ \alpha(t))\, \frac{\partial}{\partial x^i}\bigg|_{\alpha(t)}\,. \tag{2.68}$$



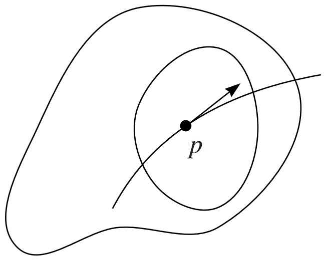
Because of $(*)$ we have $\alpha'(t) = V(\alpha(t))$ and therefore we can compare to identify the coefficients

$$\frac{\mathrm{d}(x^i \circ \alpha)}{\mathrm{d}t} = V(x^i \circ \alpha)\,, \tag{2.69}$$

for $i = 1, \ldots, n$. This is a system of first order differential equations. From existence and uniqueness of solutions of such systems (e.g. Königsberger Volume II, 4.2) with the given initial conditions (e.g. $x^1(\alpha(0))$, $i = 1, \ldots$ n) (which is equivalent to $((x^1 \circ \alpha)(0), \ldots, (x^1 \circ \alpha)(0)) = \varphi(\alpha(0)) = \varphi(p)$) we get:

Theorem 4

Let $V \in VM$ and $p \in M$ then there exists an interval $I = I(p) \subset \mathbb{R}$ around 0 and a unique integral curve α: $I \mapsto M$ of V with $\alpha(0) = p$.



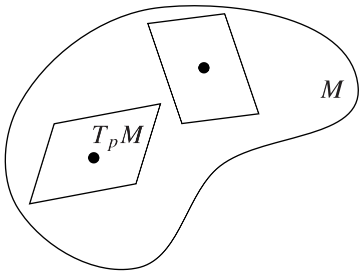
Corollary

For any $v \in T_pM$ there is a smooth curve α with $\alpha(0) = p$ and $\alpha'(0) = v$. Proof: Extend v locally to a vector field ans use theorem 4.

Kapitel 3

Riemannian metrics

We want to introduce the basic structure to study geometry on Riemannian manifolds. Let M be a smooth manifold; in every point there exists a tangent space.



A **Riemannian metric g on M** is a mapping which associates to each $p \in M$ a scalar product g_p: $T_pM \times T_pM \mapsto \mathbb{R}$ (this is a bilinear symmetric form that is positive definite) such that g_p "varies smoothly with p". More precisely: If φ: $U \mapsto \mathbb{R}^n$, $q \mapsto \varphi(q) = (x^1(q), \dots, x^n(q))$ are local coordinates at p (chart). Then the functions g_{ij}: $U \mapsto \mathbb{R}$,

$$g_{ij}(q) := g_p \left(\left.\frac{\partial}{\partial x^i}\right|_q, \left.\frac{\partial}{\partial x^j}\right|_q \right) . \tag{3.1}$$

are C^∞. In other words, the positive definite symmetric matrices of g_q (with respect to the natural basis $\{\partial/\partial x^i|q|i = 1, \dots, n\}$) for $q \in U$, $(g_{ij}(q)) \in \mathbb{R}^{n\times n}$ with $q \in U$, has C^∞ entries. **Other Notations:** Instead of g_p one writes also $\langle \bullet, \bullet \rangle_p$ (respectively, intead of g we write $\langle \bullet, \bullet \rangle$). A **Riemannian manifold** is a smooth manifold together with a Riemannian metric (M, g).

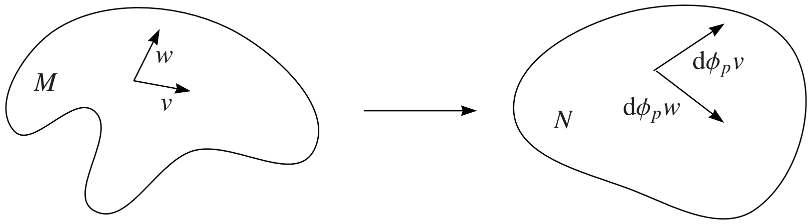
Remark

- A Riemannian **metric** is **not** a **metric** in the sense of metric spaces (distance functions). It is just called metric for historical reasons, but it can be use to construct a distance function.
- A pseudo-Riemannian metric is a smooth map $p \mapsto g_p$ with g_p a nondegenerate symmetric bilinear form e.g. Lorentz metrics $\operatorname{diag}(1, \dots, 1, -1)$, which is used in special/general relativity theory.

A smooth map ϕ: $(M, \langle \bullet, \bullet \rangle) \mapsto (N, \langle\langle \bullet, \bullet \rangle\rangle)$ is a **Riemannian isometry** if ϕ is a diffeomorphism and $\mathrm{d}\phi_p$: $(T_pM, \langle \bullet, \bullet \rangle_p) \mapsto (T_{\phi(p)}N \langle\langle \bullet, \bullet \rangle\rangle_{\phi(p)})$ is a linear isometry for all $p \in M$, i.e.

$$\langle\langle \mathrm{d}\phi_p(u), \mathrm{d}\phi_p(v) \rangle\rangle_{\phi(p)} = \langle u, v \rangle_p \, , \tag{3.2}$$

for all u, $v \in T_pM$. (This means that the differential does not change the length of a vector. From ϕ being a diffeomorphism it follows that $\mathrm{d}\phi_p$: $T_pM \mapsto T_{\phi(p)}N$ is a vectorspace isomorphism by the chain rule.) A smooth map ϕ: $M \mapsto N$ is a **local isometry** if every point $p \in M$ has a neighbourhood W such that $\phi|_W$: $W \mapsto \phi(W) \subset N$ is a Riemannian isometry. If ϕ: $M \mapsto (N, \langle\langle \bullet, \bullet \rangle\rangle)$ (with manifold M and Riemannian manifold $(N, \langle\langle \bullet, \bullet \rangle\rangle)$) is an immersion one can "pull back" the Riemannian metric on N to one on M such that ϕ becomes a local isometry, namely $\langle \bullet, \bullet \rangle := \phi^* \langle\langle \bullet, \bullet \rangle\rangle$ by definition $\langle u, v \rangle_p := \langle\langle \mathrm{d}\phi_p(u), \mathrm{d}\phi_p(v) \rangle\rangle_{\phi(p)}$ for all $p \in M$.



Is $\langle \bullet, \bullet \rangle_p$ positive definite? Yes, it is!

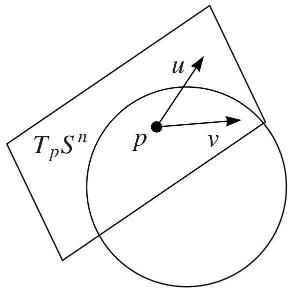
$$0 = \langle u, u \rangle_p = \|u\|_p^2 = \|d\phi_p(u)\|^2 \Rightarrow d\phi_p(u) = 0 \Rightarrow u = 0\,, \tag{3.3}$$

since ϕ is an **immersion**.

Examples

1) Classical geometries:

   - Euclidian geometry: ($\mathbb{R}^n$, standard scalar product)
   - Spherical geometry:
     Embed S^n in $\mathbb{R}^{n+1}$. Instead of lines take great circles and draw triangles. Think of the tangent vectors as sitting in $T_p\mathbb{R}^{n+1} = \{p\} \times \mathbb{R}^{n+1} \simeq \mathbb{R}^{n+1}$, $(u, v) \in T_pS^n \subset T_p\mathbb{R}^{n+1}$ with the pullback metric $\langle u, v \rangle := u \cdot v$.

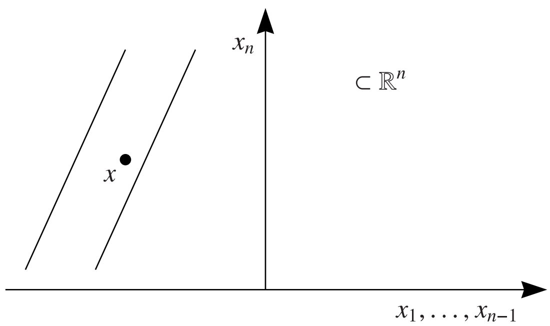


   - Hyperbolic geometry
   - Projective geometry

2) $(\mathbb{R}^n, \langle \bullet, \bullet \rangle)$ with the standard scalar product $\langle \bullet, \bullet \rangle$ is a Riemannian manifold.

3) $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$
   Since this is a subset there exists and immersion, namely the standard inclusion map $S^n \hookrightarrow \mathbb{R}^n$. S^n with the pullback metric from $\mathbb{R}^{n+1}$ is a Riemannian metric.

4) Hyperbolic space: $H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n > 0\}$

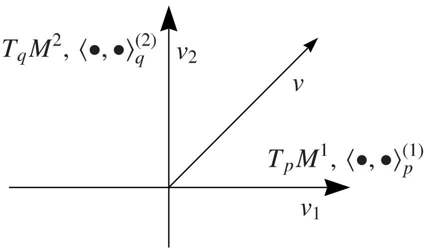


   Riemannian metric: Use $\{\mathrm{id}|_{H^n}\}$ a an atlas:

$$g_{ij}(x) := \begin{cases} 1/(x^n)^2 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} . \tag{3.4}$$

4) Riemannian products: Produce new Riemannian manifolds from given ones! Consider the Riemannian manfolds $(M_1, \langle \bullet, \bullet \rangle^{(1)})$, $(M_2, \langle \bullet, \bullet \rangle^{(2)})$. Then

$$T_{(p,q)}(M_1 \times M_2) = T_p M_1 \oplus T_q M_2 = (v_1, v_2)\,. \tag{3.5}$$



$$\pi_i : M_1 \times M_2 \mapsto M_i\,, \quad \mathrm{d}\pi_i|_{(p,q)}(v) = v_i\,, \tag{3.6}$$

for $i = 1, 2$. Define the Riemannian product metric as

$$u = (u_1, u_2),\ v = (v_1, v_2) \in T_{(p,q)}(M^1 \times M^2),\ \langle u, v \rangle_{(p,q)} := \langle u_1, v_1 \rangle_p^{(1)} + \langle u_2, v_2 \rangle_q^{(2)}\,. \tag{3.7}$$

Note "Pythagoras"

$$\|v\|^2_{(p,q)} = (\|v_1\|_p^{(1)})^2 + (\|v_2\|_q^{(2)})^2\,, \tag{3.8}$$

and

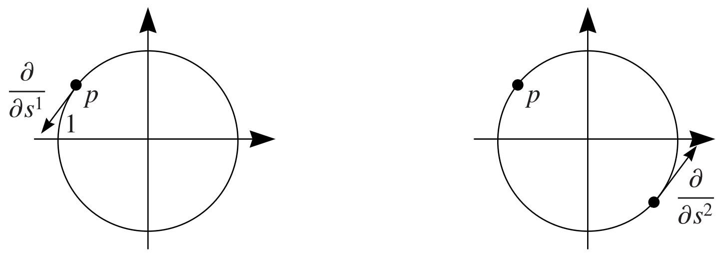
$$\langle (u_1, 0), (0, v_2) \rangle_{(p,q)} = 0\,, \quad (u_1, 0) \in T_p M^{(1)}\,, \quad (0, v_2) \in T_q M^{(2)}\,, \tag{3.9}$$

i.e. $T_p M^{(1)} \perp T_q M^{(2)}$. The product metric is constructed in such a way that these two manifolds are orthogonal.

4a) $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ with standard Riemannian metrics

4b) Flat torus $T^2 = S^1 \times S^1$ with $M_i = S^1 = \{x \in \mathbb{R}^2 | \|x\| = 1\} \in \mathbb{R}^2$
Endow each factor with the Riemannian metric induced from $\mathbb{R}^2$. Let $\partial/\partial s^1$ (respectively $\partial/\partial s^2$) denote the unit tangent vector fields on S^1.

Hence, Riemannian geometry is a vast generalization of "classical geometry".



Then

$$T_{(p,q)}T^2 = T_{(p,q)}(S^1 \times S^1) = \mathbb{R}\, \frac{\partial}{\partial s^1}\bigg|_p \oplus \mathbb{R}\, \frac{\partial}{\partial s^2}\bigg|_q\,, \tag{3.10}$$

$$u = a_1\, \frac{\partial}{\partial s^1}\bigg|_p + a_2\, \frac{\partial}{\partial s^2}\bigg|_q \in T_{(p,q)}T^2\,, \quad \mathrm{d}\pi_i(u) = a_i \frac{\partial}{\partial s^i}\,, \quad i = 1, 2\,. \tag{3.11}$$

By calculate the product metric in the basis

$$\left\{ \left(\frac{\partial}{\partial s^1}\bigg|_p, 0 \right), \left(0, \frac{\partial}{\partial s^2}\bigg|_q \right) \right\}, \tag{3.12}$$

and obtain

$$g_{\|}(p,q) = \left\| \mathrm{d}\pi_1 \left(\frac{\partial}{\partial s^1}\bigg|_p, 0 \right) \right\|^2 + \left\| \mathrm{d}\pi_2 \left(\frac{\partial}{\partial s^1}\bigg|_p, 0 \right) \right\|^2 = 1\,. \tag{3.13}$$

Moreover $g_{12}(p,q) = g_{21}(p,q) = 0$, and $g_{22}(p,q) = 1$. This implies

$$(g_{ij}(p,q)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.14}$$

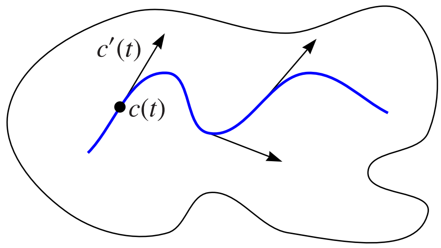
which is the same metric as for the standard Riemannian metric on $\mathbb{R}^2$. The reason for this is that T^2 and $\mathbb{R}^2$ are locally isometric Riemannian manifolds, but not globally isometric ones. (Otherwise T^2 and $\mathbb{R}^2$ would be diffeomorphic, in particular homemorphic. But this is a contradiction since T^2 is compact but $\mathbb{R}^2$ not.) The rotation torus is a different realization. Its Riemannian metric from is the pullback metric from the immersion in $\mathbb{R}^3$. They are topologically the same but not as Riemannian manifolds. $T^2 = S^1 \times S^1$ is the flat torus defined by the product structure of two circles.

3.1 The length of smooth curves

What are Riemannian metrics good for? The basic reason is the concept of **curvature**.
Let $c\colon I \mapsto (M, \langle \bullet, \bullet \rangle)$ be a smooth curve. The tangent vector field of c is

$$c'(t) \equiv \frac{\mathrm{d}c}{\mathrm{d}t}(t) := \mathrm{d}c|_t \left(\frac{\partial}{\partial t}\right) \in T_{c(t)}M \;\forall\, t \in I\,. \tag{3.15}$$

a smooth vector field along c.



The **length** of $c\colon [a,b] \mapsto (M, \langle \bullet, \bullet \rangle)$ (with respect to the given Riemannian metric) is defined as

$$L(c) := \int_a^b \sqrt{\left\langle \frac{\mathrm{d}c}{\mathrm{d}t}, \frac{\mathrm{d}c}{\mathrm{d}t} \right\rangle_{c(t)}}\, \mathrm{d}t = \int_a^b \|c'(t)\|_{c(t)}\, \mathrm{d}t = \int_a^b \sqrt{g_{c(t)}(c'(t), c'(t))}\, \mathrm{d}t\,, \tag{3.16}$$

which is the same formula as in $\mathbb{R}^n$.

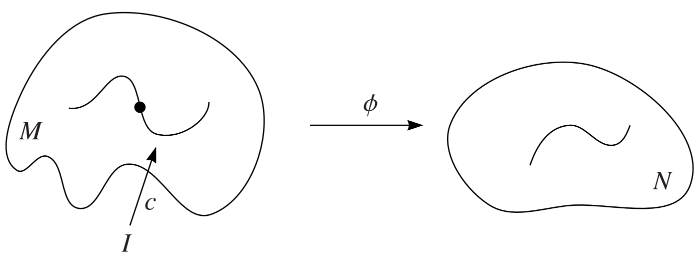
3.1.1 Properties of the definition for the length

1) $L(c)$ is independent of parameterization (i.e. a property of the image set $c(I) \subset M$).

2) More precisely: Let $s = s(t)\colon I = [a,b] \mapsto J = [s(a), s(b)]$, $t \mapsto s(t)$ be a diffeomorphism. With the transformation formula for integrals one obtains:

$$L_{(t)}(c) = \int_a^b \left\|\frac{\mathrm{d}c}{\mathrm{d}t}\right\| \mathrm{d}t \underset{\text{chain rule}}{=} \int_a^b \left\|\frac{\mathrm{d}c(s(t))}{\mathrm{d}s}\right\| \left|\frac{\mathrm{d}s}{\mathrm{d}t}\right| \mathrm{d}t = \int_{s(a)}^{s(b)} \left\|\frac{\mathrm{d}c}{\mathrm{d}s}\right\| \mathrm{d}s = L_{(s)}(c)\,. \tag{3.17}$$

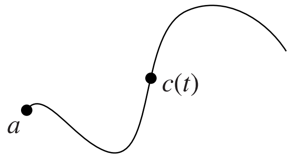
2) If $\phi\colon (M, \langle \bullet, \bullet \rangle) \mapsto (N, \langle\langle \bullet, \bullet \rangle\rangle)$ is a Riemannian isometry and $c\colon I \mapsto M$ a smooth curve of length l, then $\phi \circ c$ is also a smooth curve of length l: $L(\phi(c)) = l = L(c)$.



Use $\|\mathrm{d}\phi_{c(t)} c'(t)\|_{\phi(c(t))} = \|c'(t)\|_{c(t)}$.

3) Every curve with $c'(t) \neq 0$ for all $t \in I$ can be parameterized by arc-length. Namely we set

$$s(t) := \int_a^t \left\| \frac{\mathrm{d}c}{\mathrm{d}\tau} \right\|_{c(\tau)} \mathrm{d}\tau \,. \tag{3.18}$$



In particular one then has (with the chain rule):

$$\left\| \frac{\mathrm{d}c}{\mathrm{d}s} \right\| = \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\| \cdot \left| \frac{\mathrm{d}t}{\mathrm{d}s} \right| = 1 \,. \tag{3.19}$$

Note: Parameterization by arc-length is equivalent to saying that c is a local isometry of $[0, L(c)] \mapsto I$, i.e. one-dimensional Riemannian manifolds are locally isometric.

3.2 Existence of Riemannian manifolds

Theorem 1

One every n-dimensional smooth manifold there exists a Riemannian metric.

Proof

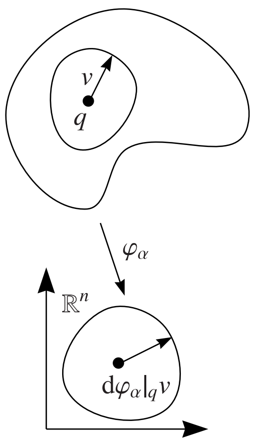
The proof has to main steps:

1) Local construction for each chart of an atlas M:

Consider φ_α: $U_\alpha \mapsto \mathbb{R}^n$, $q \mapsto (x_\alpha^1(q), \ldots, x_\alpha^n(q))$. A Riemannian metric on U_α is given by $n(n+1)/2$ C^∞-functions. Locally a Riemannian metric looks like

$$\begin{pmatrix} g_{11}^{(\alpha)}(q) & \ldots & g_{1n}^{(\alpha)}(q) \\ \vdots & \ddots & \vdots \\ g_{n1}^{(\alpha)}(q) & \ldots & g_{nn}^{(\alpha)}(q) \end{pmatrix} , \tag{3.20}$$

and it is a positive definite, symmetric matrix for all $q \in U_\alpha$. One way to produce such a family of matrices on U_α is by the pullback construction. **One possibility** is to pick the standard scalar product on $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$: $\langle \bullet, \bullet \rangle$ (i.e. if $\{i_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$ then $\langle e_i, e_j \rangle = \delta_{ij}$). Then set for all u, $v \in T_qM$, $q \in U_\alpha$: $g_\alpha(u, v) := \langle \mathrm{d}\varphi_\alpha|_q u, \mathrm{d}\varphi_\alpha|_q v \rangle$.



Since

$$\mathrm{d}\varphi_\alpha|_q \left(\left. \frac{\partial}{\partial x_\alpha^i} \right|_q \right) = e_i \,, \quad i = 1, \ldots, n \,, \tag{3.21}$$

this definition is equivalent to

$$g_{ij}^{(\alpha)}(q) := g_\alpha\left(\left.\frac{\partial}{\partial x_\alpha^i}\right|_q, \left.\frac{\partial}{\partial x_\alpha^j}\right|_q\right) = \langle e_i, e_j\rangle = \delta_{ij}\,. \tag{3.22}$$

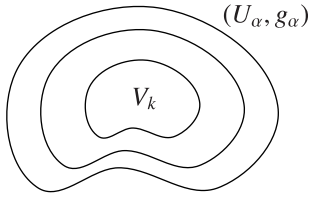
Example: For $n = 2$ we need locally

$$\begin{pmatrix} g_{11}(q) & g_{12}(q) \\ g_{21}(q) & g_{22}(q) \end{pmatrix}\,. \tag{3.23}$$

This matrix is positive definite if it obeys the Hurwitz criteria: $g_{11}(q) > 0$ and $g_{11}(q)g_{22}(q) - g_{12}(q)^2 > 0$.

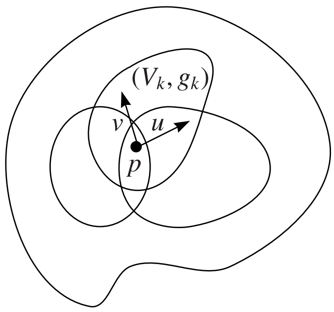
2) Global construction:

Paste together the "local metrics" using a partition of unity. Let $\mathcal{A} = \{U_\alpha\}_{\alpha\in A}$ be the given atlas for M and $(V_k)_{k\in I}$ a locally finite refinement and C^∞-functions f_k: $M \mapsto \mathbb{R}$ such that firstly $f_k \geq 0$ on $\overline{V}_k$, $f_k = 0$ on $M \setminus \overline{V}_k$ and secondly $(\sum_{k\in I} f_k)(p) = 1$ for all $p \in M$ $(M = \bigcup_{\alpha\in A} U_\alpha = \bigcup_{k\in I} V_k)$. For each $k \in I$ pick some $\alpha \in A$ such that $V_k \subset U_\alpha$ and set $g_k := g_\alpha|_{V_k}$ (we restrict the metric to a subset).



For $p \in M$ and u, $v \in T_pM$ we set:

$$\boxed{\langle u, v\rangle_p := \sum_{k\in I} f_k(p)g_k(p)(u, v)\,.} \tag{3.24}$$



We have to **show** that $\langle\bullet, \bullet\rangle_p$ is a scalar product for each p:

- $\langle u, v\rangle_p = \langle v, u\rangle_p$ ✓
- bilinearity ✓
- $\langle u, u\rangle_p \geq 0$, since $f_k \geq 0$, and $g_k(u, u) \geq 0$
- $\langle u, u\rangle_p = 0$ is equivalent to

$$\sum_k f_k(p)g_k(p)(u, u) = 0 \Rightarrow f_k(p)g_k(p)(u, u)\,, \tag{3.25}$$

for all k. From the property (2) of f_k we have that there is a $k_0 \in I$ such that $f_{k_0}(p) > 0$. Hence $g_{k_0}(u, u) = 0$ and $u = 0$.

Kapitel 4

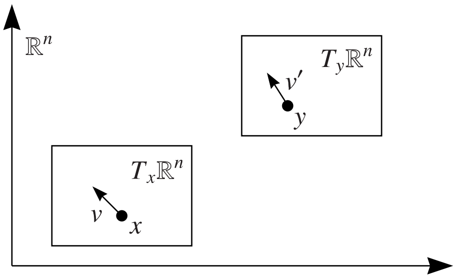
Affine connections and parallel translation

4.1 Motivation

In $\mathbb{R}^n$ one can "compare" different tangent spaces by **translation**. The tangent space is given by $T_x\mathbb{R}^n = \{x\} \in \mathbb{R}^n$ for $x \in \mathbb{R}^n$ and consider additionally $T_y\mathbb{R}^n = \{y\} \times \mathbb{R}^n$. (One can then compare tangent vectors.)

$$(x, y) \in T_x\mathbb{R}^n \xrightarrow{T_{x-y}} (y, v) \in T_y\mathbb{R}^n\,. \tag{4.1}$$

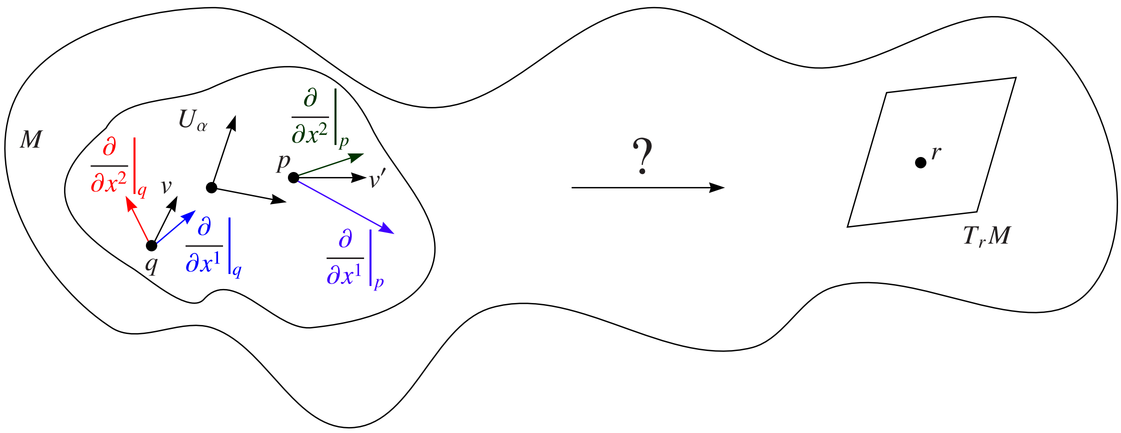
Can one construct a tool that allows us to compare these different tangent spaces?



The consequence is that $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n} T_x\mathbb{R}^n$ is a global product. This is not true for general manifolds: TM is not a product globally, it is only a product locally. For (U, φ) a chart then $TU = U_\alpha \times \mathbb{R}^n$, since we have local basefields $(\partial/\partial x^1, \ldots, \partial/\partial x^n)$.

$$v \in T_qU_\alpha \Rightarrow v = \sum_i v(x^i) \left.\frac{\partial}{\partial x^i}\right|_q . \tag{4.2}$$

Vectors are parallel if they have the same component with respect to the base field. There is no way to compare tangent space with another tangent space at r, because we do not have the base fields there. (On $\mathbb{R}^n$ we have the globally defined base fields $\{e_1, \ldots, e_n\}$.) The affine connection is tool to compare points that are arbitrarily far away.



If

$$v = \sum_i a_i \left.\frac{\partial}{\partial x^i}\right|_q , \quad v' = \sum_i a_i \left.\frac{\partial}{\partial x^i}\right|_p , \tag{4.3}$$

then v is parallel to v'.

4.1.1 The affine connection

An **affine connection** (or **covariant derivative**) D on a differentiable manifold M is a map $D\colon \mathcal{V}M \times \mathcal{V}M \mapsto \mathcal{V}M$, $(X,Y) \mapsto D_X Y$ such that for all X, Y, $Z \in \mathcal{V}M$ and f, $g \in C^\infty(M)$ one has the following properties:

1) linearity in the first argument: $D_{fX+gY}Z = fD_X Z + gD_Y Z$
2) additivity in the second argument: $D_X(Y+Z) = D_X Y + D_X Z$
3) $D_X(fY) = fD_X Y + (Xf)Y$

The **goal** now is to introduce the notion of **parallel-translation** (along a curve).

Example

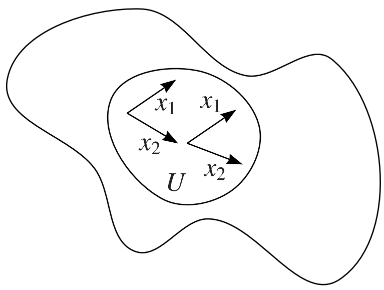
A simple nontrivial example is $M = \mathbb{R}^n$ with the vector field $Y = \sum_{i=1}^n v^i \partial/\partial x^i (\equiv (v1^1, \ldots, v^n))$. (This vector field is defined globally on $\mathbb{R}^n$ with the base fields v^i.)

$$D_X Y = \sum_{i=1}^n X(v^i) \frac{\partial}{\partial x^i} . \tag{4.4}$$

Note: If $Y = \text{const.} = (a^1, \ldots, a^n)$ then $D_X Y = 0$.

4.1.2 Local description of an affine connection

Pick some chart (U, φ) which leads to local base fields $X_i := \partial/\partial x^i$ for $i = 1, \ldots, n$.



If

$$X = \sum_{i=1}^n v^i X_i , \quad Y = \sum_{j=1}^n w^j X_j , \tag{4.5}$$

then

$$D_X Y = D_{\sum_{i=1}^n v^i X_i} \left(\sum_{j=1}^n w^j X_j \right) = \sum_{i=1}^n v^i D_{X_i} \left(\sum_{k=1}^n w^j X_j \right) = \sum_{i,j=1}^n v^i w^j D_{X_i} X_j + \sum_{i,j=1}^n v^i X_i(w^j) X_j . \tag{4.6}$$

We write $D_{X_i} X_j$ (which is $\in \mathcal{V}M$) in the form

$$D_{X_i} X_j =: \sum_{k=1}^n \Gamma_{ij}{}^k X_k , \tag{4.7}$$

with $\Gamma_{ij}{}^k \in C^\infty(U)$. Hence

$$D_X Y = \sum_{k=1}^n \left\{ \sum_{i,j=1}^n v^i w^j \Gamma_{ij}{}^k + X(w^k) \right\} X_k . \tag{4.8}$$

This formula shows that $(D_X Y)(p)$ is determined by the values $v^i(p)$, $w^j(p)$, and $X_p(w^k)$. In particular one needs to know the vector field Y in direction of X. The **consequence** of this observation is that one can define the "derivative" along a curve in the direction of that curve. More precisely: If Y is a vector field along a curve C, i.e. $Y(c(t)) = \sum_k w^k(t) X_k(c(t))$, where $X_k(c(t))$ is the tangent vector at the point $c(t)$. Then we define the covariant derivative along the tangent vector of the curve as

$$D_{\dot{c}} Y := \sum_{k=1}^{n} \left\{ \sum_{i,j=1}^{n} \dot{x}^i(t) w^j(t) \Gamma_{ij}{}^k(c(t)) + \dot{w}^k(t) \right\} X_k \,. \tag{4.9}$$

Here we use that the tangent vector $\dot{c}$ of c is given by

$$\dot{c} = \sum_{i=1}^{n} a^i X_i \,, \tag{4.10}$$

whereas the basis theorem said (with $(\varphi \circ c)(t) = (x^1(t), \ldots, x^n(t))$)

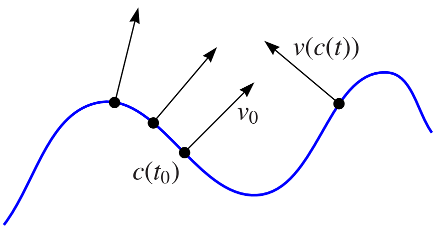
$$\begin{aligned} a^i &= \dot{c}(x^i) = \dot{c}(u^i \circ \varphi) = \mathrm{d}c\left(\frac{\partial}{\partial t}\right)(u^i \circ \varphi) = \mathrm{d}c\left(\frac{\partial}{\partial t}\right)(u^i \circ \varphi) = \frac{\partial}{\partial t}(u^i \circ \varphi \circ c) = \\ &= \frac{\partial}{\partial t}(x^i(t)) = \frac{\mathrm{d}}{\mathrm{d}t} x^i(t) = \dot{x}^i \,. \end{aligned} \tag{4.11}$$

Example

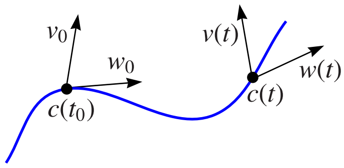
Define in $\mathbb{R}^2$ the two vector fields $X = (v^1, v^2)$ and $Y = (w^1, w^2)$ with v^i, $w^j \in C^\infty(\mathbb{R}^2)$. The derivative of Y in the direction of X is given by $(X(w^1), X(w^2))$, whereas X is the directional derivative. In $\mathbb{R}^2$, if Y is a field of "parallel" vectors then Y is constant. Then these directional derivatives vanish.
On a general manifold a vector field Y **along a curve** $c \colon I \mapsto M$ is **parallel** (with respect to D) if and only if $D_{\dot{c}} Y = 0$.

Theorem 2

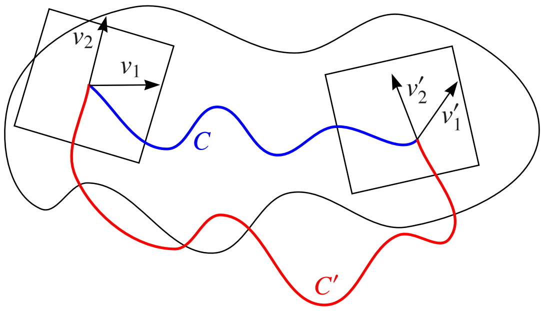
Let M be a differentiable manifold with an affine connection D. Let $c \colon I \mapsto M$ be a smooth curve and $v_0 \in T_{c(t_0)} M$. Then there exists a unique parallel vector field V along c such that $V(c(t_0)) = v_0$.



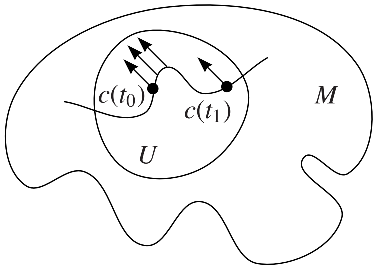
We say that $V(t) := V(c(t))$ is the **parallel translate** of v_0 along c. The map denoted by $c\|_{t_0}^{t} \colon T_{c(t_0)} M \mapsto T_{c(t)} M$, $v_0 \mapsto V(t)$ is called **parallel translation along** c.



The property that parallel translated vector fields can depend on the curve is called holonomy.



Proof



Be $V = \sum_{j=1}^{n} w^j X_j$.

1.) Let $t_1 \in I$ such that $c([t_0, t_1])$ is contained in a chart domain (U, φ). In this chart the equation

$$D_{\dot{c}} V = \sum_{k=1}^{n} \left\{ \sum_{i,j=1}^{n} \dot{x}^i(t) w^j(t) \Gamma_{ij}{}^k(c(t)) + \dot{w}^k(t) \right\} X_k = 0\,, \tag{4.12}$$

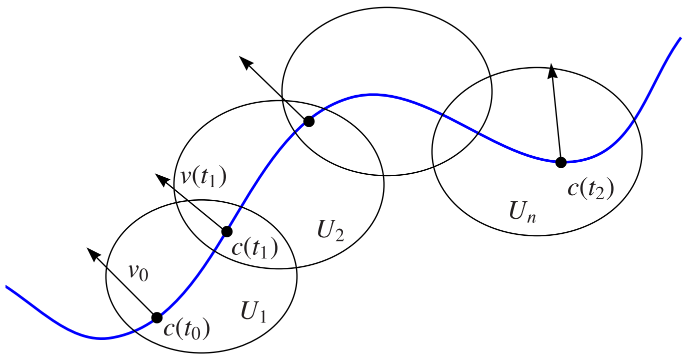
is equivalent to

$$0 = \sum_{i,j=1}^{n} \Gamma_{ij}{}^k w^j \frac{\mathrm{d}x^i}{\mathrm{d}t} + \frac{\mathrm{d}w^k}{\mathrm{d}t}\,, \tag{4.13}$$

for $k = 1, \ldots, n$. For given initial conditions $w(t_0) = (w^1(t_0), \ldots, w^n(t_0))$ this system of linear differential equations has a unique solution $w(t) = (w^1(t), \ldots, w^n(t))$ defined on $[t_0, t_1]$. $w(t_0)$ is equivalent to v_0 and $w(t)$ is equivalent to $v(t)$, $t \in [t_0, t_1]$.

2) Extension to all of c:

Let $t_2 \in I$ be arbitrary. Since $c([t_0, t_2])$ is a compact subset of M it can be covered by finitely many charts. In every chart there is a unique solution/parallel field by step (1). From transitivity is follows that there is a unique parallel field v on $c[t_0, t_1]$.

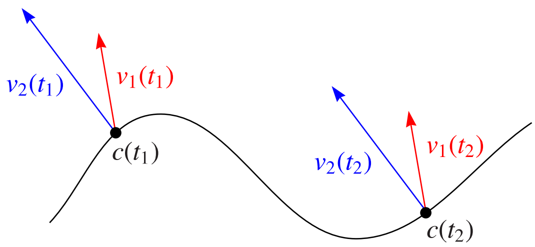


(This topology argument would not work if the differential equation was a nonlinear equation. Then a unique solution is only defined in an ϵ-subset. These ϵ-subsets may get smaller and smaller with the result than one could never reach t_2.)

4.2 The Levi-Civita connection

So far we have considered manifolds M equipped with Riemannian metrics g_{ij} and affine connections $\Gamma_{ij}{}^k$.

An affine connection D on a Riemannian manifold $(M, \langle \bullet, \rangle)$ is called **compatible** with $\langle \bullet, \bullet \rangle$ if for every differentiable curve c and every pair of **parallel** vector fields v_1, v_2 along c: $I \mapsto M$ one has $\langle v_1(t), v_2(t) \rangle_{c(t)} =$ const.



Hence parallel translation along c is a linear isometry $c\|_{t_1}^{t_2}$: $T_{c(t_1)}M \mapsto T_{c(t_2)}M$ for t_1, $t_2 \in I$.

4.2.1 Equivalent formulations of "compatible"

Theorem 2

Let $(M, \langle \bullet, \bullet \rangle)$ be a Riemannian manifold and D an affine connection.

1) D is compatible with $\langle \bullet, \bullet \rangle$ if and only if for **arbitrary** vector fields V, W along an arbitrary curve c one has for all $t \in I$

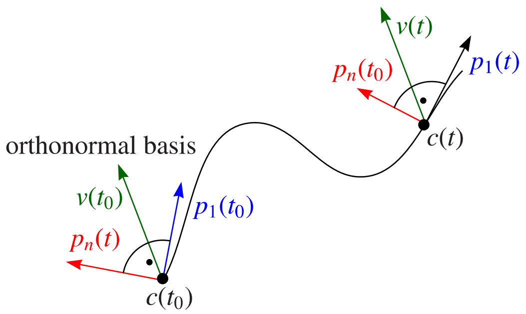
$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle V, W\rangle = \langle D_i V, W\rangle + \langle V, D_j W\rangle\,.} \tag{4.14}$$

2) D is compatible with $\langle \bullet, \bullet \rangle$ if and only if for all X, Y, $Z \in \mathcal{V}M$ one has

$$\boxed{X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle\,.} \tag{4.15}$$

Proof

1) It is clear that from (4.14) follows compatibility (just choose V, W parallel).



If D is compatible we choose an orthonormal basis $\{P_1(t_0), \ldots, P_n(t_0)\}$ of $T_{c(t_0)}M$. By assumption parallel translation is a linear isometry between each of the two tangent spaces. This yields an orthonormal basis for all $t \in I$: $\{P_1(t), \ldots, P_n(t)\}$. We then can write

$$V(t) = \sum_{i=1}^{n} v_i(t) P_i(t)\,, \quad W(t) = \sum_{j=1}^{n} w_j(t) P_j(t)\,, \tag{4.16}$$

whereas $v_i(t)$ and $w_j(t)$ are C^∞-functions. Hence

$$D_{\dot c} V = \sum_{i=1}^{n} \dot c(v_i) P_i(t) + \sum_{i=1}^{n} v_i D_{\dot c} P_i\,. \tag{4.17}$$

Since the P_i are parallel translated fields it holds that $D_{\dot c} P_i = 0$ and the second sum vanishes. Furthermore $\dot c(v_i) = \dot v_i = \mathrm{d}v_i/\mathrm{d}t$. The computation of the left-hand side of (4.14) yields

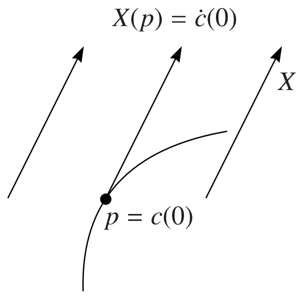
$$\frac{\mathrm{d}}{\mathrm{d}t}\langle V, W\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \sum_{i=1}^{n} v_i P_i, \sum_{j=1}^{n} w_j P_j \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \sum_{i=1}^{n} v_i w_i \right\rangle = \sum_{i=1}^{n} (\dot v_i w_i + v_i \dot w_i)\,, \tag{4.18}$$

whereas the computation of the right-hand side leads to

$$\langle D_{\dot c} V, W\rangle + \langle V, D_{\dot c} W\rangle = \left\langle \sum_{i=1}^{n} \dot v_i P_i, \sum_{j=1}^{n} w_j P_j \right\rangle + \left\langle \sum_{i=1}^{n} v_i P_i, \sum_{j=1}^{n} \dot w_j P_j \right\rangle = \sum_{i=1}^{n} (\dot v_i w_i + v_i \dot w_i)\,. \tag{4.19}$$

We realize that both sides are equal.

2) (4.14) follows from (4.15). Just specialize: Take $x = \dot c$. We have still to show the inverse direction. Pick $p \in M$ and a curve c with $c(0) = p$ and $\dot c(0) = X(p)$.



Then

$$X(p)\langle Y, Z\rangle = \dot{c}(0)\langle Y, Z\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \langle Y(c(t)), Z(c(t))\rangle_{c(t)} \overset{(*)}{=} \langle D_{X(p)}Y, Z\rangle_p + \langle Y, D_{X(p)}Z\rangle_p \,. \tag{4.20}$$

Since p is arbitrary (4.15) holds. □

Question: Is there a connection compatible with a given metric? To answer this question definitely we need one **more** definition: An affine connection D is called **symmetric** if $D_X Y - D_Y X = [X, Y] = XY - YX$ for all X, $Y \in \mathcal{V}M$.

Remark

In local coordinates (U, φ) we have for a symmetric connection and base fields $\partial/\partial x^i = X_i$ $(i = 1, \ldots, n)$:

$$D_{X_i}X_j - D_{X_j}X_i = \sum_{k=1}^n {\Gamma_{ij}}^k X_k - \sum_{k=1}^n {\Gamma_{ji}}^k X_k = \sum_{k=1}^n ({\Gamma_{ij}}^k - {\Gamma_{ji}}^k) X_k \,. \tag{4.21}$$

The right-hand side leads to

$$[X_i, X_j] = \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i} = 0 \,, \tag{4.22}$$

because locally this is the ordinary derivative

$$\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}(f) = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x^j \partial x^i} = \frac{\partial^2 (f \circ \varphi^{-1})}{\partial x^i \partial x^j} \,, \tag{4.23}$$

whenever $f \circ \varphi^{-1} \in C^2$. This is alright with our assumption $f \circ \varphi^{-1} \in C^\infty$. So we have ${\Gamma_{ij}}^k = {\Gamma_{ji}}^k$ for all i, j, and k.

Theorem 3

On every Riemannian manifold $(M, \langle \bullet, \bullet \rangle)$ there is a unique affine connection D such that

1) D is symmetric and
2) D is compatible with $\langle \bullet, \bullet \rangle$.

This unique connection is called **the** Levi-Civita connection of $(M, \langle \bullet, \bullet \rangle)$.

Proof

Let us assume that D **exists**. What are its properties then? **Trick:** D is compatible by using (4.15) (take cyclic permutation):

$$X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle \,, \tag{4.24a}$$

$$Y\langle Z, X\rangle = \langle D_Y Z, X\rangle + \langle Z, D_Y X\rangle \,, \tag{4.24b}$$

$$Z\langle X, Y\rangle = \langle D_Z X, Y\rangle + \langle X, D_Z Y\rangle \,. \tag{4.24c}$$

By using the symmetry of D this leads to

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = \langle [X, Z], Y\rangle + \langle [Y, Z], X\rangle + \langle [X, Y], Z\rangle + 2\langle Z, D_Y X\rangle \,. \tag{4.25}$$

Hence we get the **Kozul-formula**:

$$\langle Z, D_Y X\rangle = \frac{1}{2}\left\{X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle [X,Z],Y\rangle - \langle [Y,Z],X\rangle - \langle [X,Y],Z\rangle\right\}\ . \tag{4.26}$$

(4.26) shows that D is **uniquely** determined by $\langle\bullet,\bullet\rangle$: If D, $\widetilde{D}$ are two connections with the properties (1), (2) then by (4.26) we have $\langle Z, \widetilde{D}_Y X\rangle = \langle Z, D_Y X\rangle$ for all X, Y, $Z \in \mathcal{V}M$. This is equivalent to $\langle Z, \widetilde{D}_Y X - D_Y X\rangle = 0$ and hence $\widetilde{D}_Y X - D_Y X = 0$ for all X, Y (since $\langle\bullet,\bullet\rangle$ is non-degenerate). This leads to $\widetilde{D} = D$. The **existence** is clear by using (4.26) as a **definition** and checking that D satisfies the axioms (AC 1), (AC 2), and (AC 3). □

4.2.2 The local form of the Levi-Civita connection

Given a chart (U,φ) we have base fields $X_i := \partial/\partial x^i$ $(i = 1, \ldots, n)$ on U. Let $g_{ij} := \langle X_i, X_j\rangle$ and $D_{X_i}X_j =: \sum_{l=1}^n {\Gamma_{ij}}^l X_l$ with the **Christoffel-symbols** ${\Gamma_{ij}}^l$. The Kozul-formula is equivalent to the following equation:

$$\begin{aligned}\langle X_k, D_{X_i}X_j\rangle &= \sum_{l=1}^n {\Gamma_{ij}}^l \langle X_k, X_l\rangle = \sum_{l=1}^n {\Gamma_{ij}}^l g_{kl} = \frac{1}{2}\left\{X_j\langle X_i,X_k\rangle + X_i\langle X_j,X_k\rangle - X_k\langle X_i,X_j\rangle\right\} = \\ &= \frac{1}{2}\left\{\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right\}\ ,\end{aligned} \tag{4.27}$$

whereas the commutators $[\bullet,\bullet]$ of the base fields vanish, since the test functions are assumed to be C^∞. The matrix (g_{ij}) has the inverse $(g_{ij})^{-1} = (g^{ij})$ and hence we obtain by multiplying with g^{mk}:

$$\boxed{{\Gamma_{ij}}^m = \frac{1}{2}\sum_{k=1}^n g^{mk}\left\{\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right\}\ .} \tag{4.28}$$

This again shows that the Christoffel symbols (Levi-Civita connection) is completely determined by the Riemannian metric. Furthermore one realizes the symmetry in (i,j).

Remarks

The same result holds for pseudo-Riemannian manifolds. (One just has to claim non-degeneracy of the bilinear form $\langle Z, X\rangle$.) This is important in special/general relativity, where one has pseudo-Riemannian manifolds. Let us consider one example, namely ($\mathbb{R}^n$, standard scalar product) i.e. $g_{ij} = \delta_{ij}$ and hence ${\Gamma_{ij}}^k \equiv 0$. This is Euclidian geometry. So the ${\Gamma_{ij}}^k$ measures how the geometry of a (pseudo-)Riemannian manifold differs from Euclidian geometry.

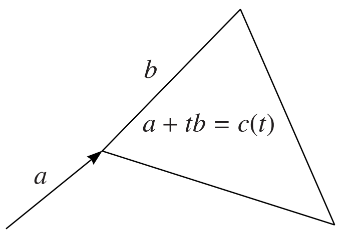
Kapitel 5

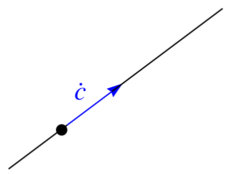
Geodesic lines

Be $(M, \langle \bullet, \bullet \rangle)$ a Riemannian manifold with Levi-Civita connection D. The goal is to find/define an analogue of a "straight line" in M. There are two characteristics for a straight line in $\mathbb{R}^n$:

1) It is the shortest connection between two points (measuring of lengths).
2) $\ddot{c} = 0$, which is a differential equation.

Hence, straight lines can be described by two different concepts: length (variational problem) and a differential equation.





In Euclidian space it holds that ${\Gamma_{ij}}^k = 0$. The fact that the tangent vector field is parallel just means $D_{\dot{c}}\dot{c} = \ddot{c} = 0$.

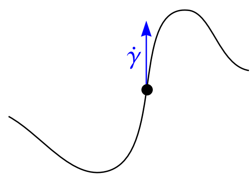
5.1 Definition of geodesic lines

Definition

A geodesic line in $(M, \langle \bullet, \bullet \rangle)$ is a differentiable curve $\gamma \colon I \mapsto M$ such that $D_{\dot{\gamma}(t)}\dot{\gamma}(t) = 0$ for all $t \in I$ i.e. the tangent vector field $\dot{\gamma}(t) \equiv \mathrm{d}/\mathrm{d}t(\gamma(t)) = \mathrm{d}\gamma(\partial/\partial t)$ is parallel along γ.

Remark

This generalizes the concept of a straight line in Euclidian geometry: $D_{\dot{\gamma}}\dot{\gamma} = \ddot{\gamma}$.



Some consequences of this definition are:

1) $\|\dot{\gamma}\|_{\gamma(t)} = \text{const.} =: k$

   Proof:

$$\|\dot{\gamma}\|^2 = \langle \dot{\gamma}, \dot{\gamma} \rangle \Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\langle \dot{\gamma}, \dot{\gamma} \rangle = \langle D_{\dot{\gamma}}\dot{\gamma}, \dot{\gamma} \rangle + \langle \dot{\gamma}, D_{\dot{\gamma}}\dot{\gamma} \rangle = 0\,, \tag{5.1}$$

   since $D_{\dot{\gamma}}\dot{\gamma} = 0$. In other words, geodesics have constant speed.

2) A geodesic is parameterized proportional to arc-length:

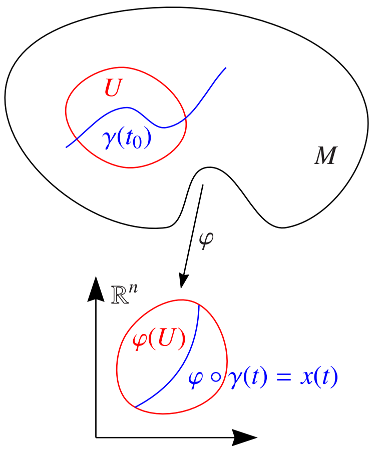
$$s(t) = \int_{t_0}^{t} \|\dot\gamma\| \, \mathrm{d}t = k(t - t_0)\,, \quad t \geq t_0\,. \tag{5.2}$$

If $k = 1$, γ is called **normal**. In this case γ: $I \mapsto \gamma(I) \subset M$ is a local isometry.

3) Note that "being a geodesic" depends on the parameterization and not on the image $\gamma(I)$.
Consider as an example γ_i: $\mathbb{R} \mapsto \mathbb{R}^2$ (for $i = 1, 2$) of the form $\gamma_1(t) := (t, 0)$, $\dot\gamma_1(t) = (1, 0)$ and $\gamma_2(t) = (t^3, 0)$, $\dot\gamma_2(t) = (3t^2, 0)$. The second curve has no constant speed, hence it cannot be a geodesic, although the images of both curves is the same, namely $\mathbb{R}$.

5.2 Differential equations for geodesics

Let γ: $I \mapsto M$ be a geodesic in $(M, \langle \bullet, \bullet \rangle)$ and $\gamma(t_0) \in (U, \varphi)$ (chart at $\gamma(t_0)$) with $\varphi \circ \gamma(t) = (x^1(t), \ldots, x^n(t)) =: x(t)$.



We then have

$$\dot\gamma(t) = \sum_{i=1}^{n} \dot x^i(t) \frac{\partial}{\partial x^i}(\gamma(t))\,. \tag{5.3}$$

The general formula for parallel fields (section 3.2) yields

$$D_{\dot\gamma}\dot\gamma = \sum_{k=1}^{n} \left\{ \ddot x^k + \sum_{i,j=1}^{n} \Gamma_{ij}{}^{k} \dot x^i \dot x^j \right\} \frac{\partial}{\partial x^k} = 0\,. \tag{5.4}$$

Locally $D_{\dot\gamma}\dot\gamma = 0$ is equivalent to a system of second order **ordinary differential equations**

$$\boxed{\ddot x^k(t) = -\sum_{i,j=1}^{n} \Gamma_{ij}{}^{k}(x(t)) \dot x^i(t) \dot x^j(t)\,, \quad k = 1, \ldots, n\,.} \tag{5.5}$$

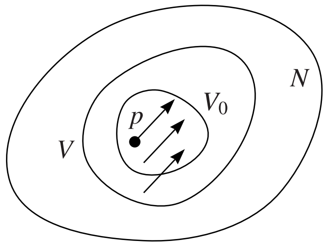
The right-hand side describes deviation from Euclidian geometry depending on $\Gamma_{ij}{}^{k}$. To solve (5.5) introduce a new parameterization by $\dot x^k =: y^k$ and hence

$$\boxed{\dot y^k = -\sum_{i,j=1}^{n} \Gamma_{ij}{}^{k} y^i y^j\,.} \tag{5.6}$$

In order to study (5.5) we consider the tangent bundle of M: $TM = \{(q, v) | q \in M, v \in T_q M\}$. If (U, φ) is a chart on M then any vector $v \in T_q M$, $q \in U$ can be written as $v = \sum_{i=1}^{n} y^i \partial/\partial x^i$ (theorem 1 in (1.3)). One can thus take $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ as local coordinates on TU. Now for any differentiable curve $t \mapsto c(t)$ in M we have a curve $t \mapsto (c(t), \dot c(t) = \mathrm{d}/\mathrm{d}t(c(t)))$ in the tangent bundle TM. Locally this is given by $t \mapsto (x^1(t), \ldots, x^n(t), \dot x^1(t), \ldots, \dot x^n(t))$. In particular if γ is a geodesic and $\varphi \circ \gamma(t) = (x^1(t), \ldots, x^n(t))$ then the curve $t \mapsto (x^1(t), \ldots, x^n(t), \dot x^1(t), \ldots, \dot x^n(t))$ satisfies the system (5.6).
We now use the following result about differential equations (see for example Boothby, chapter 4):

Proposition 1

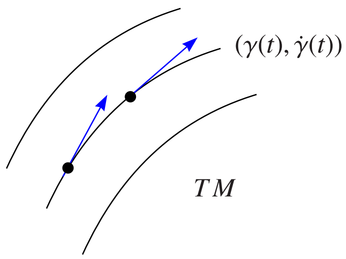
If X is a C^∞ vector field on an open set V in a smooth manifold N and $p \in V$ then there exists an open set $V_0 \subset V$, $p \in V_0$, a number $\delta = \delta(p) > 0$ and a C^∞ map ϕ: $(-\delta, \delta) \times V_0 \mapsto V$ such that the curve $t \mapsto \phi(t, q)$ is the unique integral curve of X with $\phi(0, q) = q$ for all $q \in V_0$.



The map ϕ_t: $V_0 \mapsto V$, $\phi_t(q) := \phi(t, q)$ is called **flow** of X **on** V.

Lemma 1

There exists a unique vector field G on TM whose integral curves are of the form $t \mapsto (\gamma(t), \dot{\gamma}(t))$, where γ is a geodesic in M.



Proof

We want to proof uniqueness (assuming existence): Let (U, φ) be a chart of M. Then the integral curves of G on the tangent bundle TU are given by $t \mapsto (\gamma(t), \dot{\gamma}(t))$, whereas γ is a geodesic. From uniqueness of integral curves ($\Leftrightarrow$ solution of differential equations) for given initial conditions G is unique:

$$\widetilde{G}((\gamma(t), \dot{\gamma}(t))) = \frac{\mathrm{d}}{\mathrm{d}t}(\gamma(t), \dot{\gamma}(t)) = G((\gamma(t), \dot{\gamma}(t))) \,. \tag{5.7}$$

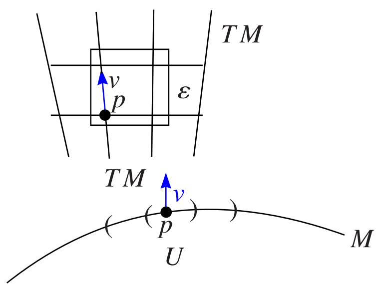
Existence: We define G locally by

$$y^k(t) = \dot{x}^k(t) \,, \quad \dot{y}^k = -\sum_{i,j=1}^{n} {\Gamma_{ij}}^k y^i y^j \,. \tag{5.8}$$

Then uniqueness shows that G is well defined globally on TM. □
The vector field $G \in \mathcal{V}(TM)$ is called **geodesic field on** TM. Its flow is called **geodesic flow**.
Now apply proposition 1 to $G \in \mathcal{V}(TM)$ at the point $(p, 0) \in TM$ and get:

Theorem 1

For each point $p \in M$ there exists an open set O in TU (where (U, φ) is a chart of M at p) with $(p, 0) \in O$, a number $\delta = \delta(p) > 0$ and a C^∞-map ϕ: $(-\delta, \delta) \times O \mapsto TU$ such that $t \mapsto \phi(t, q, v)$ is the unique integral curve of G with initial conditions $\phi(0, q, v) = (q, v)$ for each $(q, v) \in O$.
Question: What does that mean for geodesics in M? We can write O as $O = \{(q, v) \in TU | q \in V, v \in T_q M$ with $\|v\| < \varepsilon_1\}$, where $V \subset U$ is an open neighborhood of p.



Let π: $TM \mapsto M$, $(p, v) \mapsto p$ be the canonical projection. We set $\gamma := \pi \circ \phi$. Then we reformulate theorem 1.

Theorem 2

Given $p \in M$ there is an open set $V \subset M$, $p \in V$. There are numbers $\delta = \delta(p) > 0$ and $\varepsilon_1 = \varepsilon_1(p) > 0$ and C^∞-map $\gamma = \pi \circ \phi$: $(-\delta, \delta) \times O \mapsto M$, with $O = \{(q,v) | q \in V, v \in T_qM, \|v\| < \varepsilon_1\}$ such that the curve $t \mapsto \gamma(t,q,v)$ is the unique geodesic on M with $\gamma(0,q,v) = q$ and $\dot{\gamma}(0,q,v) = v$ (for each $q \in V$ and $v \in T_qM$ with $\|v\|_q < \varepsilon$).

5.3 The exponential map

We want to get rid of the dependence on δ.

Lemma 2

Let $a > 0$, $a \in \mathbb{R}$. If the geodesic $\gamma(t,q,v)$ is defined on the interval $(-\delta,\delta)$ then the geodesic $\gamma(t,q,av)$ is defined on $(-\delta/a, \delta/a)$ and $\gamma(t,q,av) = \gamma(at,q,v)$.

Proof

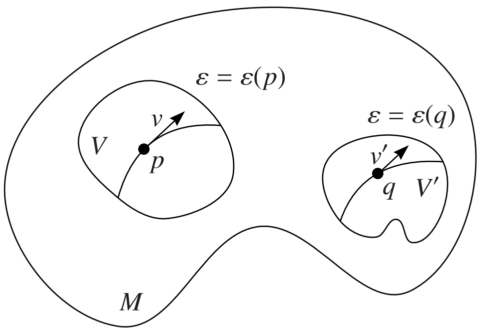
Consider the curve h: $(-\delta/a, \delta/a) \mapsto M$, $t \mapsto \gamma(at,q,v)$. Then $h(0) = q(= \gamma(0,q,v))$. Furthermore $\dot{h}(t) = (\mathrm{d}/\mathrm{d}t)h(t) = a\dot{\gamma}(at,q,v)$ and $\dot{h}(0) = av = a\dot{\gamma}(0,q,v)$. From the geodesic equation it follows that

$$D_{\dot{h}}\dot{h} = D_{a\dot{\gamma}}a\dot{\gamma} = a^2 D_{\dot{\gamma}}\dot{\gamma} = 0\,, \tag{5.9}$$

hence h is a geodesic. From uniqueness of geodesics with given initial condition we have

$$\gamma(at,q,v) \underset{\text{definition}}{=} h(t) \underset{\text{uniqueness}}{=} \gamma(t,q,av)\,. \qquad \Box \tag{5.10}$$

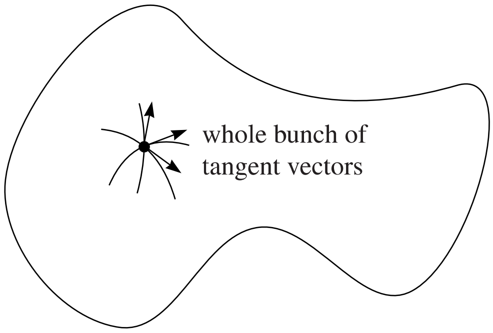
By theorem 2 $\gamma(t,q,v)$ is defined for $q \in V(= V(p))$. It holds that $|t| < \delta(= \delta(p))$ and $v \in T_qM$, $\|v\|_q < \varepsilon_1(= \varepsilon_1(p))$. Lemma 2 implies that $\gamma(t,q,\delta/2v)$ is defined for $|t| < 2$. We set $\varepsilon := \delta \cdot \varepsilon_1/2(= \varepsilon(p))$. Then the geodesic $\gamma(t,q,v)$ is defined for all $q \in V$, $v \in T_qM$, $\|v\|_q < \varepsilon$ and $|t| < 2$.



All this proves:

Theorem 3

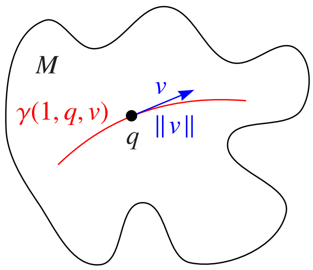
For every point $p \in (M, \langle \bullet, \bullet \rangle)$ there exists a neighborhood V of p, $\varepsilon = \varepsilon(p) > 0$ and a C^∞-map γ: $(-2,2) \times \{(q,v) | q \in V, v \in T_qM, \|v\| < \varepsilon\} \mapsto M$ such that for fixed (q,v), $t \mapsto \gamma(t,q,v)$ is the unique geodesic in M with initial conditions $\gamma(0,q,v) = q$ and $\dot{\gamma}(0,q,v) = v$.



Let $p \in M$ and $O \subset TM$ open as in theorem 3. The **exponential map** (on O) is exp: $O(\subset TM) \mapsto M$:

$$\exp(q, v) := \gamma(1, q, v) = \gamma\left(1, q, \frac{\|v\|}{\|v\|}v\right) = \gamma\left(\|v\|, q, \frac{v}{\|v\|}\right). \tag{5.11}$$

This geodesic is parameterized by arc-length.



Remark

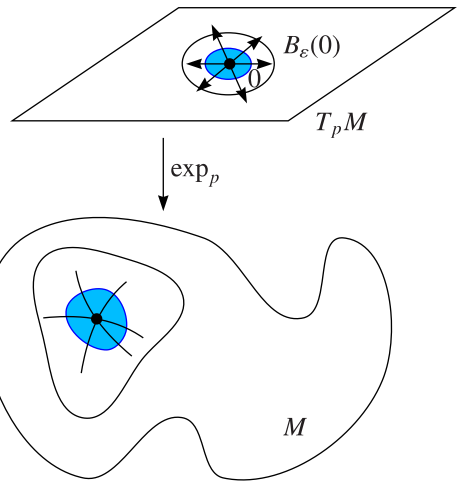
1) exp is C^∞ since γ is C^∞ by theorem 2.

2) Often used is the restriction of exp: $\exp_p := \exp(p, \bullet)$: $B_\varepsilon(0) \subset T_pM \mapsto M$ (where $B_\varepsilon(0) = \{v \in T_pM | \|v\| < \varepsilon\}$ (open ball of radius ε)).

   Note: $\exp_p$ is smooth and $\exp_p(0) = p (= \gamma(1, p, 0) = p)$.

Theorem 4

For every point $p \in (M, \langle \bullet, \bullet \rangle)$ there exists $r = r(p) > 0$ such that the map $\exp_p$: $B_r(0) \subset T_pM \mapsto \exp_p(B_r(0)) \subset M$ with $B_r(0) = \{v \in T_pM | \|v\|_p < r\}$ is a diffeomorphism onto an open neighborhood of p.



Proof

We us the inverse function theorem for manifolds (theorem 2 in 1.4).

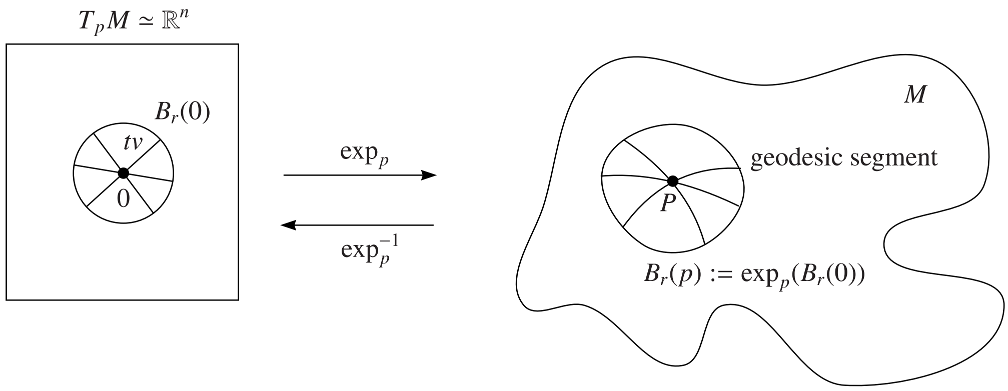
$$\mathrm{d}\exp_p|_0 : T_0(B_r(0))(\simeq T_pM) \mapsto T_{\exp_p(0)}M = T_pM\,. \tag{5.12}$$

Let $v \in T_pM$. To compute this differential pick a curve $c = tv$ with $c'(0) = v$ and $c(0) = 0$. Then

$$\mathrm{d}\exp_p|_0(v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 (\exp_p \circ c)(t) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp_p(tv) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma(1, p, tv) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma(t, p, v) = v\,, \tag{5.13}$$

i.e. $\mathrm{d}\exp_p|_0 = \mathrm{id}_{T_pM}$ hence is a vector space isomorphism. From this follows the claim. □

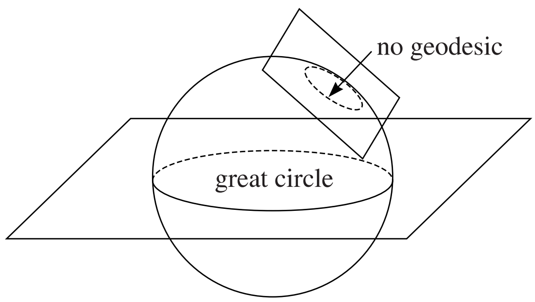
A neighborhood U of p is a **geodesic normal neighborhood** of p if $\exp_p|_V$: $V \mapsto U := \exp_p(V)$ is a diffeomorphism. The set $B_r(p) = \exp_p(B_r(0))$ is a geodesic ball at p of radius r. The chart $\exp_p^{-1}$: $U = \exp_p(V) \mapsto V \subset \mathbb{R}^n$ is called **geodesic normal coordinates**.



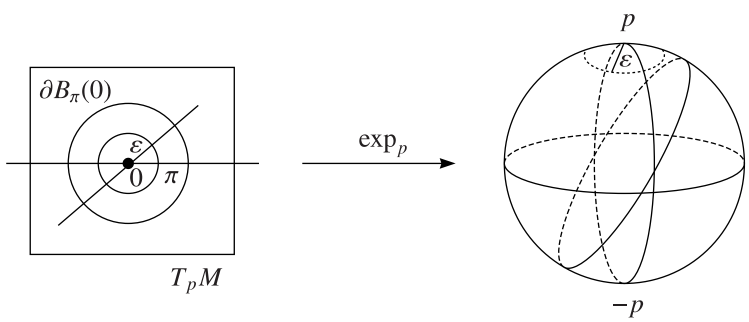
Examples

1) $M = (\mathbb{R}^n, \text{canonical metric})$: The geodesics are straight lines: $t \mapsto at + b$ with $a,\, b \in \mathbb{R}^n$. If we identify $T_p\mathbb{R}^n \simeq \mathbb{R}^n$ then $\exp_p\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is just the identity for all p.

2) $M = (S^n, \text{canonical metric})$, where S^n is the unit sphere in $\mathbb{R}^{n+1}$ and the canonical metric is the induced metric from $\mathbb{R}^{n+1}$. Geodesics are great circles i.e. intersections of S^n with two-dimensional vector subspaces.



The exponential map maps the straight line in T_pM to a great circle on the manifold M:



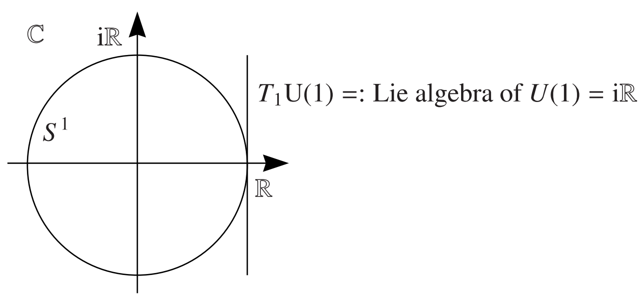
Hence $\exp_p |B_\pi(0)$ (open ball) is a diffeomorphism.

$$\exp_p(\partial B_\pi(0)) = \{-p\} \quad \exp_p(\partial B_{2\pi}(0)) = \{p\}\,. \tag{5.14}$$

3) The name “exponential map” comes from Lie theory. We consider as an example

$$G = \mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | {}^tA\overline{A} = E\},. \tag{5.15}$$

Hence $\mathrm{U}(1) = \{z \in \mathbb{C} | z\overline{z} = |z|^2 = 1\}$. The Lie algebra of the group is the tangent space of the unit element of the group.

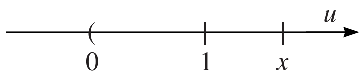


$$\exp : T_1\mathrm{U}(1) \mapsto \mathrm{U}(1),\ \mathrm{i}t \mapsto \exp(\mathrm{i}t)\,, \tag{5.16}$$

which is a homeomorphism. One can learn a lot about the group by studying the linearization $T_1\mathrm{U}(1)$. The translation between Lie algebra and Lie group is done by the exponential map. Let us look at another Lie group, namely $G = \mathrm{O}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}) | {}^tAA = E\}$ with the unit matrix E. The Lie algebra $T_E\mathrm{O}(n)$ of G is the set of skew-symmetric matrices. $B \in T_E\mathrm{O}(n)$ can be realized as the tangent vector of a curve $B = A^t(0)$, ${}^tA(s)A(s) = E$, $A(0) = E$. By differentiation of ${}^tA(s)A(s) = E$ one obtains ${}^tA'(s)A(s) + {}^tA(s)A'(s) = 0$ and for $s = 0$ it follows that $BE + EB = 0$, hence ${}^tB = -B$. For $B \in T_E\mathrm{O}(n)$ set

$$C(s) := \exp(sB) \equiv \mathrm{e}^{sB} := E + (sB) + \frac{1}{2!}(sB)^2 + \dots\,, \tag{5.17}$$

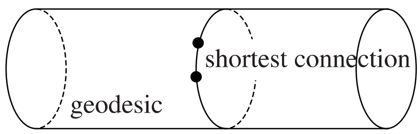
with the native exponential series. Exercise: $C(s)$ exists (series converges) by using ${}^tB = -B$ and show that $C(s) \in \mathrm{O}(n)$. **Fact:** All this Lie-exponential maps are Riemannian exponential maps. The metrics are certain Riemannian metrics on the groups. Consider $G = (\mathbb{R}_{>0}, \cdot)$, whereas $(\cdot)$ is standard multiplication. The tangent space is given by $T_1\mathbb{R}_{>0} = (\mathbb{R}, +)$. G is a nonlinear object and $T_1\mathbb{R}_{>0}$ is a linear one.



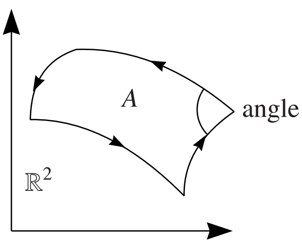
The Riemannian metric has to be compatible with the group operation. Define a length which depends on where the vector $u \in T_x\mathbb{R}_{>0}$ is: $\|u\|_x := x^{-1}|u|$ (left-invariant metric). This defines a Riemannian metric on the space of positive real numbers. Be γ a unit speed geodesic (which exists by the general theory) with $\gamma(0) = 1$ and $\|\dot\gamma(0)\|_1 = 1$. We write $(\gamma(t), \dot\gamma(t)) =: (a(t), b(t))$. The necessary condition is $1 = \|\dot\gamma(t)\|_{\gamma(t)} = a(t)^{-1}|b(t)|$. This implies $a(t) = |b(t)| > 0$ and hence $\gamma(t) = \dot\gamma(t)$ so $\gamma(t) = \mathrm{e}^t$. The definition of the exponential map delivers the long-known property e: $(\mathbb{R}, +) \mapsto (\mathbb{R}_{>0}, \cdot)$, $\mathrm{e}^{x+y} = \mathrm{e}^x \cdot \mathrm{e}^y$.

5.4 Minimality properties of geodesics

In Euclidian geometry ($\mathbb{R}^n$, canonical metric) straight lines are shortest connections. This cannot be true in general Riemannian manifolds. For example take a cylinder, which is locally isometric to the real space.

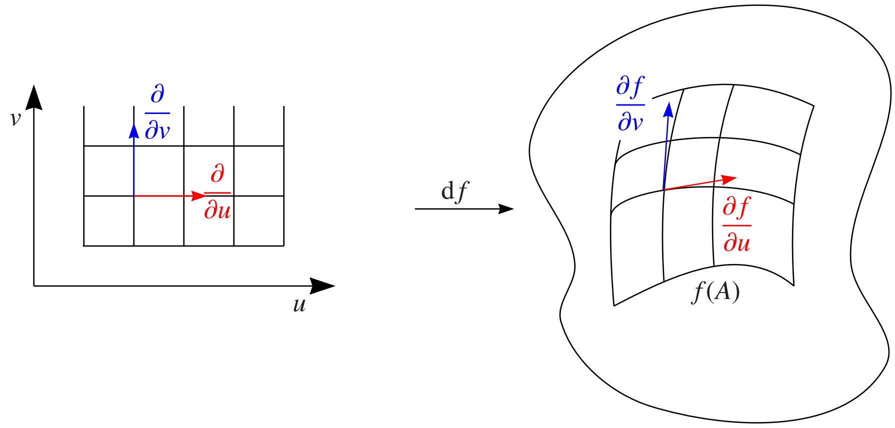


We will show that this property holds on a general Riemannian manifold if the points are close to each other. The **goal** will be to show that geodesics are "locally" shortest connections. For this we will need a technical tool, namely the concept of **vector fields along surfaces**. Let A be an open, connected subset of $\mathbb{R}^2$ with piecewise smooth boundary (and interior angles $< \pi$), e.g. a rectangle.



A **parameterized surface** is a differentiable map f: $A \mapsto M$, $(u, v) \mapsto f(u, v)$, where M is a smooth manifolds and $A \in \mathbb{R}^2$. A **vector field** along f is a differentiable map V: $A \mapsto TM$ such that $V(u, v) \in T_{f(u,v)}M$. In particular the parameter lines $u \mapsto f(u, v_0)$, $v \mapsto f(u_0, v)$ define vector fields along f:

$$\frac{\partial f}{\partial u}(u,v) := \mathrm{d}f|_{(u,v)}\left(\left.\frac{\partial}{\partial u}\right|_{(u,v)}\right)\,,\quad \frac{\partial f}{\partial v}(u,v) := \mathrm{d}f|_{(u,v)}\left(\left.\frac{\partial}{\partial v}\right|_{(u,v)}\right) \in T_{f(u,v)}M\,. \tag{5.18}$$



Define the covariant derivative of a vector field V along f (with respect to some affine connection D) by

$$\frac{D}{\partial u} V(u, v_0) := D_{\frac{\partial f}{\partial u}(u,v_0)} V(u, v_0)\,, \quad \frac{D}{\partial v} V(u_0, v) := D_{\frac{\partial f}{\partial v}(u_0,v)} V(u_0, v)\,, \tag{5.19}$$

where one of the two is the covariant derivative of a vector field along a curve.

Lemma 3

Let M be a differentiable manifold and D a **symmetric** affine connection on M. Then one has for a parameterized surface $f\colon A \mapsto M$:

$$\frac{D}{\partial v}\left(\frac{\partial f}{\partial u}\right) = \frac{D}{\partial u}\left(\frac{\partial f}{\partial v}\right) . \tag{5.20}$$

Proof

In localized coordinates (U, φ) in a neighborhood of a point in $f(A) \subset M$ we have

$$\varphi \circ f(u, v) = (x^1(u, v), \ldots, x^n(u, v))\,, \tag{5.21}$$

and hence

$$\frac{D}{\partial v}\left(\frac{\partial f}{\partial u}\right) = \frac{D}{\partial v}\left\{\sum_{i=1}^{n} \frac{\partial x^i}{\partial u} \frac{\partial}{\partial x^i}\right\} = \sum_{i=1}^{n} \frac{\partial^2 x^i}{\partial v \partial u} \frac{\partial}{\partial x^i} + \sum_{i,j=1}^{n} \frac{\partial x^i}{\partial u} \frac{\partial x^j}{\partial v} D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i} . \tag{5.22}$$

The symmetry of D implies

$$D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}\,, \tag{5.23}$$

which shows the claim.

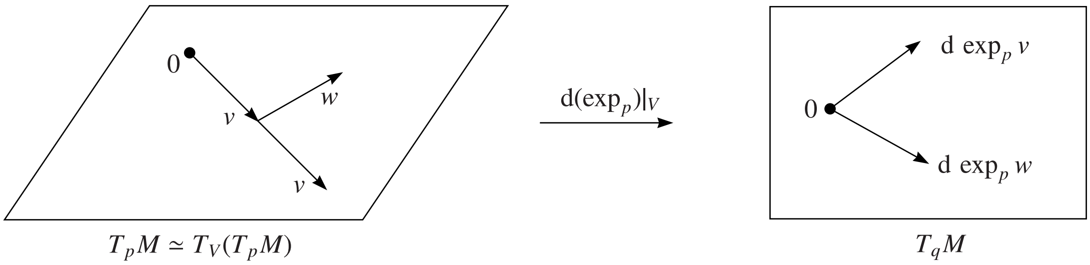
5.4.1 The Gauß lemma

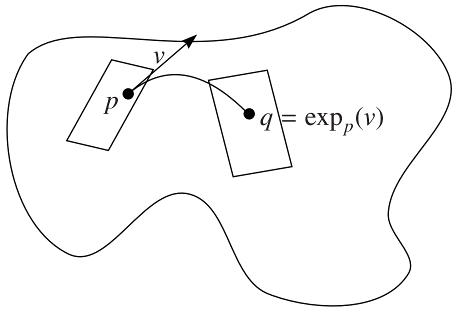
Theorem 5

Let $(M, \langle \bullet, \bullet \rangle)$ be a Riemannian manifold. Let $p \in M$ and $v \in T_pM$ such that $q := \exp_p v$ is defined. For $w \in T_V(T_pM) \simeq T_pM$ holds

$$\langle \mathrm{d}\exp_p|_V v, \mathrm{d}\exp_p|_V w\rangle_q = \langle v, w\rangle_p . \tag{5.24}$$

In particular, if $v \perp w$ this implies $\mathrm{d}\exp_p v \perp \mathrm{d}\exp_p w$.





Proof

Decompose $w = w_T + w_N$, where w_T is a component of w in direction of v and w_N is a component normal to v, hence

$$w_T := \left\langle w, \frac{v}{\|v\|} \right\rangle \frac{v}{\|v\|}\,, \quad w_N := w - w_T\,. \tag{5.25}$$

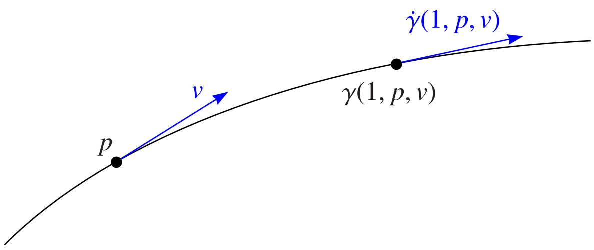
The differential is a linear map between the vector spaces $T_v(T_pM) \simeq T_pM$ and $T_{\exp_p v}M = T_qM$, and for this reason we can treat the two cases separately:

$$\mathrm{d}\exp_p|_V(w) = \mathrm{d}\exp_p|_V(w_T) + \mathrm{d}\exp_p|_V(w_N)\,. \tag{5.26}$$

- Case 1: $w = w_T$

  We have by using $\exp_p(tv) = \gamma(t, p, v)$:

$$\begin{aligned}\mathrm{d}\exp_p|_V(v) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp_p(v + tv) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \exp_p((1+t)v) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma((1+t), p, v) = \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \gamma(t, \gamma(1,p,v), \dot\gamma(1,p,v)) = \dot\gamma(1,p,v)\,.\end{aligned} \tag{5.27}$$



$$\begin{aligned}\mathrm{d}\exp_p|_V(w_T) &= \mathrm{d}\exp_p|_V\left(\left\langle w_T, \frac{v}{\|v\|}\right\rangle, \frac{v}{\|v\|}\right) = \left\langle w_T, \frac{v}{\|v\|}\right\rangle \frac{1}{\|v\|}\mathrm{d}\exp_p|_V(v) = \\ &= \left\langle w_T, \frac{v}{\|v\|}\right\rangle \frac{1}{\|v\|}\dot\gamma(1,p,v)\,.\end{aligned} \tag{5.28}$$

From this we obtain

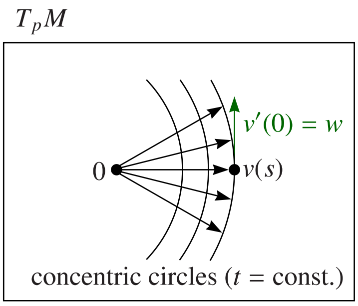
$$\begin{aligned}\left\langle \mathrm{d}\exp_p|_V v, \mathrm{d}\exp_p|_V w_T\right\rangle_q &= \left\langle w_T, \frac{v}{\|w\|_q}\right\rangle_p \frac{1}{\|v\|_p}\langle\dot\gamma(1,p,v), \dot\gamma(1,p,v)\rangle_q = \\ &= \left\langle w_T, \frac{v}{\|v\|_q}\right\rangle_p \frac{1}{\|v\|_p}\langle v, v\rangle_p = \langle w_T, v\rangle_p\,,\end{aligned} \tag{5.29}$$

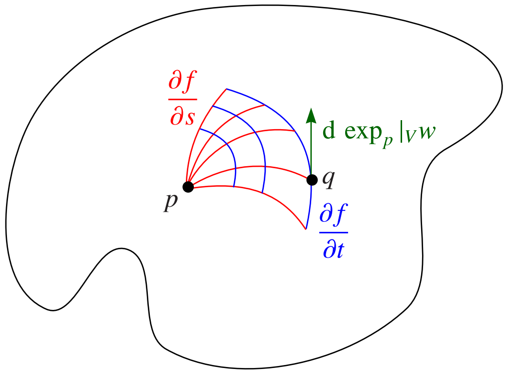
since geodesics have constant speed.

- Case 2: $w = w_N \perp v$

  By assumption we have $q = \exp_p v$. This implies that there is a $\varepsilon > 0$ such that $\exp_p$ is defined for $u = tv(s)$, where $v(s)$ is a curve in T_pM with $v(0) = v$, $\|v(s)\| = \text{const.}$ and $v'(0) = w \perp v$.

$$\{0 \leq t \leq 1\,, \quad -\varepsilon < s < \varepsilon\} := A\,. \tag{5.30}$$





Consider the parameterized surface $f\colon A \mapsto M$, $f(t,s) := \exp_p(tv(s))$. Note: $f(t, s_0)$ is a geodesic $\forall\ s_0$ and $f(1,0) = q$.

$$\left.\frac{\partial f}{\partial s}\right|_{(1,0)} \overset{\text{put } t=1}{\text{chain rule}} \mathrm{d}\exp_p|_V(v'(0)) = \mathrm{d}\exp_p|_V(w)\,, \qquad \left.\frac{\partial f}{\partial t}\right|_{(1,0)} \overset{\text{put } s=0}{=} \mathrm{d}\exp_p|_V(v)\,. \tag{5.31}$$

It remains to show that

$$\left\langle \left.\frac{\partial f}{\partial s}\right|_{(1,0)}, \left.\frac{\partial f}{\partial t}\right|_{(1,0)} \right\rangle_q = 0\,. \tag{5.32}$$

We first show that

$$\left\langle \left.\frac{\partial f}{\partial s}\right|_{(t,s)}, \left.\frac{\partial f}{\partial t}\right|_{(t,s)} \right\rangle_{f(t,s)}\,, \tag{5.33}$$

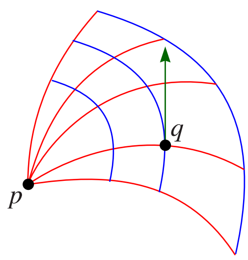
is independent of t. By compatibility of the Levi-Civita connection of the manifold this can be written as

$$\frac{\partial}{\partial t}\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle = \left\langle \frac{D}{\partial t}\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right\rangle + \left\langle \frac{\partial f}{\partial s}, \frac{D}{\partial t}\frac{\partial f}{\partial t}\right\rangle = \left\langle \frac{D}{\partial s}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle = \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t}\right\rangle = 0\,, \tag{5.34}$$

since the differential vector $\partial f/\partial t$ along a geodesic is constant.

Hence we have for arbitrary t

$$\left\langle \left.\frac{\partial f}{\partial s}\right|_{(1,0)}, \left.\frac{\partial f}{\partial t}\right|_{(1,0)} \right\rangle_q = \left\langle \left.\frac{\partial f}{\partial s}\right|_{(t,0)}, \left.\frac{\partial f}{\partial t}\right|_{(t,0)} \right\rangle_{f(t,0)} \overset{\text{set } t=0}{=} \left\langle \left.\frac{\partial f}{\partial s}\right|_{(0,0)}, \left.\frac{\partial f}{\partial t}\right|_{(0,0)} \right\rangle_{f(0,0)=p}\,. \tag{5.35}$$

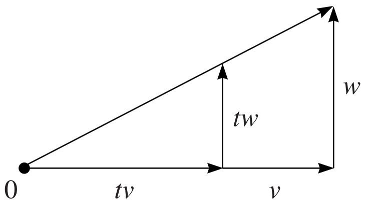


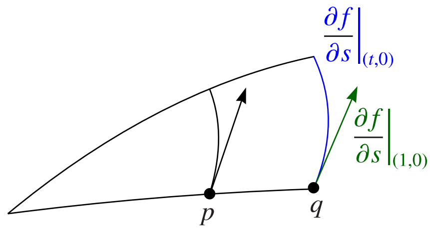
But

$$\left.\frac{\partial f}{\partial s}\right|_{(t,0)} = (\mathrm{d}\exp_p)|_{tv(s)} tv'(s)\Big|_{s=0} = \mathrm{d}\exp_p|_{tv}(tw)\,, \tag{5.36}$$

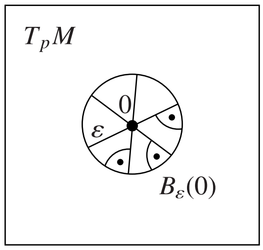
and hence

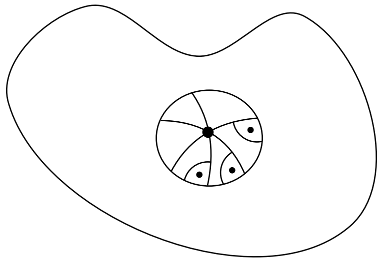
$$\left.\frac{\partial f}{\partial s}\right|_{(0,0)} = \lim_{t\mapsto 0} \mathrm{d}\exp_p|tv(tw) = \mathrm{d}\exp_p|_0 0 = 0\,. \tag{5.37}$$



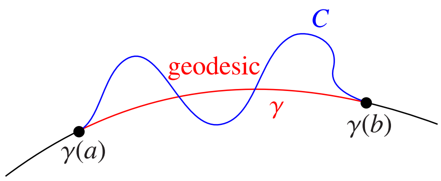


5.4.2 The geometric interpretation of the Gauss lemma



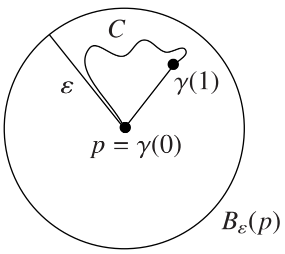


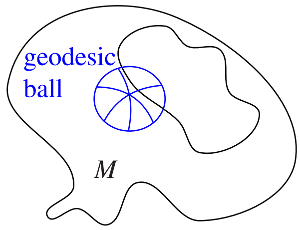
If $B_\varepsilon(p) = \exp_p(B_\varepsilon(0))$ is a normal geodesic ball with center p and radius ε then its boundary $S_\varepsilon(p) := \partial B_\varepsilon(p) = \exp_p(\partial B_\varepsilon(0))$ is a "hypersurface" in M **orthogonal** to all geodesics issuing from p. $S_\varepsilon(p)$ is called geodesic sphere with center p and radius ε.
A segment $\gamma|_{[a,b]}$ of a geodesic $\gamma\colon I \mapsto M$ (with $[a,b] \subseteq I$) is called **minimizing** if $L(\gamma|_{[a,b]}) \le L(c)$, where L is the length and c is an arbitrary curve in M joining $\gamma(a)$ to $\gamma(b)$.



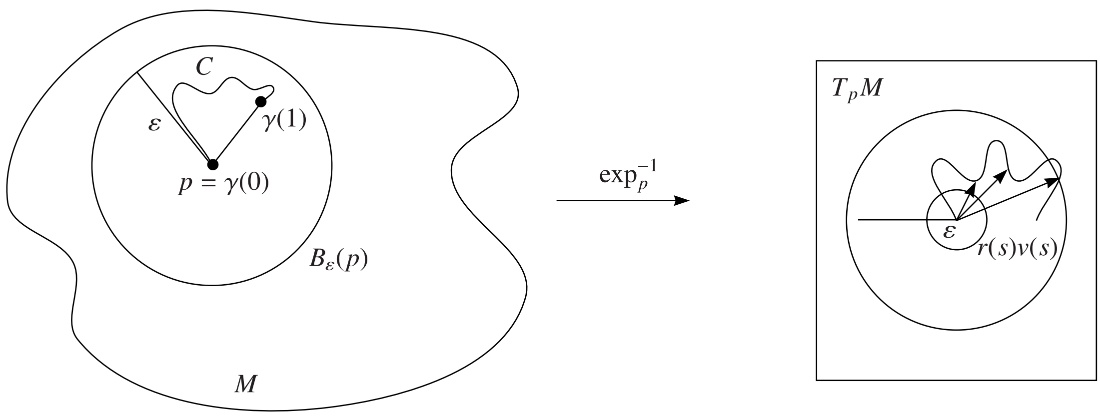
Theorem 6 (geodesics are locally minimizing)

Let $p \in M$, U a normal geodesic neighborhood of p and $B \subset U$ a normal geodesic ball with center p. Let $\gamma\colon [0,1] \mapsto B$ be a geodesic segment with $\gamma(0) = p$. If $c\colon [0ß,1] \mapsto M$ is an arbitrary piecewise differentiable curve with $c(0) = \gamma(0)$ and $c(1) = \gamma(1)$ then $L(\gamma) \le L(c)$ and if $L(\gamma) = L(c)$ then $\gamma([0,1]) = c([0,1])$.

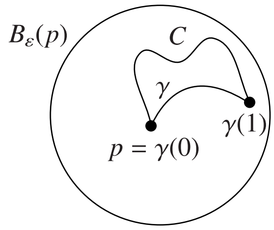




Proof



- Case 1: $c([0,1]) \subset \overline{B}$



$\exp_p$ is a diffeomorphism from $B_\varepsilon(0) \subset T_pM$ to B. Using Euclidian polar coordinates at 0 in T_pM we can write $c(t) = \exp_p(r(t)v(t))$, $t \in [0,1]$ with $v(t)$ being a curve in T_pM with $\|v(t)\| = 1$ and $r\colon [0,1] \mapsto \mathbb{R}$ positive and piecewise smooth. Then $f(r,t) := \exp_p(rv(t))$, $r \geq 0$, $t \in [0,1]$ is a parameterized surface which contains the curve c. Up to finitely many points (c is only piecewise smooth) we have

$$\frac{\mathrm{d}c}{\mathrm{d}t} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t} \,. \tag{5.38}$$

Then by the Gauß lemma (and its proof) it follows that

$$\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial r} \right\rangle = 0 \,. \tag{5.39}$$

Hence

$$\left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\|^2 = \left\| \frac{\partial f}{\partial r} \right\|^2 |r'|^2 + \left\| \frac{\partial f}{\partial t} \right\|^2 = |r'|^2 + \left\| \frac{\partial f}{\partial t} \right\|^2 \overset{(1)}{\geq} |r'|^2 \,, \tag{5.40}$$

since $\|\partial f/\partial r\| = \|v(t)\|$ (by using polar coordinates). Because of the fact that polar coordinates are not smooth at the origin, the lower integration limit is chosen as ε:

$$\int_\varepsilon^1 \left\| \frac{\mathrm{d}c}{\mathrm{d}t} \right\| \mathrm{d}t \geq \int_\varepsilon^1 |r'| \, \mathrm{d}t \overset{(2)}{\geq} \int_\varepsilon^1 r' \, \mathrm{d}t = r(1) - r(\varepsilon) \,. \tag{5.41}$$

As $r(\varepsilon) \mapsto 0$ (for $\varepsilon \mapsto 0$) we get $L(c) \geq r(1) = L(\gamma)$.

Equality: If $L(c) = L(\gamma)$ then we have equality in (1) und (2) i.e.

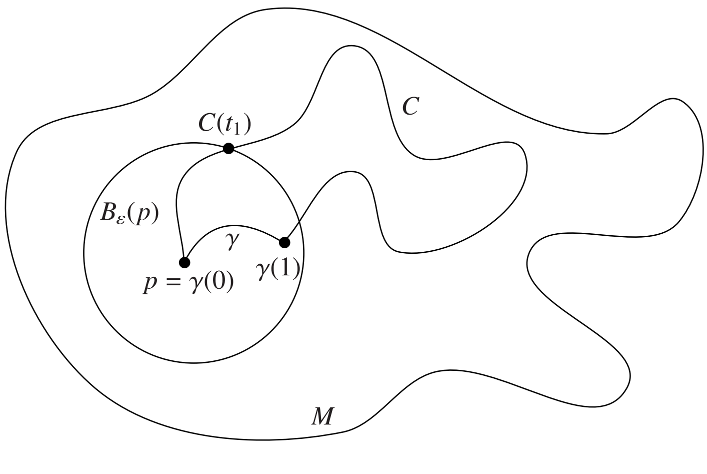
$$(1): \left\| \frac{\partial f}{\partial t} \right\| = 0 \Leftrightarrow 0 = \frac{\partial f}{\partial t} = \mathrm{d}\exp_p(rv') \Leftrightarrow rv'(t) = 0 \Leftrightarrow v'(t) = 0 \Leftrightarrow v(t) = \text{const.} = v(1) \,. \tag{5.42}$$

This means the $v(t)$ is constant and that there is a fixed direction.

$$(2): |r'(t)| = f'(t) > 0 \,, \tag{5.43}$$

i.e. $r(t)$ is monotone so C is a monotone reparameterization of γ and in particular $c([0,1]) = \gamma([0,1])$.

- Case 2: $C([0,1]) \not\subset \overline{B} = \overline{B_\varepsilon(p)}$

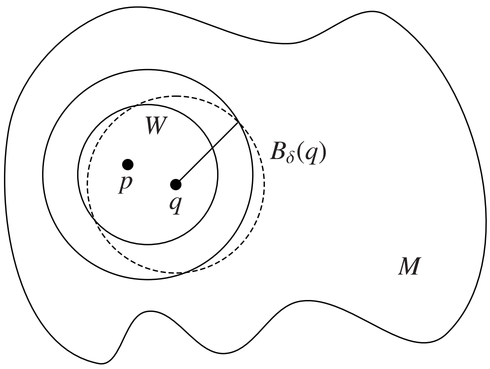


Let ε be the radius of B and $t_1 \in [0,1]$ the fint parameter value for which $c(t_1) \in \partial B$. Then $L(c) \geq L(c|_{[0,t_1]}) \overset{\text{case 1}}{\geq} \varepsilon \geq L(\gamma)$. □

Question: Does the converse hold? **Answer:** Yes! To show this we need a refinement of theorem 4.

Theorem 7 (totally normal neighborhoods)

For every point $p \in M$ there is a neighborhood W of p and $\delta = \delta(p) > 0$ such that for all $q \in W$ $\exp_q$: $B_\delta(0) \subset T_qM \mapsto \exp_q(B_\delta(0)) \supset W$, in other words W is a normal geodesic neighborhoods for each of its points. (W is called totally normal.)

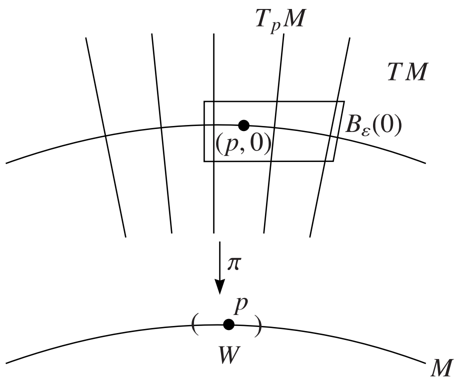


Proof

Let $\varepsilon = \varepsilon(p) > 0$, V a neighborhood of p and $O = \{(q,w)|q \in V, w \in T_qM, \|w\| < \varepsilon\}$ a neighborhood of $(p,0) \in TM$ as in Theorem 3. Trick: Define a differentiable map F: $O \mapsto M \times M$, $F(q,w) = (q, \exp_q w)$. Further let (U, φ) be a chart at p such that $V \subseteq U$. For $F(p,0) = (p,p) \in M \times M$ we then have the chart $(U \times U, \varphi \times \varphi)$. With respect to this chart the Jacobi-matrix of $\mathrm{d}F|_{(p,0)}$ is given by

$$\begin{pmatrix} E & 0 \\ E & E \end{pmatrix}, \tag{5.44}$$

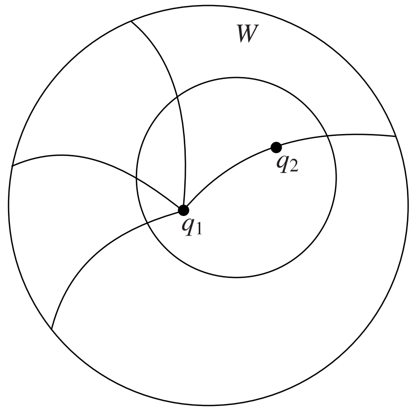
with E being the $(n \times n)$ unit matrix. (Exercise: Use $\mathrm{d}\exp_p|_0 = \mathrm{id}$.) We conclude that F is a local diffeomorphism of an open neighborhood $\widetilde{O} \subset O$ of $(p,0)$ in TM onto a neighborhood $\widetilde{W} = F(\widetilde{O})$ of $(p,p) \in M \times M$ (inverse function theorem for manifolds).



We can again choose $\widetilde{O}$ as $\widetilde{O} = \{(q,v)|q \in \widetilde{V}, v \in T_qM, \|v\| < \delta = \delta(p)\}$ where $\widetilde{V}$ is a neighborhood of p. We further choose a neighborhood W of $p \in M$ such that $W \times W \subseteq \widetilde{W} = F(\widetilde{O})$ (exists by definition of product topology). For W and δ as above we have the assertions of the theorem: Let $q \in W$ and $B_\delta(0) \subset T_qM$. Then $\{q\} \times W \subset \{q\} \times \exp_q B_\delta(0)$ by definition of F. In particular, $W \subset \exp_q B_\delta(0)$. □

Remark

From the theorem and the minimality property of geodesics we have: For any two points q_1, $q_2 \in W$ (= total normal) there is **a unique minimizing** geodesic of length $\leq \delta$ which connects q_1 to q_2.



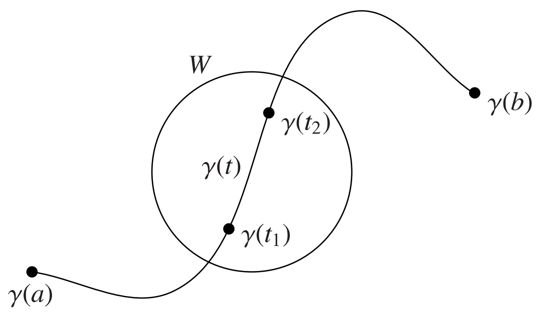
One can reach point the point q_2 by some geodesic line and since this is minimizing, every point can play the role of the center of a geodesic ball.

Corollary

Let γ: $[a, b] \mapsto M$ be a piecewise smooth curve in M parameterized by arc-length. If $L(\gamma) \leq L(c)$ for any piecewise smooth curve c joining $\gamma(a)$ to $\gamma(b)$ then γ is a geodesic.

Proof

Let $t \in [a, b]$ and W a totally normal neighborhood of $\gamma(t)$.



Then there is a closed interval $I = [t_1, t_2] \subseteq [a, b]$ such that $t \in I$, $\gamma(I) \subset W$. $\gamma|_I$ is piecewise smooth and minimizing. By theorem 6/7 we have $L(\gamma|_I)$ = length of the radial geodesic segment in W joining $\gamma(t_1)$ to $\gamma(t_2)$. Since γ is parameterized by arc-length we conclude by theorem 6 (equality case) that $\gamma|_I$ is a geodesic segment in a neighborhood of t. As t is arbitrary the claim follows. ☐

Application

Riemannian isometries map geodesics to geodesics. Consider a map ϕ: $(M, \langle \bullet, \bullet \rangle) \mapsto (N, \langle\langle \bullet, \bullet \rangle\rangle)$ with Levi-Civita connections D^M and D^N, respectively. From $D^M_{\dot\gamma}\dot\gamma = 0$ it follows that $(D^N_{\phi\circ\gamma})^\cdot(\phi \circ \gamma)^\cdot = 0$ (see exercises).

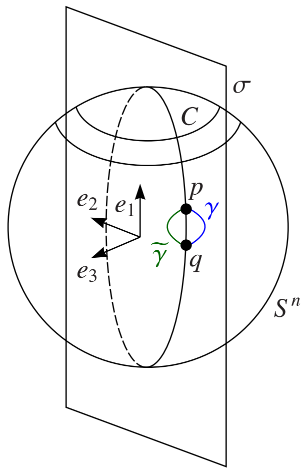
Proof

Be ϕ: $M \mapsto N$ an isometry and γ a geodesic in M. This implies $L(\phi \circ \gamma) = L(\gamma)$. Hence if γ is minimizing then $\phi \circ \gamma$ is also minimizing. By the corollary the assertion follows. ☐

Consider geodesics on the uni sphere $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$ in $\mathbb{R}^{n+1}$ equipped with the induced Riemannian metric from $\mathbb{R}^{n+1}$. The geodesics are great circles and parameterized by arc-length.

Proof

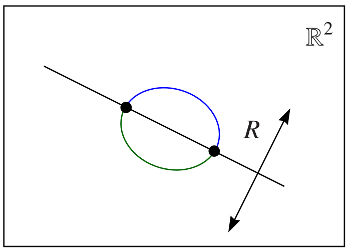
Let c be a great circle i.e. the image of c is $\mathrm{Im}(c) = \sigma \cap S^n$, where σ is a two-dimensional vector subspace of $\mathbb{R}^{n+1}$ (plane) through 0.

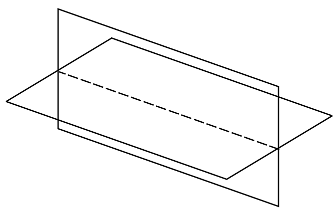


Pick $p \in \mathrm{Im}(c)$ and $q \in \mathrm{Im}(c)$ in a normal neighborhood of p. Let γ be the unique geodesic between p and q. The reflection R of $\mathbb{R}^{n+1}$ which fixes σ pointwise induces an isometry $\widehat{R}$ of S^n. (Reflections: Decompose $\mathbb{R}^n$ in $U \oplus U^\intercal$ ($U^\intercal$ is the orthogonal component) and define $R(u+u^\intercal) = u - u^\intercal$, whereas R is an involution: $R^2 = \mathrm{id}$.) One can assume $\sigma = [e_1, e_2]$ i.e. R: $\mathbb{R}^{n+1} \mapsto \mathbb{R}^{n+1}$, $(x_1, \dots, x_{n+1}) \mapsto (x_1, x_2, -x_3, \dots, -x_{n+1})$. σ is the fix-point set of the isometry: $\sigma = \mathrm{Fix}(R) := \{y \in \mathbb{R}^{n+1} | R(y) = y\}$. Then $\widetilde{\gamma} := \widehat{R} \circ \gamma$ is also a geodesic (segment) of S^n (by "application") between p and q. (Note: R fixes p and q.) But by the choice of p, q (in a normal neighborhood) there is a **unique** geodesic segment between p and q. This implies that $\widetilde{\gamma} = \widehat{R} \circ \gamma = \gamma$. Hence $\mathrm{Im}(\gamma) \subset \mathrm{Fix}(\widehat{R}) = \sigma \cap S^n = \mathrm{Im}(c)$. □

Example

Geodesics in $\mathbb{R}^n$ are straight lines parameterized by arc-length.





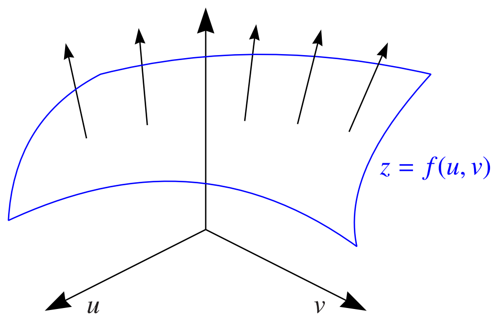
Consider for example $\mathbb{R}^2$ and pick two points. Claim that a straight connecting these two points is a geodesic. A reflection with respect to this straight line is an isometry and the image of the straight line under this reflection is again the same straight line. Hence, is must be a geodesic. In higher dimension use a sequence of reflections.

Kapitel 6

Curvature

6.1 Some history

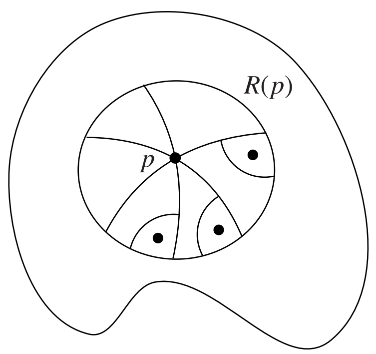
- In 1827 Gauß wrote a paper about surfaces (i.e. two-dimensional submanifolds) of $\mathbb{R}^3$. He defined the (Gauß-)curvature using the embedding $S \subset \mathbb{R}^3$. The curvature is defined of some measure of changes of these normal vectors.



The most important result was the theorema egregium: K is a concect of the inner geometry of the surface, i.e. depends only on the induced Riemannian metric. (The curvature can be defined as soon as one knows the Riemannian metric. The embedding in some ambiant space is not necessary to define the curvature.)

- Betraud-Puiseux (1848): formula for the length of a geodesic circle

$$L(S_R(p)) = 2\pi R \left\{ 1 - K(p)\frac{R^2}{6} + \mathcal{O}(R^2) \right\} . \tag{6.1}$$



The interesting thing is that the Gauß curvature appears in this approximation of the length of the circle. Hence

$$K(p) = \lim_{R \mapsto 0} \frac{2\pi R - L(S_R(p))}{\frac{\pi}{3} R^3} . \tag{6.2}$$

The Gauß curvature compares the length of the Euclidian and the geodesic circle, i.e. the Gauß curvature measures the deviation of $L(S_R(p))$ from the Euclidian circle of radius R.

- 1854: Riemann defines the general notion of curvature for arbitrary Riemannian manifolds (M, g). His idea was that the curvature measures the deviation of geometry of (M, g) from Euclidian geometry.

For more see Spivak volume II.

6.2 The Riemann curvature tensor

Let $(M, \langle\bullet,\bullet\rangle)$ be a Riemannian manifold with Levi-Civita connection D. Then the **Riemann curvature tensor** of M with respect to D is the map R: $\mathcal{V}M \times \mathcal{V}M \times \mathcal{V}M \mapsto \mathcal{V}M$, $(X, Y, Z) \mapsto R(X, Y)Z$ is defined as follows:

$$R(X,Y)Z := D_Y D_X Z - D_X D_Y Z + D_{[X,Y]} Z\,. \tag{6.3}$$

(In literature one sometimes finds a definition, which is the negative one of the above.)

Example

Consider $(\mathbb{R}^n, \text{can})$ and $Z = (z^1, \dots, z^n) \in \mathcal{V}\mathbb{R}^n$. The Levi-Civita connection is the directional derivative:

$$D_X Y = (Xz^1, Xz^2, \dots, Xz^n)\,, \quad D_Y D_X Z = (YXz_1, YXz_2, \dots, YXz^n)\,. \tag{6.4}$$

From the definition of the Lie bracket $[X, Y] = XY - YX$ one obtains $R(X, Y)Z \equiv 0\ \forall\ X,\ Y,\ Z \in \mathcal{V}\mathbb{R}^n$. This is the first indication that this map measures the deviation from Euclidian geometry, if it is nonzero for some manifold different from $\mathbb{R}^n$.

Remark

With respect to basefields $\partial/\partial x^i$ induced by some chart (U, φ) with $\varphi(p) = (x^1(p), \dots, x^n(p))$ one has

$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0\,, \tag{6.5}$$

for all i, j; hence the base fields commute (applied to functions $\in C^\infty$). As a result of that

$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k} = D_{\frac{\partial}{\partial x^j}} D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - D_{\frac{\partial}{\partial x^i}} D_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}\,. \tag{6.6}$$

The curvature tensor measures the commutativity of the second covariant derivatives of vector fields.

6.2.1 Tensor fields

We set $\mathcal{V}_0 M := C^\infty(M)$, $\mathcal{V}_r M := \mathcal{V}M \times \ldots \times \mathcal{V}M$ (with r factors). Note that $\mathcal{V}_r M$ is a $C^\infty M$-module. A tensorfield on M of type (s, r) is an r-linear map T: $\mathcal{V}_r M \mapsto \mathcal{V}_s M$ over the ring of $C^\infty M$ i.e.

$$\begin{aligned} T(V_1, \dots, V_{i-1}, fV + gW, V_{i+1}, \dots, V_r) = &fT(V_1, \dots, V_{i-1}, V, V_{i+1}, \dots, V_r) \\ &+ gT(V_1, \dots, V_{i-1}, W, V_{i+1}, \dots, V_r)\,. \end{aligned} \tag{6.7}$$

That is a tensor field on the manifold of the type (s, r). There is a more general way of defining tensor fields by taking the dual spaces, which leads to the notion of covariant and contravariant tensor fields (multilinear algebra).

Theorem 1

The Riemann curvature tensor is a (1,3) tensorfield on M.

Proof

Compute! Illustration: We show that $R(X, Y)fZ = fR(X, Y)Z$ for all $f \in C^\infty M$ and X, Y, $Z \in \mathcal{V}M$. By definition

$$D_Y D_X(fZ) = D_Y(fD_X Z + (Xf)Z) = (Yf)D_X Z + fD_Y D_X Z + (YXf)Z + (Xf)D_Y Z\,. \tag{6.8}$$

Hence

$$D_Y D_X(fZ) - D_X D_Y(fZ) = f(D_Y D_X Z - D_X D_Y Z) + (YXf - XYf)Z = \\ = f(D_Y D_X Z - D_X D_Y Z) - [X,Y]fZ\,, \tag{6.9}$$

and moreover

$$D_{[X,Y]} fZ = f D_{[X,Y]} Z + ([X,Y]f)Z\,, \tag{6.10}$$

which yields

$$R(X,Y)fZ = f(D_Y D_X - D_X D_Y)Z - ([X,Y]f)Z + f D_{[X,Y]} Z + ([X,Y]f)Z = \\ = f(D_Y D_X - D_X D_Y)Z + f D_{[X,Y]} Z = fR(X,Y)Z\,, \tag{6.11}$$

etc. □

(One just needs the properties of the affine connection here. There is no metric involved.)

6.2.2 Symmetries of the Riemann curvature tensor

Theorem 2

Let $(M, \langle \bullet, \bullet \rangle)$ be a Riemannian manifold with Levi-Civita connection D and curvature tensor R. Then for all X, Y, Z, $T \in \mathcal{V}M$ holds

1) First Bianchi identity: $R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$ (cyclic permutation)
2) Skew symmetry: $R(X,Y)Z = -R(Y,X)Z$
3) $\langle R(X,Y)Z, T\rangle = -\langle R(X,Y)T, Z\rangle$
4) $\langle R(X,Y)Z, T\rangle = \langle R(Z,T)X, Y\rangle$

(The information encoded in the Riemann curvature tensor can be reduced because of these symmetries.)

Proof

1) This reduces to the proof of the Jacobi identity of the Lie bracket:

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0\,. \tag{6.12}$$

Exercise!

2) This directly follows from the definition of R.

3) „⇒“: This is equivalent to $\langle R(X,Y)W, W\rangle = 0\ \forall\ X,\ Y,\ W$. („⇐“ Set $W = Z + T$.) But

$$\langle R(X,Y)W, W\rangle = \langle D_Y D_X W - D_X D_Y W + D_{[X,Y]} W, W\rangle \tag{6.13}$$

and

$$\langle D_Y D_X W, W\rangle = Y\langle D_X W, W\rangle - \langle D_X W, D_Y W\rangle\,, \tag{6.14}$$

which follows from compatibility (theorem 2 in section 3.3). Moreover

$$\langle D_{[X,Y]} W, W\rangle = \frac{1}{2}[X,Y]\langle W, W\rangle\,, \tag{6.15}$$

which again is a result of compatibility. Hence

$$\langle R(X,Y)W, W\rangle = Y\langle D_X W, W\rangle - \langle D_X W, D_Y W\rangle - X\left(\frac{1}{2}Y\langle W,W\rangle\right) \\ + \langle D_Y W, D_X W\rangle + \frac{1}{2}[X,Y]\langle W,W\rangle = \\ = \frac{1}{2}[Y,X]\langle W,W\rangle + \frac{1}{2}[X,Y]\langle W,W\rangle = 0\,. \tag{6.16}$$

4) The proof of this is similar to (3).

6.3 The curvature tensor in local coordinates

Consider a chart (U,φ) with base fields $X_i := \partial/\partial x^i$ $(i = 1, \ldots, n)$. Since by definition $R(X,Y)Z$ is again a vector field, it can be written as a linear combination of base fields with some coefficients:

$$R(X_i,X_j)X_k = \sum_{l=1}^{n} R_{ijk}{}^{l} X_l\,, \quad R_{ijk}{}^{l} \in C^\infty U\,. \tag{6.17}$$

This defines components of R in local coordinates (compare to the analoges g_{ij} and $\Gamma_{ij}{}^{k}$). For arbitrary vector fields X, Y, $Z \in \mathcal{V}M$ we have

$$X = \sum_{i=1}^{n} u^i X_i\,, \quad Y = \sum_{j=1}^{n} v^j X_j\,, \quad Z = \sum_{k=1}^{n} v^k X_k\,, \tag{6.18}$$

and by theorem 1

$$R(X,Y)Z = R\left(\sum_i u^i X_i, \sum_j v^j X_j\right) \sum_k w^k X_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} R_{ijk}{}^{l} u^i v^j w^k X_l\,. \tag{6.19}$$

The Einstein summation convention which neglects the sums (the summation is done over indices that appear twice) simplifies these local expressions a lot. However, we will not use it here, since we will not do too much computations with the local expressions.

Remark (tensor property of R)

The above formula shows that $(R(X,Y)Z)(p) \in T_pM$ depends only on the values of X, Y, Z at p. (in contrast e.g. to $D_X Y$) which is not a tensor field.

6.3.1 Formula for the local expression of the Riemann curvature tensor

With the vanishing Lie bracket one obtains

$$\begin{aligned} R(X_i,X_j)X_k &= D_{X_j}(D_{X_i}X_k) - D_{X_i}(D_{X_j}X_k) + D_{[X_i,X_j]}X_k = \\ &= D_{X_j}\left(\sum_m \Gamma_{ik}{}^{m} X_m\right) - D_{X_i}\left(\sum_m \Gamma_{ik}{}^{m} X_m\right) = \\ &= \sum_m \left[X_j(\Gamma_{ik}{}^{m})X_m + \Gamma_{ik}{}^{m} D_{X_j} X_m\right] - \sum_m (i \leftrightarrow j) = \\ &= \sum_m \left[\frac{\partial}{\partial x^j}\Gamma_{ik}{}^{m} X_m + \Gamma_{ik}{}^{m} \sum_l \Gamma_{jm}{}^{l} X_l\right] - \sum_m (i \leftrightarrow j)\,. \end{aligned} \tag{6.20}$$

Hence

$$\boxed{R_{ijk}{}^{l} = \frac{\partial}{\partial x^j}\Gamma_{ik}{}^{l} + \sum_{m=1}^{n} \Gamma_{ik}{}^{m}\Gamma_{jm}{}^{l} - \frac{\partial}{\partial x^i}\Gamma_{jk}{}^{l} - \sum_{m=1}^{n} \Gamma_{jk}{}^{m}\Gamma_{im}{}^{l}\,.} \tag{6.21}$$

Furthermore

$$\langle R(X_i,X_j)X_k, X_s\rangle = R_{ijk}{}^{l}\langle X_l, X_s\rangle = R_{ijk}{}^{l} g_{ls} = R_{ijks}\,, \tag{6.22}$$

which is a $(0,4)$-tensor field, that follows from the contraction of the indices l and s (Ricci calculus). The symmetries of the Riemann curvature tensor can be written in the local form as follows:

1) $R_{ijks} + R_{jkis} + R_{kijs} = 0$

2) $R_{ijks} = -R_{jiks}$

3) $R_{ijks} = -R_{ijsk}$

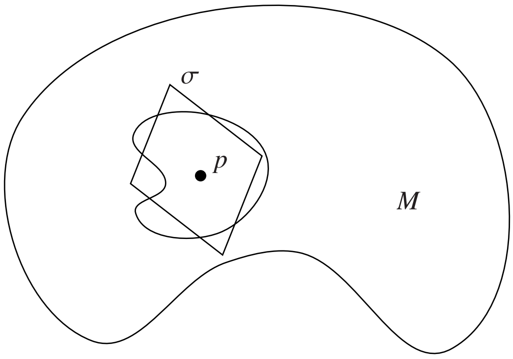
4) $R_{ijks} = R_{ksij}$

6.4 The sectional curvature

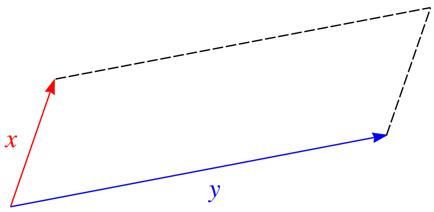
The Riemann curvature tensor induces additional tools for measuring the deviation of a manifold from Euclidian space. One of these is the so-called sectional curvature (Schnitt-Krümmung).

Remark

Locally, ${R_{ijk}}^l$ has n^4 components, but not all of these are independent of each other because of the symmetries. For $n = 2$ one has i, j, k, $l \in \{1, 2\}$. Here, symmetries imply that only $R_{1212} \neq 0$. Gauß's calculations and the local formulas for R show that $R_{1212}(p) = K(p)$ is the Gauß curvature of the two-dimensional manifold at point p.
Be $\dim(M) = n$. The idea of Riemann was the following:



Consider two-dimensional subspaces $\sigma \subset T_pM$ and its exponential images in M (which are two-dimensional submanifolds). The sectional curvature (of σ at p) is the Gauß curvature of $\exp_p(\sigma) \subset M$ at p. For more on history see Spivak Vol II.
We want to introduce some notation (linear algebra). Let $(V, \langle \bullet, \bullet \rangle)$ be a Euclidian vector space. For x, $y \in V$ set $|x \wedge y| := \sqrt{\|x\|^2\|y\|^2 - \langle x, y\rangle^2}$. (This is well-defined since $\|x\|\|y\|^2 - \langle x, y\rangle^2 \geq 0$ by the Cauchy-Schwarz unequality). Geometrically, this is the area of the parallelogram spanned by x and y.



The Gram matrix is given by

$$\begin{pmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{pmatrix} = G(x, y)\,, \tag{6.23}$$

and hence $|x \wedge y| = \sqrt{\det(G(x, y))}$. For x, y being orthonormal follows $|x \wedge y| = 1$.

Lemma 1

Let $(M, \langle \bullet, \bullet \rangle)$ be a Riemannian manifold, $p \in M$ and $\sigma \subset T_pM$ a two-dimensional subspace spanned by x, $y \in T_pM$. Then (here we use the tensor property of R)

$$K(x, y) := \frac{\langle R(x, y)x, y\rangle_p}{|x \wedge y|^2}\,, \tag{6.24}$$

is independent of the choice of the basis $\{x, y\}$ of σ.

Proof

This can be done by direct computation with choosing another basis $\widetilde{x} = ax + by$, $\widetilde{y} = cx + dy$. This leads to $K(\widetilde{x}, \widetilde{y}) = K(x, y)$, which can be done as an exercise! □

Consequence

By lemma the following definition makes sense: For $p \in M$, $\sigma \subset T_pM$, the two-dimensional subspace, we set $K(\sigma, p) := K(x, y)$ for any basis $\{x, y\}$ of σ. $K(\sigma, p) \in \mathbb{R}$ is called the **sectional curvature of $\boldsymbol{\sigma}$ in p**.

Remarks

1) One can show that the set of curvature data $\{K(\sigma,p)|\sigma\subset T_pM\}$ completely determines the curvature tensor R at $p\in M$ (see do Carmo, chapter 4, section 3.3).

2) The sectional curvature generalizes the Gauß curvature (they coincide for $n=2$).

Examples

1) For $(\mathbb{R}^n,\mathrm{can})$ all sectional curvatures are zero since $R\equiv 0$.

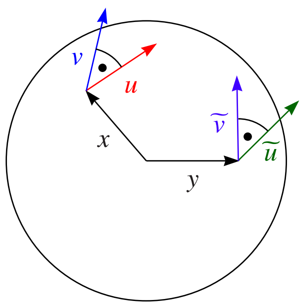
2) The sectional curvature for (S^n,can) is constant. (This is the reflection of the high symmetry of the sphere.) For showing this we use Lemma 2: Let $\phi\colon (M,\langle\bullet,\bullet\rangle)\mapsto(N,\langle\langle\bullet,\bullet\rangle\rangle)$ be a Riemannian isometry. For $\sigma\subset T_pM$ (two-dimensional subspace) $\mathrm{d}\phi_p(\sigma)\subset T_{\phi(p)}N$ is a two-dimensional subspace, since ϕ is a diffeomorphism. Furthermore $K^M(\sigma,p)=K^N(\mathrm{d}\phi_p(\sigma),\phi(p))$ i.e. the sectional curvatures are **invariant under isometries**.

   For the proof of this lemma, we need certain transformation properties. For the Levi-Civita connection it holds that $D^N_{\mathrm{d}\phi X}\,\mathrm{d}\phi Y=\mathrm{d}\phi(D^M_X Y)$ (see exercise 1, sheet 7). Moreover $[\mathrm{d}\phi X,\mathrm{d}\phi Y]^N=\mathrm{d}\phi[X,Y]^M$ and of course $\langle\langle\mathrm{d}\phi X,\mathrm{d}\phi Y\rangle\rangle=\langle X,Y\rangle$. This leads to

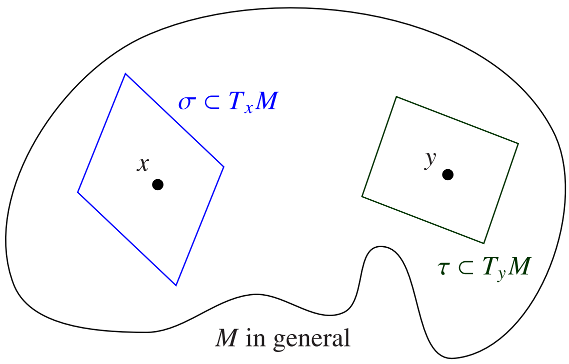
$$R^N(\mathrm{d}\phi X,\mathrm{d}\phi Y)\mathrm{d}\phi Z=\mathrm{d}\phi(R^M(X,Y)Z)\,,\tag{6.25}$$

   which implies the lemma. □

   Claim: (S^n,can) has **constant** sectional curvature.



   Be $\sigma:=[v,u]\subset T_xS^n$ and $\tau:=[\widetilde{v},\widetilde{u}]\subset T_YS^n$ (the span). Without loss of generality we choose $\{v,u\}$ and $\{\widetilde{v},\widetilde{u}\}$ as orthonormal.



   The sphere is a homogeneous space, on which isometries that map two-dimensional tangent spaces in different points; these are rotations. We set $e_1:=x$, $e_2:=u$, and $e_3:=v$. We can complete this to an orthonormal basis of $\mathbb{R}^{n+1}$: $\{e_1,e_2,e_3,\dots,e_{n+1}\}$. With **respect to this** basis we have

$$y=\begin{pmatrix}y_1\\ \vdots\\ y_{n+1}\end{pmatrix}:=f_1\,,\quad \widetilde{u}=\begin{pmatrix}\widetilde{u}_1\\ \vdots\\ \widetilde{u}_{n+1}\end{pmatrix}=:f_2\,,\quad \widetilde{v}=\begin{pmatrix}\widetilde{v}_1\\ \vdots\\ \widetilde{v}_{n+1}\end{pmatrix}=:f_3\,.\tag{6.26}$$

   Complete this to be an orthonormal basis $\{f_1,f_2,\dots,f_{n+1}\}$. Then the matrix $A:=(f_1|f_2|\dots|f_{n+1})$ is orthogonal, $A\in O(n+1)$ and by construction $A_{e_i}=f_i$ for $i=1,\dots,n+1$. Since $\phi\colon\mathbb{R}^{n+1}\circlearrowleft$, $v\mapsto Av$ is an isometry of $(\mathbb{R}^{n+1},\mathrm{can})$, ϕ leaves S^n invariant and induces and isometry of (S^n,can). Since ϕ is linear this implies $\mathrm{d}\phi = \phi$ and also $\mathrm{d}\phi(\sigma) = \mathrm{d}\phi(\mathrm{span}(u, v) = \mathrm{span}(e_1, e_2)) = \mathrm{span}(f_1, f_2) = \tau$. Hence $\phi(x) = \phi(e_1) = f_1 = y$. By lemma 2 the claim follows: $K(\sigma, x) = K(\tau, y)$. **Remark**: Later we show $K \equiv 1$.

3) n-dimensional hyperbolic space:

Consider $H^n\mathbb{R} = \{x = (x^1, \dots, x^n) \in \mathbb{R}^n | x^n > 0\}$ (the upper half space). With respect to the chart $(H^n\mathbb{R}, \mathrm{id})$ we define a Riemannian metric (the so-called hyperbolic metric)

$$(g_{ij}) = \begin{pmatrix} 1/(x^n)^2 & 0 & \dots \\ 0 & 0 & \dots \\ \vdots & \vdots & \ddots \\ \dots & 0 & 1/(x^n)^2 \end{pmatrix} . \tag{6.27}$$

The sectional curvature of $(H^n, \mathbb{R}$,with the above metric) is $K \equiv -1$ (exercise). The idea is the calculate ${\Gamma_{ij}}^k$. From this it follows ${R_{ijk}}^l$ and the sectional curvature.

Remarks

1) For arbitrary n there are manifolds with $K = 0,\ 1,\ -1$.

2) Intermediate values for K can be obtained by **conformal change** of the metric. Be (M, g) a Riemannian manifold with $\lambda \in C^\infty M$ and $\lambda(p) > 0$. This delivers a way to construct new Riemannian metrics by $\widetilde{g} := \lambda g$ $(\widetilde{g}(p)(X, Y) := \lambda(p)g(p)(X, Y))$. For constant $\lambda > 0$ this is just a rescaling and we have $\widetilde{K} = \lambda^{-1}K$, whereas $\widetilde{K}$ is the sectional curvature of $\widetilde{g}$ and K is the sectional curvature of g. To see this look at local coordinates: ${\widetilde{\Gamma}_{ij}}{}^k = {\Gamma_{ij}}^k$. This leads to ${\widetilde{R}_{ijk}}{}^l = {R_{ijk}}^l$ and hence

$$\widetilde{K} = \frac{\widetilde{g}(\widetilde{R}(x, y)x, y)}{\widetilde{g}(x, x)\widetilde{g}(y, y) - \widetilde{g}(x, y)^2} = \lambda^{-1}K . \tag{6.28}$$

Stretching the metric by λ means stretching the length of vectors by λ^2: $\|x\|_{\sim}^2 = \lambda\|x\|^2$ with $\widetilde{g}(x, x) = \lambda g(x, x)$. Be S_1^n the n-dimensional unit sphere and S_R^n the n-dimensional sphere of radius R. With $R := \sqrt{K}$ $(K = R^2)$ S_R^n has curvature $1/R^2$. Hence $K > 0$ for $(S^n, 1/K\mathrm{can})$, $K = 0$ for $(\mathbb{R}^n, \mathrm{can})$, and $-|K| = K < 0$ for $(H^n\mathbb{R}, 1/|K|\mathrm{can})$.

Theorem

A complete, simply connected, connected Riemannian manifold with constant sectional curvature K is isometric to $(\mathbb{R}^n, \mathrm{can})$ if $K = 0$, $(S_1^n, /K\mathrm{can})$ if $K > 0$, and $(J^n\mathbb{R}, 1/|K|\mathrm{can})$ if $K < 0$.

Proof

For the proof see do Carmo, Section 8.4.
"Complete" means that every geodesic in M is defined on $\mathbb{R}$ (see also chapter 7). "Simply connected" means that every loop in M is homotopic to a point, i.e. there is a continuous deformation of the loop to a point. (The sphere is simply connected which can be shown by stereographic projection.)

6.4.1 The theorem of Killing-Hopf

This holds for all Riemannian manifolds of constant curvature.

Theorem

Be (M^n, g) a Riemannian manifold of Dimension ≥ 2, $K \in \mathbb{R}$. M^n is complete and connected of constant sectional curvature K, if and only if M^n is isometric to the orbit space $\mathbb{R}^n/\Gamma$ with $\Gamma \subset \mathrm{Isom}(\mathbb{R}^n, \mathrm{can}) = E(n)$ (Euclidian transformation group, translations and rotations).

Proof

For the proof see Wolf: "Spaces of constant curvature".

Example

1) Consider $\mathbb{R}^2$ and the discrete subgroup $\mathbb{Z}^{\nvDash}$. $\mathbb{Z}^2$ is the group of translations in the two-dimensional plane with integer numbers (defines a grid on the plane). Then $\mathbb{R}^2/\mathbb{Z}^2 \simeq T^2$, whereas T^2 is the flat torus.

2) Consider S^n/Γ with $\Gamma \subset \mathrm{Isom}(S^n, \mathrm{can}) = O(n+1)$ (orthogonal group, group of all rotations) if $K > 0$.

3) Consider $H^n\mathbb{R}/\Gamma$ with $\Gamma \subset \mathrm{Isom}(H^n\mathbb{R}, \mathrm{can}) \simeq O(1,n) = \{A \in \mathrm{GL}(n+1,\mathbb{R}) | A^\intercal \mathrm{diag}(-1,1,\ldots,1)A = (-1,1,\ldots,1)\}$ if $K < 0$.

Γ acts without fixes points and with discrete orbits. Studying spaces with constant curvature reduces to studying the subgroups of Γ; hence a problem of Riemannian geometry/topology is transferred to an algebraic problem.

6.5 The Ricci curvature

Definition

Let R be the Riemann curvature tensor of a Riemannian manifold $(M, \langle \bullet, \bullet \rangle)$ and X, Y, $Z \in \mathcal{V}M$. Then for every point $p \in M$ $Y(p) \mapsto R(X(p), Y(p))Z(p)$ is an endomorphism of T_pM. Also for X, Z fixed $R(X, \bullet)Z$ is a $(1,1)$ tensor field. For an arbitrary (1,1) tensor field A, $A(p)$: $T_pM \mapsto T_pM$ is an endomorphism. We define the **trace of A** by

$$(\mathrm{Tr}(A))(p =) := \sum_{i=1}^{n} \langle A(p)e_i, e_i \rangle_p \,, \tag{6.29}$$

where $\{e_i\}_{i=1}^n$ is an orthonormal basis of T_pM ("contraction").
The Ricci tensor of the manifold M is the $(0,2)$ tensor field $\mathrm{Ricci}(X, Z) := \mathrm{Tr}(Y \mapsto R(X, Y)Z)$.

Comment

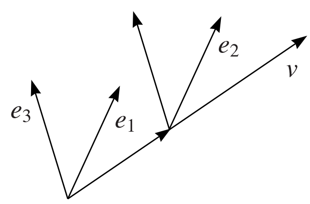
Recall from linear algebra: Be ϕ an endomorphism with matrix $M = (m_{ij})$ of an Euclidian vector space. Then $\mathrm{Tr}(\phi) := \mathrm{Tr}(M) = \sum_{i=1}^n m_{ii}$ is independent of the chosen basis. In particular for an orthonormal basis $\{e_i\}_{i=1}^n$ one has $m_{ii} = \langle \phi e_i, e_i \rangle$. For any point $p \in M$ and an orthonormal basis $\{e_i\}$ of T_pM one has

$$\mathrm{Ricci}_p(v, w) = \sum_{i=1}^{n} \langle R(v, e_i)w, e_i \rangle_p \,, \tag{6.30}$$

with v, $w \in T_pM$. In particular the Ricci tensor is symmetric. The **Ricci curvature** of M in direction $v \in T_pM$ with $v \neq 0$ is

$$r(v) := \mathrm{Ricci}\left(\frac{v}{\|v\|}, \frac{v}{\|v\|}\right) = \sum_{i=1}^{n} \left\langle R\left(\frac{v}{\|v\|}, e_i\right)\frac{v}{\|v\|}, e_i \right\rangle_p = \sum_{i=2}^{n} K\left(\frac{v}{\|v\|}, e_i\right) \,, \tag{6.31}$$

whereas $\{e_i\}$ is an orthonormal basis. This basis can be chosen such that $v/\|v\| =: e_1$. The sectional curvature is calculated in a plane that is spanned by e_1 and e_2; this calculation is also done in other planes. One then performs the mean of all sectional curvatures. Because of that one often writes a prefactor $1/(n-2)$ in front of the above definition.



The Ricci curvature is a simpler object than the Riemann curvature tensor, but it contains less information than the latter.

Remark

K is a function going from the Grassmann bundle of all two-dimensional subspaces of all tangent spaces to $\mathbb{R}$, whereas the Ricci curvature r is a function from the tangent bundle to $\mathbb{R}$. Furthermore one can introduce the scalar curvature s, which is a function from the manifold M to $\mathbb{R}$.
The **scalar curvature** of M is the C^∞-function s: $M \mapsto \mathbb{R}$

$$s(p) := \sum_{j=1}^{n} r(e_j)\,, \tag{6.32}$$

for $\{e_i\}$ being an orthonormal basis of T_pM. Hence

$$s(p) = \sum_{i,j=1}^{n} \langle R(e_j, e_i)e_j, e_i\rangle_p = \sum_{i,j=1}^{n} K([e_i, e_j], p)\,, \tag{6.33}$$

where we use $K([X, X], p) := 0$. In each point $p \in M$ this is the trace of the symmetric bilinear form Ricci_p with respect to $\langle \bullet, \bullet\rangle_p$ (in particular independent of the chosen orthonormal basis).

Some remarks

1) Theorem (do Carmo, Section 4.3): A Riemannian manifold $(M, \langle \bullet, \bullet\rangle)$ has constant sectional curvature $K(\sigma, p) = K_0$ for all $p \in M$ and for all $\sigma \subset T_pM$ if and only if $\langle R(X, Y)W, Z\rangle = K_0(\langle X, W\rangle\langle Y, Z\rangle - \langle Y, W\rangle\langle X, Z\rangle)$.

2) A Riemannian manifold (M, g) is an **Einstein space** if $\mathrm{Ricci}_p(X, Y) = \lambda g_p(X, Y)$ for all X, $Y \in \mathcal{V}M$, where $\lambda \in C^\infty M$. Examples are manifolds of constant curvature. It suffices to show that $\mathrm{Ricci}(x, x) = \lambda g(x, x)$ for $\|x(p)\| = 1$ for all p. But

$$\mathrm{Ricci}_p(x, x) = \sum_i \langle R(x, e_i)x, e_i\rangle_p = \sum_i K([x, e_i], p)g_p(x, x) = (n-1)K_0 g_p(x, x)\,. \tag{6.34}$$

The Einstein tensor of a (Pseudo-)Riemannian manifold (for which the metric is not necessarily positive definite) is $G := \mathrm{Ricci} - (s/2)g$, whereas s is the scalar curvature. G is a (0,2)-tensor. The Einstein field equations modeling gravitation are given by $G = 8\pi T$, where G is the Einstein tensor for a four-dimensional Pseudo-Riemannian manifold with signature (-,+,+,+). T is the energy-stress tensor encoding the mass distribution. The constant 8π can be found by comparing the Einstein equation to the Newtonian limit. In local coordinates it holds that

$$R_{ij} = \mathrm{Ricci}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\,. \tag{6.35}$$

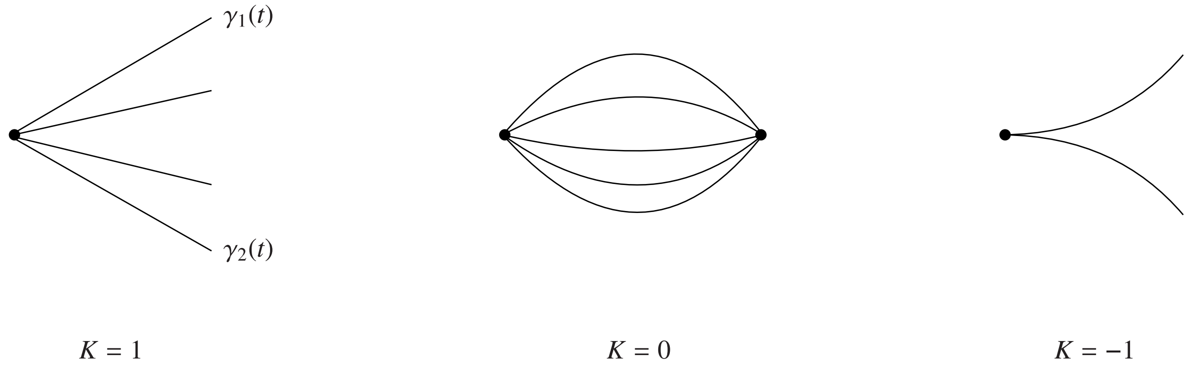
and the Einstein equations are then:

$$\boxed{\left(R_{ij} - \frac{s}{2}g_{ij}\right) = 8\pi T_{ij}\,.} \tag{6.36}$$

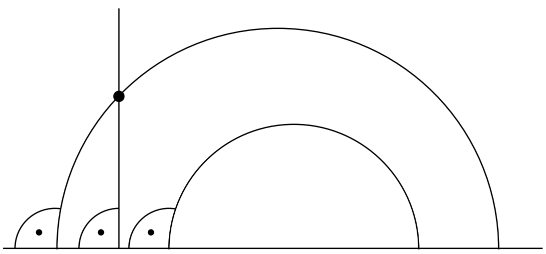
Hence, mass describes the curvature of space-time and vise versa.

6.6 Jacobi fields

Jacobi fields locally describe a relation between curvature and geodesics. In a space with $K = 0$ (for example the Euclidian plane) geodesics are given by straight lines. The distance between two straight lines that enclose a certain angle between them grows linearly. However, in a space with $K = 1$ (for example S^2 with the canonical metric) the distance between geodesics is sublinear and they tend to meet again. In a space with $K = -1$ (for example $H^2\mathbb{R}$ with corresponding hyperbolic metric) the geodesics diverge exponentially.



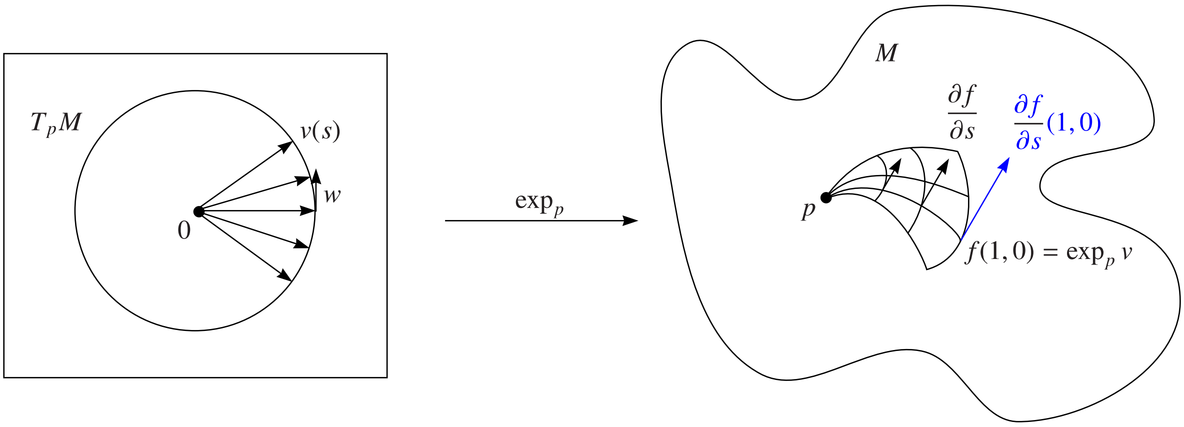
For example, in $H^2\mathbb{R}$ the geodesics were calculated to be straight lines perpendicular to the x-axis and half-circles. (This was an exercise on one of the last sheets.) The above mentioned behavior is visible in this example.



We now want to generalize this concept.

6.6.1 Jacobi equation

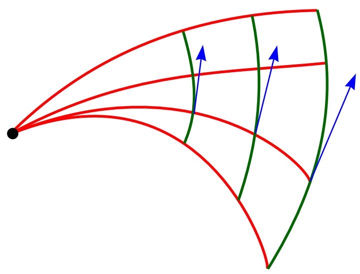
Consider a Riemannian manifod $(M, \langle\bullet,\bullet\rangle)$ and $p \in M$. We pick some $v \in T_pM$ such that $\exp_p v$ is defined. Then consider the parameterized surface $f(t,s) := \exp_p tv(s)$ for $0 \leq t \leq 1$ and $|s| < \varepsilon$, where $v(s)$ is a curve in T_pM such that $v(0) = v$ and $\|v(s)\| = 1$.



Be w the tangent vector to the curve $v(s)$: $w := v'(0) \perp v$. From the proof of the Gauß lemma (chapter 4, theorem 5) we know that

$$d\exp_p|_v(w) = \frac{\partial f}{\partial s}(1,0) \in T_{\exp_p(v)}M\,. \tag{6.37}$$

$\|d\exp_p|_v(w)\|$ measures "how quickly" the geodesic rays $t \mapsto f(t,s)$ diverge.



We consider the vector field

$$\mathrm{d}\ \exp_p|_v(tw) = \frac{\partial f}{\partial s}(t,0)\,, \tag{6.38}$$

along the geodesic $\gamma(t) := f(t,0) = \exp_p tv$ for $0 \leq t \leq 1$. As γ is a geodesic we have for all t, s:

$$\boxed{\frac{D}{\partial t}\frac{\partial f}{\partial t}(t,s) = 0\,,} \tag{6.39}$$

with the tangent vector field $\partial f(t,s)/\partial t$ along the geodesic and the covariant derivative $D/\partial t$.

Lemma

Let $f\colon A \subset \mathbb{R}^2 \mapsto M$, $(u,v) \mapsto f(u,v)$ be a parameterized surface and $V(u,v)$ a vector field along f. Then we have

$$\frac{D}{\partial v}\frac{D}{\partial u}V - \frac{D}{\partial u}\frac{D}{\partial v}V = R\left(\frac{\partial f}{\partial u},\frac{\partial f}{\partial v}\right)V\,. \tag{6.40}$$

Proof

This ia a local statement. It suffices to prove it in a chart (U,φ) with associated local base fields $X_i := \partial/\partial x^i$ for $i = 1, \ldots, n$. Hence $V = \sum_{i=1}^n v^i X_i$ with $v^i = v^i(u,v)$ is defined on the parameterized surface. Now we can calculate the covariant derivative in direction of $\partial f/\partial u$:

$$\frac{D}{\partial u}V = \frac{D}{\partial u}\left(\sum_{i=1}^n v^i X_i\right) \overset{\text{def}}{=} D_{\frac{\partial f}{\partial u}}\left(\sum_{i=1}^n v^i X_i\right) = \sum_{i=1}^n \frac{\partial v^i}{\partial u}X_i + \sum_{i=1}^n v^i\frac{D}{\partial u}X_i\,. \tag{6.41}$$

An analogue expression holds for the covariant derivative along $\partial f/\partial v$:

$$\frac{D}{\partial v}V = \ldots\,. \tag{6.42}$$

Furthermore

$$\frac{D}{\partial u}\left(\frac{D}{\partial v}V\right) = \sum_{i=1}^n \frac{\partial^2 v^i}{\partial u\partial v}X_i + \sum_{i=1}^n \frac{\partial v^i}{\partial v}\frac{D}{\partial u}X_i + \sum_{i=1}^n \frac{D}{\partial v}X_i + \sum_{i=1}^n v^i\frac{D}{\partial u}\frac{D}{\partial v}X_i\,. \tag{6.43}$$

Thus

$$\frac{D}{\partial v}\left(\frac{D}{\partial u}V\right) - \frac{D}{\partial u}\left(\frac{D}{\partial v}V\right) = \sum_{i=1}^n v^i\left(\frac{D}{\partial v}\frac{D}{\partial u}X_i - \frac{D}{\partial u}\frac{D}{\partial v}X_i\right)\,. \tag{6.44}$$

A computation of $(D/\partial v)(D/\partial u)X_i$ yields the following. We have $\partial f/\partial u = \sum_{i=1}^n(\partial x^j/\partial u)X_j$ for $\varphi\circ f(u,v) = (x^1(u,v),\ldots,x^n(u,v))$ and $\partial f/\partial v = \sum_{k=1}^n(\partial x^k/\partial v)X_k$. Then

$$\frac{D}{\partial u}X_i \overset{\text{def}}{=} D_{\frac{\partial f}{\partial u}}X_i = \sum_{j=1}^n \frac{\partial x^j}{\partial u}D_{X_j}X_i\,, \tag{6.45}$$

and

$$\begin{aligned}\frac{D}{\partial v}\frac{D}{\partial u}X_i &= \sum_{j=1}^n \frac{\partial^2 x^j}{\partial v\partial u}D_{X_j}X_i + \sum_{j=1}^n \frac{\partial x^j}{\partial u}D_{\frac{\partial f}{\partial v}}(D_{X_j}X_i) = \\ &= \sum_{j=1}^n \frac{\partial^2 x^j}{\partial v\partial u}D_{X_j}X_i + \sum_{j,k=1}^n \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}D_{X_k}(D_{X_j}X_i)\,.\end{aligned} \tag{6.46}$$

As a result of that

$$\left(\frac{D}{\partial v}\frac{D}{\partial u} - \frac{D}{\partial u}\frac{D}{\partial v}\right)(x_i) = \sum_{j,k=1}^n \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}(D_{X_k}D_{X_j}X_i - D_{X_j}D_{X_k}X_i) = \sum_{j,k=1}^n \frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}R(X_j,X_k)X_i\,, \tag{6.47}$$

since the term $D_{[X_j,X_k]}X_i$ vanishes for local base fields X_j and X_k. Using (6.44) leads us to

$$\frac{D}{\partial v}\left(\frac{D}{\partial u}V\right)-\frac{D}{\partial u}\left(\frac{D}{\partial v}V\right)=\sum_{i,j,k=1}^{n} v^i\frac{\partial x^j}{\partial u}\frac{\partial x^k}{\partial v}R(X_j,X_k)X_i=$$

$$=R\left(\sum_{j=1}^{n}\frac{\partial x^j}{\partial u}X_j,\sum_{k=1}^{n}\frac{\partial x^k}{\partial v}X_k\right)\left(\sum_{i=1}^{n}v^iX_i\right)=R\left(\frac{\partial f}{\partial u},\frac{\partial f}{\partial v}\right)V\,. \tag{6.48}$$

Now apply lemma 1 to a special parameterized surface $f(t,s)=\exp_p(tv(s))$. From $(D/\partial t)(\partial f/\partial t)=0$ and lemma 1 we get

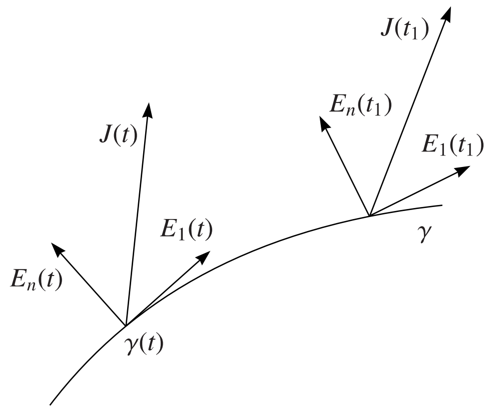
$$0=\frac{D}{\partial s}\left(\frac{D}{\partial t}\frac{\partial f}{\partial t}\right)\overset{\substack{\text{lemma 1}\\ \text{and } v=\partial f/\partial t}}{=}\frac{D}{\partial t}\left(\frac{D}{\partial s}\frac{\partial f}{\partial t}\right)+R\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}=$$

$$\overset{\substack{\text{lemma 3,}\\ \text{chapter 4}}}{=}\frac{D}{\partial t}\left(\frac{D}{\partial t}\frac{\partial f}{\partial s}\right)+R\left(\frac{\partial f}{\partial t},\frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}\,. \tag{6.49}$$

Set $J(t):=(\partial f/\partial s)(t,0)$ and $\gamma(t):=f(t,0)$. Then we obtain the Jacobi equation

$$\boxed{\frac{D}{\partial t}\frac{D}{\partial t}J(t)+R(\gamma'(t),J(t))\gamma'(t)=0\,.} \tag{6.50}$$

One can also make a more general definition: Let γ: $[0,a]\mapsto M$ be a geodesic. A vector field $J(\gamma(t))\equiv J(t)$ along γ is called **Jacobi field** it J satisfies the Jacobi equation for all $t\in[0,a]$.
A Jacobi field is completely determined by the initial values $J(0)$ and $J'(0):=(D_{\dot\gamma}J)(0)$. To see this we consider orthonormal parallel fields along γ, $E_1(t)$, ..., $E_n(t)$.



Then we can write

$$J(t)=\sum_{i=1}^{n}f_i(t)E_i(t)\,, \tag{6.51}$$

with $f_i\in C^\infty[0,a]$ and we obtain

$$J'(t)=D_{\dot\gamma}I(t)=\sum_{i=1}^{n}f_i'(t)E_i(t)\,,\quad J''(t)=\sum_{i=1}^{n}f_i''(t)E_i(t)\,. \tag{6.52}$$

We also set $a_{ij}(t):=\langle R(\gamma'(t),E_i(t))\gamma'(t),E_j(t)\rangle_{\gamma(t)}$. Then

$$R(\gamma',J)\gamma'=\sum_{j=1}^{n}\langle R(\gamma',J)\gamma',E_j\rangle E_j=\sum_{i,j=1}^{n}a_{ij}(t)f_i(t)E_j(t)\,. \tag{6.53}$$

With that, the Jacobian equation is equivalent to

$$f_j''(t)+\sum_{i=1}^{n}a_{ij}(t)f_i(t)=0\,,\quad j=1,\,\dots,\,n\,, \tag{6.54}$$

and this is a **linear** system of second order differential equations. For given initial conditions $(f_j(0),f_j'(0))$ for $j=1$, ..., n ($\Leftrightarrow$ $(J(0),J'(0))$) because of $J(0)=\sum_{j=1}^{n}f_j(0)E_j(0)$ and $J'(0)=\sum_{j=1}^{n}f_j'(0)E_j(=)$. There is a unique solution $f_j(t)$ for $j=1$, ..., n which is equivalent to a unique Jacobi field $J(t)$ defined on $[0,a]$. The conclusion is that the set of Jacobi fields along a given geodesic γ is a $2n$-dimensional vector space. In particular there are $2n$ linearly independent Jacobi fields along a geodesic ($n=\dim M$).

Remark

The tangent field of a geodesic $J_1(t) := \gamma'(t)$ is a Jacobi field. This follows from $(D/\partial t)\gamma' = 0$ (which holds because γ is a geodesic) and from $R(\gamma', J)\gamma' = R(\gamma', \gamma')\gamma' = 0$, which holds because of the antisymmetry of the Riemann curvature tensor with respect to the first two arguments. Therefore, we make the *Ansatz* $J_2(t) := a(t)\gamma'(t)$. Hence

$$\frac{D}{\partial t}\frac{D}{\partial t}J_2(t) = \frac{D}{\partial t}\left(a'\gamma'(t) + a\frac{D}{\partial t}\gamma'\right) = \frac{D}{\partial t}(a'\gamma') = a''\gamma' \,, \tag{6.55}$$

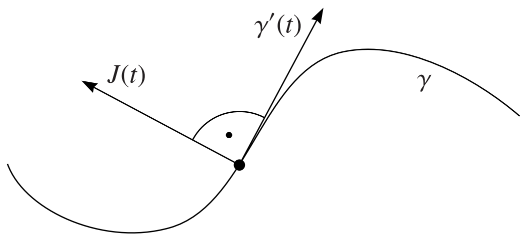
and

$$R(\gamma', J_2(t))\gamma' = a(t)R(\gamma', \gamma')\gamma' = 0 \,, \tag{6.56}$$

because of the multilinearity of the Riemann curvature tensor. As a result of that, J_2 is a Jacobi field if and only if $a''(t) = 0$ and hance $a(t) = a + bt$ with $a, b \in \mathbb{R}$. The **conclusion** is that $J_1(t) := \gamma'(t)$ and $J_2(t) := t\gamma'(t)$ span a two-dimensional subspace of the vector space $\mathrm{Jac}(\gamma)$. It suffices to understand the $(2n-2)$-dimensional subspace of $\mathrm{Jac}(\gamma)$ of all Jacobi fields orthogonal to γ'.

Examples

We consider Jacobi fields for manifolds of **constant sectional curvature**. Let $(M, \langle\bullet, \bullet\rangle)$ be a Riemannian manifolds with constant sectional curvature K_0. Let $\gamma\colon [0, a] \mapsto M$ be a normal geodesic (i.e. γ is parameterized by arc-length). Further let $J(t)$ be a Jacobi field along γ such that $J(t) \perp \gamma'(t)$ for all t.



According to a remark in section 5.2 one has the following formula for an arbitrary vector field X along γ:

$$\langle R(\gamma', J)\gamma', X\rangle = K_0(\langle\gamma', \gamma'\rangle\langle J, X\rangle - \langle\gamma', X\rangle\langle J, \gamma'\rangle) = K_0\langle J, X\rangle \,. \tag{6.57}$$

Since γ is a normal geodesic, $\langle\gamma', \gamma'\rangle = 1$. Furthermore we consider a orthogonal Jacobian field, hence $\langle J, \gamma'\rangle = 0$. Since X is arbitrary, we find $R(\gamma', J)\gamma' = K_0 J$. So the Jacobi equation just reduces to

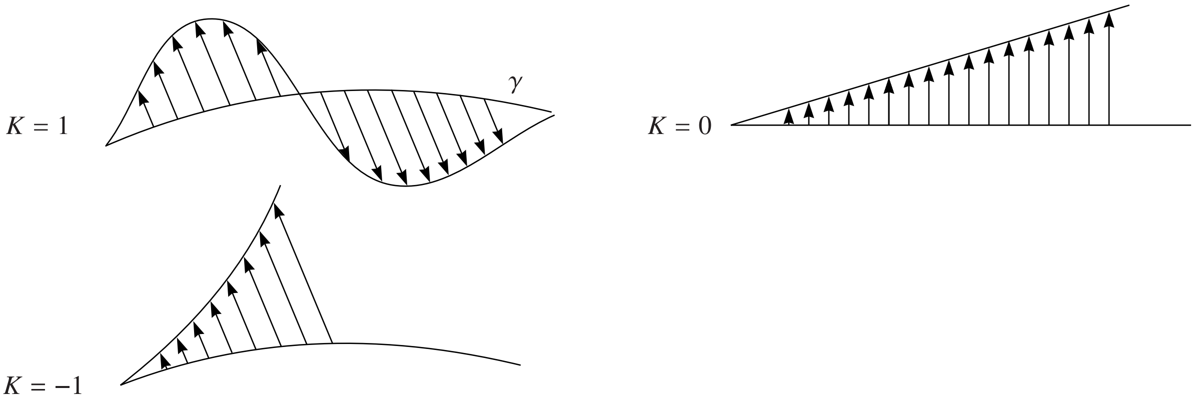
$$\boxed{J'' + K_0 J = 0 \,, \quad \frac{D}{\partial t} :=' } \tag{6.58}$$

This is the differential equation describing a harmonic oscillator. Let $E(t)$ be a parallel field along γ with $\|E(t)\|_{\gamma(t)} = 1$ for all t and $\langle\gamma'(t), E(t)\rangle = 0$. Then the solutions of (6.58) with initial conditions $J(0) = 0$ and $J'(0) = E(0)$ are given by:

$$J(t) = \begin{cases} \dfrac{\sin(t\sqrt{K_0})}{\sqrt{K_0}}E(t) & \text{if} \quad K_0 > 0 \\ tE(t) & \text{if} \quad K_0 = 0 \\ \dfrac{\sinh(t\sqrt{-K_0})}{\sqrt{-K_0}}E(t) & \text{if} \quad K_0 < 0 \end{cases} \,. \tag{6.59}$$

Proof

Just compute!



Remark

Using $E = E_1, \ldots, E_{n-1}$ linearly independent parallel fields along γ (orthogonal to γ') yields and $(n-1)$ dimensional subspace of $\mathrm{Jac}(\gamma)$. We constructed a Jacobi field along a geodesic $\gamma(t) = \exp_p(tv)$ using a parameterized surface $f(t,s) = \exp_p(tv(s))$ with $\|v(s)\| = 1 = \|v\|$ and $t \in [0,1]$, $|s| < \varepsilon$. Then $J(t) = \partial f/\partial s(t,0)$ is a Jacobi field with

$$J(0) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 (\exp_p \cdot \cdot 0v(s)) = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 p = 0\,. \tag{6.60}$$

We now show that every Jacobi field along a geodesic (with $J(0) = 0$) is obtained in this way.

Theorem 1

Let $\gamma\colon [0,a] \mapsto M$ be a normal geodesic and J a Jacobi field along γ with $J(0) = 0$. Let $J'(0) \equiv (D/\partial t J)(0) =: w$ and $\gamma'(0) =: v$ ($\|v\| = 1$). Consider w as an element of $T_{av}(T_{\gamma(0)}M)$ and let $v(s)$ be a curve in $T_{\gamma(0)}M$ with $v(0) = av$ and $v'(0) = aw$ and define the parameterized surface $f(t,s) := \exp_p(t/av(s))$ with $0 \le t/a \le 1$, $|s| < \varepsilon$ and $p = \gamma(0)$. The claims are as follows:

a) $\overline{J}(t) = \partial f/\partial s(t,0)$ is a Jacobi field along γ.

b) $\overline{J}(t) = J(t)\ \forall\, t \in [0,a]$.

Proof

a) The fact that $\overline{J}$ is a Jacobi field follows from the construction of a parameterized surface and from the fact that for $\bar{t} := t/a$ holds $D/\partial\bar{t} = aD/\partial t$.

b) To show that $J = \overline{J}$ it suffices to show that $J(0) = \overline{J}(0)$ and $J'(0) = \overline{J}'(0)$. Now by definition $\overline{J}(0) = \partial f/\partial s(0,0) = 0$. Furthermore

$$\begin{aligned}\overline{J}'(t) &= \frac{D}{\partial t}\left(\frac{\partial f}{\partial s}(t,0)\right) \overset{\text{chain rule}}{=} \frac{D}{\partial t}\left\{\mathrm{d}\exp_p|_{t/av(0)}\left(\frac{t}{a}v'(0)\right)\right\} \overset{\substack{v(0)=av\\ v'(0)=aw}}{=} \frac{D}{\partial t}\left\{\mathrm{d}\exp_p|_{tv}(tw)\right\} \overset{\text{linearity}}{=} \\ &= \frac{D}{\partial t}\left\{t\mathrm{d}\exp_p|_{tv}(w)\right\} = \frac{\mathrm{d}t}{\mathrm{d}t}\mathrm{d}\exp_p|_{tv}(w) + t\frac{D}{\partial t}(\mathrm{d}\exp_p|_{tv}(w)) = \exp_p|_{tv}(w) + t\frac{D}{\partial t}(\mathrm{d}\exp_p|_{tv}(w))\,.\end{aligned} \tag{6.61}$$

Evaluation at $t = 0$ yields

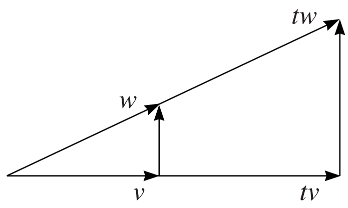
$$\overline{J}'(0) = \mathrm{d}\exp_p|_0(w) = \mathrm{id}|_{T_pM}(w) = w = J'(0)\,. \qquad \square \tag{6.62}$$

Remark

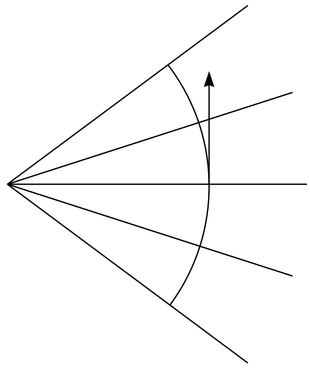
1) The calculation in the proof gives the following formula for a Jacobi field J along a normal geodesic $\gamma\colon [0,a] \mapsto M$ with $J(0) = 0$:

$$J(t) = \mathrm{d}\exp_p|_{t\gamma'(0)}(tJ'(0))\,, \tag{6.63}$$

with $t \in [0,a]$. Jacobi fields describe the linearization of the exponential map. This is an explanation why the Jacobi equation is a linear differential equation.



2) Analogous constructions/results also hold for Jacobi fields with initial condition $J(0) \neq 0$. The simplest example is $\mathbb{R}^2$ with the canonical metric.



6.7 Jacobi fields and sectional curvature

Theorem 2

Let $p \in M$, γ: $[0, a] \mapsto M$ be a normal geodesic with $\gamma(0) = p$, $\gamma'(0) = v$ and $w \in T_v(T_pM) \simeq T_pM$ with $\|w\|_p = 1$. Further let $J(t) := \mathrm{d}\exp_p|_{tv}(tw)$ for $0 \leq t \leq a$ be a Jacobi field along γ. Then the Taylor series of $\|J(t)\|_{\gamma(t)}\|^2$ at $t = 0$ is given by

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\langle R(v, w)v, w\rangle_p t^4 + \wr(t^4)\,, \tag{6.64}$$

with the notation $\wr(t^k)/t^k \mapsto 0$ for $t \mapsto 0$.

Proof

We have $J(0) = 0$, $J'(0) = w$ since $J''(0) = (-R(\gamma', J)\gamma')(0) = 0$. We compute the terms in the Taylor series:

$$\|J(t)\|^2 =: f(t) = f(0) + f'(0) + \frac{1}{2!}f''(0)t^2 + \frac{1}{3!}f'''(0)t^3 + \frac{1}{4!}f^{(4)}(0)t^4 + \ldots\,. \tag{6.65}$$

0) $\|J(0)\|^2 = \langle J, J\rangle(0) = 0$

1) $\langle J, J\rangle'(0) = 2\langle J', J\rangle(0) = 0$

   This is short-hand notation for

$$\langle J, J\rangle' = \frac{\mathrm{d}}{\mathrm{d}t}\langle J(t), J(t)\rangle_{\gamma(t)} = \gamma'(t)\langle J(t), J(t)\rangle_{\gamma(t)} = \langle D_{\gamma'(t)}J(t), J(t)\rangle_{\gamma(t)} + \langle J(t), D_{\gamma'(t)}J(t)\rangle_{\gamma(t)}\,. \tag{6.66}$$

   We will use this notation in our further computations in this proof.

2) $\langle J, J\rangle''(0) = 2(\langle J', J'\rangle + 2\langle J'', J\rangle)(0) = 2\|J'(0)\|^2 = 2\|w\|^2 = 2$

3) $\langle J, J\rangle'''(0) = 4\langle J'', J'\rangle(0) + 2\langle J''', J\rangle(0) + 2\langle J'', J'\rangle(0) = 0$ since $J''(0) = 0$

4) $(\langle J, J\rangle^{(4)}(0) = 4\langle J''', J'\rangle + 4\langle J'', J''\rangle + 2\langle J^{(4)}, J\rangle + 2\langle J''', J'\rangle + 2\langle J''', J'\rangle + 2\langle J'', J''\rangle)(0) = 8\langle J''', J'\rangle(0)$, whereas $6\langle J'', J''\rangle(0) = \langle J^{(4)}, J\rangle(0) = 0$.

   From the Jacobi equation follows

$$J''' = -\frac{D}{\partial t}R(\gamma', J)\gamma'\,. \tag{6.67}$$

   For $w \in \mathcal{V}M$ arbitrary we have

$$\left\langle \frac{D}{\partial t}R(\gamma', J)\gamma', w \right\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\langle R(\gamma', J)\gamma', w\rangle - \langle R(\gamma', J)\gamma', w'\rangle\,, \quad ' \equiv \frac{D}{\partial t} \equiv D_{\gamma'}\,. \tag{6.68}$$

By the symmetry of the Riemann curvature tensor R we obtain:

$$\langle R(\gamma', w)\gamma', J\rangle - \langle R(\gamma', J)\gamma', w'\rangle = \left\langle \frac{D}{\partial t}R(\gamma', w)\gamma', J\right\rangle + \langle R(\gamma', w')\gamma', J'\rangle - \langle R(\gamma', J\gamma', w'\rangle\,. \quad (6.69)$$

For $t = 0$ we have (again with the symmetry of R)

$$\langle -J'''(0), w(0)\rangle = \left\langle \frac{D}{\partial t}R(\gamma', J)\gamma', w\right\rangle(0) = \langle R(\gamma', w)\gamma', J'\rangle \overset{\text{symmetry}}{=} \langle R(\gamma', J')\gamma'(0), w(0)\rangle\,. \quad (6.70)$$

As w is arbitrary we get $J'''(0) = -(R(\gamma', J')\gamma')(0)$. Inserting this in (4) we find

$$\langle J, J\rangle^{(4)}(0) = -8\langle R(\gamma'(0), J'(0))\gamma'(0), J'(0)\rangle_{\gamma(0)} = -8\langle R(v, w)v, w\rangle_p\,, \quad (6.71)$$

and this leads to

$$\frac{f^{(4)}(0)}{24}t^4 = -\frac{1}{3}\langle R(v, w)v, w\rangle_p t^4\,, \quad (6.72)$$

which completes the proof. ☐

Corollary

If γ is normal (i.e. $\|v\| = 1$) and $\langle w, v\rangle = 0$ then

a) $\langle R(v, w)v, w\rangle_p$ is the sectional curvature of the plane $\sigma = [v, w]$, hence

$$\|J(t)\|^2 = t^2 - \frac{1}{3}K(\sigma, p)t^4 + \wr(t^3)\,. \quad (6.73)$$

b) $\|J(t)\| = t - \frac{1}{6}K(\sigma, p)t^3 + \wr(t^3)$

Proof

a) This is clear by definition and theorem 2.

b) That follows from comparing coefficients of the corresponding Taylor series. From

$$f(t) = a + bt + ct^2 + dt^3 + \wr(t^4)\,, \quad (6.74)$$

it follows

$$f^2(t) = (a + bt + \ldots)^2 = a^2 + 2abt + (2ac + b^2)t^2 + \ldots\,, \quad (6.75)$$

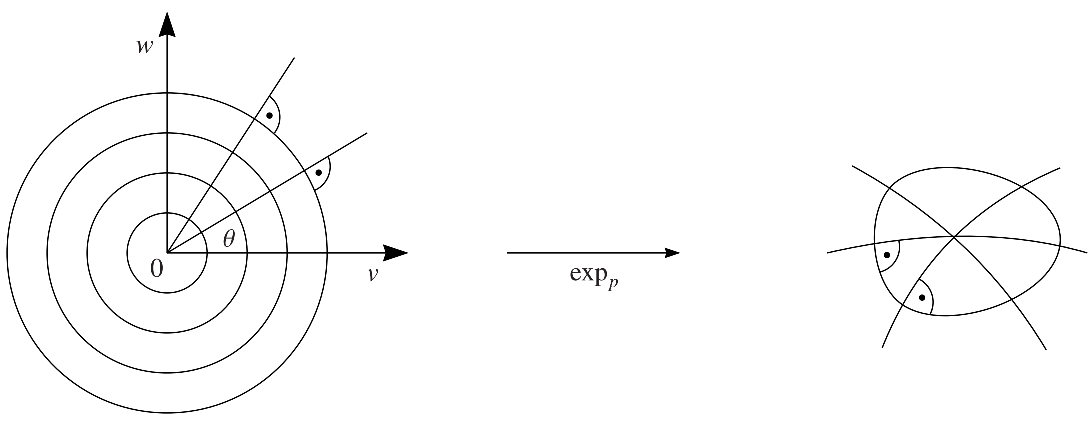
in particular in our situation we have $a = 0$, $b = 1$, $c = 0$ and $d = -K/6$. ☐

Remark

In the flat space one has a linear growth of Jacobi fields. For positive sign of the sectional curvature the growth will be a little bit smaller than proportional to t and for negative sign it will be a little bit larger.

6.7.1 Application to geodesic circles

Consider $p \in M$ and v, $w \in T_pM$ with $\|v\| = \|w\| = 1$ and $v \perp w$. Inside the Euclidian plane spanned by the vectors v and w one can define polar coordinates:

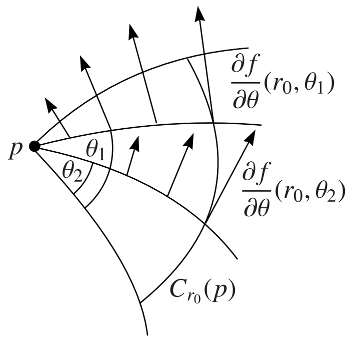


We define

$$f(r,\theta) := \exp_p(r(v\cos\theta + w\sin\theta))\,, \tag{6.76}$$

for sufficient small r. For $r = r_0$ fixed define $C_{r_0}(p) := \{f(r_0,\theta)|\theta \in [0,2\pi]\}$. $C_{r_0}(p)$ is a **geodesic circle** in M with center p and radius r_0. We want to compute the length of C_r:

$$L(C_r) = \int_0^{2\pi} \left\| \frac{\partial f}{\partial \theta} \right\| \mathrm{d}\theta\,. \tag{6.77}$$



The observation is that for every fixed θ the vector field $(\partial f/\partial\theta)(r)$ is the value of the Jacobi field $J_\theta(r)$ along the geodesic $\gamma_\theta(r) := \exp_p(rv(\theta))$. Hence (now that r is fixed, $\sigma = [v,w]$):

$$L(C_r(p)) = \int_0^{2\pi} \|J_\theta(r)\|\,\mathrm{d}\theta \overset{\text{Corollary 2}}{=} \int_0^{2\pi} \left(r - \frac{1}{6}K(\sigma,p)r^3 + \wr(r^3) \right) \mathrm{d}\theta =$$

$$= 2\pi \left(r - \frac{1}{6}K(\sigma,p)r^3 + \wr(r^3) \right) = 2\pi r \left(1 - \frac{1}{6}K(\sigma,p)r^2 + \wr(r^2) \right)\,. \tag{6.78}$$

This implies the formula of Betrand-Puiseux. The length of a geodesic circle is less than in the Euclidian situation if the sectional curvature if greater than zero and it is more if the sectional curvature is smaller than zero.

$$K(\sigma,p) = \lim_{r\mapsto 0} \frac{3}{\pi r^3}(2\pi r - L(C_r))\,, \tag{6.79}$$

hence the sectional curvature is a measure for deviation of the geometry on a manifold from Euclidian geometry.

Kapitel 7

Riemannian manifolds considered as metric spaces

7.1 Riemannian metric and distance function

So far we have considered local concepts as geodesics, the exponential map, and curvature. The setting was that we had some manifold (a global object) and to pick an arbitrary point and look what the properties are in the neighborhood of this point. Now we also want to consider the global geometry of a manifold. This can be done by including topological assumptions. A (simple) global question is: Given two points $p, q \in M$ is there a continuous curve or a piecewise geodesic curve or a geodesic between the two points? This holds locally (normal geodesic neighborhood).

Lemma

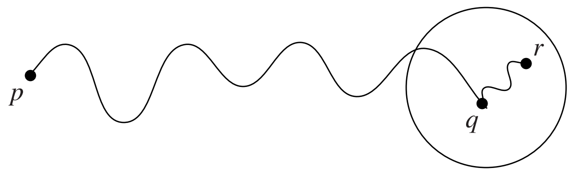
If a smooth manifold (locally Euclidian suffices) is connected, then M is also path-connected. This means that for any two points $p, q \in M$ there exists a continuous curve $c\colon [0,1] \mapsto M$ with $c(0) = p$, $c(1) = q$. (Recall: A topological space X is **connected** if X **cannot** be decomposed into two disjoint, nonempty, open subsets. Equivalently $\emptyset$ and X are the only subsets which are open and closed.)

Proof

Let $p \in M$. Set $A := \{q \in M | \exists \text{ a continuous curve between } p \text{ and } q\}$. The properties of A are:

1) $A \neq \emptyset$, since $p \in A$. To see this define a curve that joins p with itself: Take $c\colon [0,1] \mapsto A$, $c(t) := p$.

2) A is open: For $q \in A$ take $r \in B_\varepsilon(q)$ (normal neighborhood of q). Then for any $r \in B_\varepsilon(q)$, $r \in A$.



3) A is closed ($\Leftrightarrow M \setminus A$ is open). If $q \in M \setminus A$, $r \in B_\varepsilon(q)$ as before, then $r \in M \setminus A$ (otherwise $q \in A$).

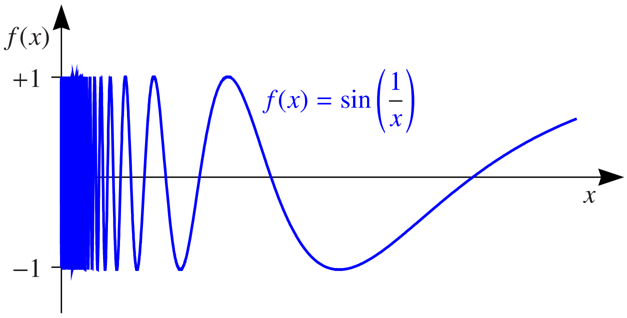
(1), (2), and (3) imply (by definition of connectedness) $A = M$. □

Remark

Path connected implies connected, but the inverse is not true, in general.

Example

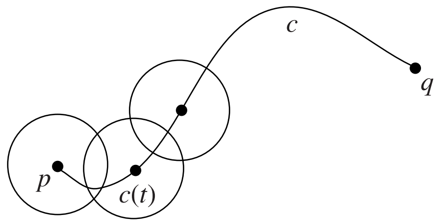
Let $X := (\{0\} \times [-1,1]) \cup \{(x, \sin(1/x)) \in \mathbb{R}^2 | x > 0\}$ endowed with the subspace topology. This space is connected but not path connected.



For the proof see: Sihger-Thorpe: “Elementary Topology and Geometry” on page 53.
From now on we want to assume that every manifold M **is connected**. An immediate consequence of that is: Let M be a connected Riemannian manifold. Then any two points p, $q \in M$ can be connected by a piecewise geodesic (in particular by a piecewise smooth path).

Proof

Since M is path-connected we can choose a continuous curve c: $[0, 1] \mapsto M$ with $c(0) = p$ and $c(1) = q$. If the image $c([0, 1]) \subset M$ is compact in M, which hence can be covered by finitely many (normal) neighborhoods U_k $(k = 1, \ldots, m)$, then pick geodesic segments in each neighborhood to replace c. □

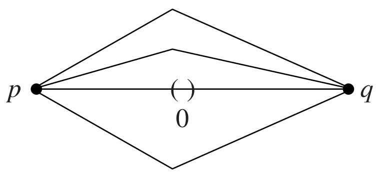


Remark

Note that in general there is no smooth geodesic segment between two points in a Riemannian manifold.

Example

Consider $\mathbb{R}^2 \setminus \{0\}$ equipped with the canonical metric.



Definition

Let $(M, \bullet, \bullet)$ be a Riemannian manifold which is connected. We define d: $M \times M \mapsto R_{\geq 0}$, $d(p, q) := \inf\{L(c_{p,q}) | c_{p,q}$ is piecewise smooth curve joining p to $q\}$ (which is not the minimum in general, see previous example $\mathbb{R}^2 \setminus \{0\}$!).

Theorem 1 (length-metric)

(M, d) is a metric space, i.e. for all points p, q, r holds:

1) $d(p, q) = d(q, p) \geq 0$
2) $d(p, q) \leq d(p, r) + d(q, r)$
3) $d(p, q) = 0 \Leftrightarrow p = q$

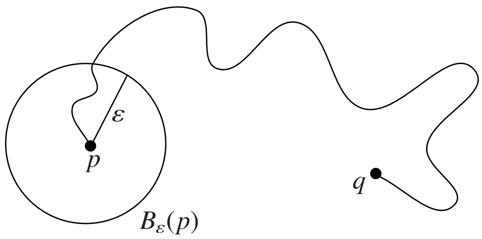
Proof

1) For c: $[0,1] \mapsto M$ with $c(0) = p$, $c(1) = q$ set $\overline{c}$: $[0,1] \mapsto M$, $\overline{c}(t) := c(1-t)$. This leads to $\overline{c}(0) = c(1) = q$ and $\overline{c}(1) = c(0) = p$, whereas $L(c) = L(\overline{c})$. This implies $d(q,p) = d(p,q)$.

2) The set $\Omega^r_{p,q}$ of curves between p and q which pass through r is a subset of the set $\Omega_{p,q}$ of **all** curves joining p to q. This implies

$$d(p,q) = \inf_{c\in\Omega_{p,q}} L(c) \le \inf_{c\in\Omega^r_{p,q}} L(c) \le \inf_{c\in\Omega_{p,r}} L(c) + \inf_{c\in\Omega_{r,q}} L(c) = d(p,r) + d(r,q)\,. \tag{7.1}$$

3) – "⇔" Be $p = q$. For $c(t) = p$ (constant path) we have $L(c) = 0$, which means $d(p,p) = 0$.
   – "⇒" Suppose $p \neq q$. Pick a normal neighborhood $B_\varepsilon(p)$ of p such that $q \notin B_\varepsilon(p)$.



Then if c is an arbitrary curve joining p to q we have $L(c) \ge \varepsilon$ (theorem 6, chapter 4). This implies $d(p,q) \ge \varepsilon > 0$. This is a contradiction! □
(This construction also works in more general contexts (length spaces).)

Definition

The **diameter** of a Riemannian manifold is defined as $\mathrm{diam}(M) := \sup d(p,q) \le \infty$ for p, $q \in M$.

Corollary

1) The topology of the metric space (M,d) is equivalent to the topology given by the definition of M as a manifold (i.e. U is open in $M \Leftrightarrow U$ is open in (M,d)).

2) For any $p_0 \in M$ the function d_{p_0}: $M \mapsto R_{\ge 0}$, $d_{p_0}(p) := d(p_0,p)$ is continuous (this is true for every metric space).

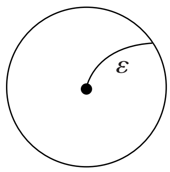
3) If M is compact then $\mathrm{diam}(M) < \infty$.

Proof

1) By theorem 6/chapter 4 there exist (open) normal balls $B_\varepsilon(p)$ of sufficient small radius ε, which are identical to metric balls of radius r (with respect to d):

$$B^{(d)}_\varepsilon(p) = \{q \in M | d(p,q) < \varepsilon\} = B_\varepsilon(p) = \exp_p(B_\varepsilon(0))\,, \tag{7.2}$$

where $B_\varepsilon(0) = \{v \in T_pM | \|v\| < \varepsilon\}$.



2) $|d_{p_0}(q) - d_{p_0}(p)| = |d(p_0,q) - d(p_0,p)| \le d(p,q)$ (which follows from the triangle inequality)

3) Fix some $p_0 \in M$. Then $d(p,q) \le d(p,p_0) + d(p_0,q) \le 2\sup d_{p_0}(r) \le D < \infty$, $r \in M$
   The finiteness follows from the fact that every continuous function on a compact set has a maximum.

7.2 Theorem of Hopf-Rinow

Definition

A Riemannian manifold $(M, \langle\bullet,\bullet\rangle)$ is **geodesically** complete if for all $p \in M$, $\exp_p$ is defined on T_pM i.e. for all $p \in M$ every geodesic $\gamma_\nu(t)$ with $\gamma_\nu(0) = p$, $\gamma'_\nu(0) = v$ is defined on all of $\mathbb{R}$.

Theorem 2 (Hopf-Rinow 1931)

Let $(M, \langle \bullet, \bullet, \rangle)$ a connected Riemannian manifold and $p \in M$. Then the following assertions are equivalent:

1) $\exp_p$ is defined on T_pM
2) Every closed subset of M with bounded diameter is compact.
3) The metric space (M, d) is complete i.e. every Cauchy sequence converges. (A sequence $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if for every $\delta > 0$ there exists an $N = N(\delta)$ such that $f(x_n, x_m) < \delta$ for all n, $m \geq N$.)
4) $(M, \langle \bullet, \bullet \rangle)$ is geodesically complete.

Moreover any of the equivalent assertions (1) to (4) implies:

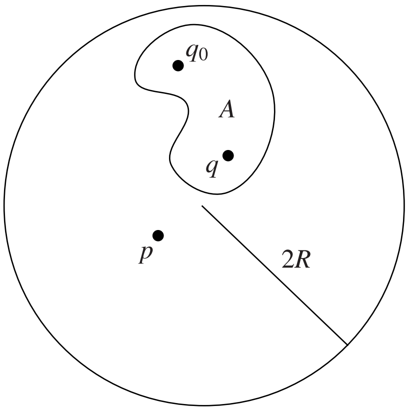
5) For every point $p \in M$ there exists (at least) one geodesic γ which joins p to q such that $L(\gamma) = d(p, q)$ i.e. γ realizes a shortest path between p and q.

Example

For $\mathbb{R}^2 \setminus \{0\}$ and two points p, q lying on the x-axis the infimum is the distance between the two points, but there exists no geodesic joining the two points.

Proof

- (1) $\Rightarrow$ (2): Let $A \subset M$ be closed with $\operatorname{diam}(A) \leq c < \infty$. Pick some point $q_0 \in A$. Then we have for any $q \in A$:



$$d(p, q) \leq d(p, q_0) + d(q_0, q) \leq d(p, q_0) + c =: R\,. \tag{7.3}$$

This implies

$$A \subset \{q \in M | d(p, q) \leq 2R\} =: \overline{B_{2R}(p)} = \exp_p(\overline{B_{2R}(0)})\,, \tag{7.4}$$

with $\exp_p(\overline{B_{2R}(0)}) = \{v \in T_pM | \|v\|_p \leq 2R\}$, whereas this is compact. Hence, $\overline{B_{2R}(p)}$ is compact and hance A is compact, since it is a closed subset of a compact set.

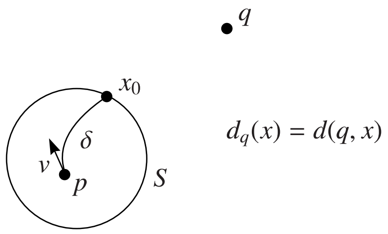
- (2) $\Rightarrow$ (3): Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then $(p_n)_{n\in\mathbb{N}}$ is bounded. Hence by (2) $(p_n)_{n\in\mathbb{N}}$ is contained in a compact subset of M. Thus, $(p_n)_{n\in\mathbb{N}}$ has a convergent subsequence. As $(p_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, $(p_n)_{n\in\mathbb{N}}$ converges also.

- (3) $\Rightarrow$ (4): Let γ: $I \mapsto M$ be a normal geodesic. We show: I is open and closed and not $\emptyset$ in $\mathbb{R}$ (hence, as $\mathbb{R}$ is connected, $I = \mathbb{R}$). The fact that I is open and not $\emptyset$ follows from the (local) existence and uniqueness theorem for geodesics (theorem 3 in section 4.4). We have: If $\gamma(t_0)$ is defined then $\gamma(t_0 + t)$ is also defined for sufficiently small t. Hence, I is open. To prove that I is closed, we define $(t_n)_{n\in\mathbb{N}}$ as a (monotone decreasing) sequence in I which converges: $t_* := \lim_{n\mapsto\infty} t_n$. For the limit it has to hold that $t_* \in I$. We have (for $m \geq a$)

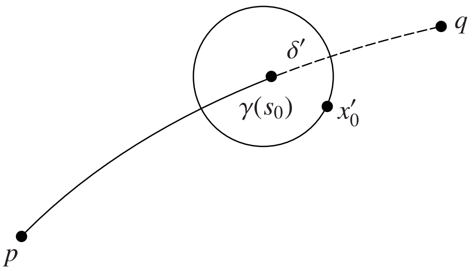
$$d(\gamma(t_n), \gamma(t_m)) \leq L(\gamma|_{[t_n,t_m]}) = |t_n - t_m|\,, \tag{7.5}$$

since the geodesic is normal (parameterized by arc-length). Since $(t_n)_{n\in\mathbb{N}}$ converges, it is Cauchy and hence $(\gamma(t_n))_{n\in\mathbb{N}}$ is also Cauchy. By (3) $(\gamma(t_n))_{n\in\mathbb{N}}$ converges, so $p_* := \lim_{n\mapsto\infty} \gamma(t_n)$. Let $W(p_*)$ be a total normal neighborhood of p_* in M. Then by theorem 7 in section 4.5 there exists some $\delta > 0$ such that every normal geodesic starting in $W(p_*)$ is defined on $(-\delta, \delta)$. Now we choose n $(\geq N_0 = N_0(\delta))$ large enough that $|t_n - t_*| < \delta/2$ and $\gamma(t_n) \subset W(p_*)$. Then the geodesic γ is defined for all t with $|t - t_*| < \delta$, hence in particular for t_*. Hence $t_* \in I$.

- (4) $\Rightarrow$ (1): Geodesically complete means that the exponential map is defined for all p. (1) said that the exponential map is defined for a certain p, hence this is clear.
- (1) $\Rightarrow$ (5):
  - Step (1): We define a candidate for a geodesic joining the two points p and q. Let $r := d(p, q)$ and for $0 < \delta << \leq r$ let $B_\delta(p)$ be a normal geodesic ball with center p and radius δ with geodesic sphere $S := S_\delta(p) = \partial B_\delta(p)$ as its compact boundary. Let x_0 be a point on S where the continuous function $d_q|_S$ assumes its minimum (we use that S is compact).

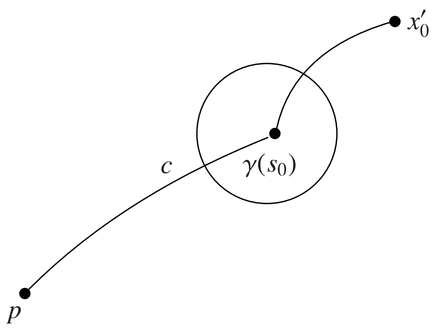


Then there exists a $v \in T_pM$ with $\|v\| = 1$ such that $x_0 = \exp_p(\delta v)$. Define $\gamma(s) := \exp_p(sv)$ for $s \in \mathbb{R}$ (use (1)).
  - Step (2) (γ is the minimal connection from p to q):
We will show that $q = \gamma(r)$, hence $d(p, \gamma(r)) = 0$. (From this follows that $d(\gamma(0), \gamma(r)) = r = d(p, q)$, so γ is minimal.) The idea is to consider the following interval: $A := \{s \in [0, r], d(\gamma(s), q) \overset{(*)}{=} r - s\}$. It remains to show that $A = [0, r]$. A is **closed** since $d(q, \bullet) = d_q$ is continuous. So $A \neq \emptyset$, since $s = 0 \in A$. Let $s_0 := \max\{t \in [0, r] | t \in A\}$. In order to show this we assume $s_0 < r$ (which will lead to a contradiction).



Consider a normal ball $B_{\delta'}(\gamma(s_0))$ at $\gamma(s_0)$ with δ' being sufficiently small so that $q \notin B_{\delta'}(\gamma(s_0))$. Let x_0' be a point on $S' = \partial B_{\delta'}(\gamma(s_0))$, where d$q$ assumes a minimum. The claim is that x_0' lies at the intersection of the geodesic and the ball, hence $x_0' = \gamma(s_0 + \delta')$ (+). Proof of this claim: First, we have
    - $d(\gamma(s_0), q) = \delta' + \min_{x \in S'} d(x, q) = \delta' + d(x_0', q)$
    - By assumption, equation $(*)$ and definition of s_0 we have $d(\gamma(s_0), q) = r - s_0$. So we get $r - s_0 = \delta' + d(x_0', q)$ (++).

Using the triangle inequality for d we obtain
    1) $d(p, x_0') \geq d(p, q) - d(q, x_0') \overset{(++)}{=} r - (r - s_0 - \delta') = s_0 + \delta'$
    2) Also for the piecewise smooth curve c $d(p, x_0) \leq d(p, \gamma(s_0)) + d(\gamma(s_0), x_0') = s_0 + \delta'$



(1) and (2) yields $d(p, x_0') = s_0 + \delta'$. We conclude that c is minimizing and hence a geodesic. (Hence, there is no corner at $\gamma(s_0)$.) Thus $x_0' = \gamma(s_0 + \delta')$. From (+) and (++) be obtain

$$r - (s_0 + \delta') \overset{(++)}{=} d(x_0', q) \overset{(+)}{=} d(\gamma(s_0 + \delta'), q)\,, \tag{7.6}$$

i.e. $(*)$ holds for $s_0 + \delta' > s_0$, which is a contradiction to the definition of s_0. Hence $s_0 = r \in A$. □

Corollary 1

A compact, connected manifold is complete.

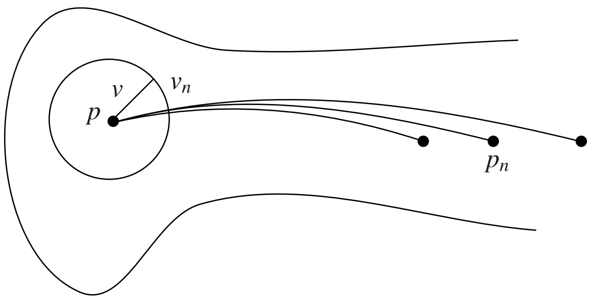
Proof

(M, d) is a complete metric space (in a compact space every Cauchy sequence converges!).

Corollary 2

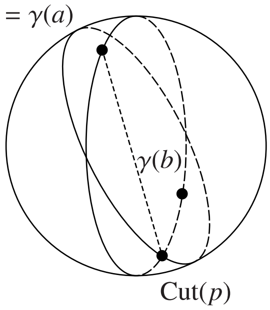
Let M be a connected and complete Riemannian manifold. If M is not compact then there is a geodesic ray $\gamma\colon [0, \infty) \mapsto M$, which is minimizing for all t: $d(\gamma(0), \gamma(t)) = t$ for all $t \geq 0$.

Proof



Idea: There exists a sequence $(p_n)_{n\in\mathbb{N}}$ with $d(p, p_n) \mapsto \infty$, since the manifold is not compact. One can find a sequence of unit vectors $v_n \in T_pM$ and there exists a convergent subsequence with unit vector v. Set $\gamma(t) = \exp_p(tv)$.

7.3 Cut-locus (“Schnittort”) of a complete Riemannian manifold



Lemma 2

Let M be a connected, complete Riemannian manifold and $\gamma\colon [a, b] \mapsto M$ a normal geodesic.

1) If there is **no** geodesic between $\gamma(a)$ and $\gamma(b)$, which is shorter than γ, then γ is minimizing.

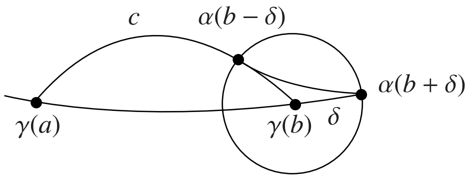
2) If there is a geodesic $c \neq \gamma$ between $\gamma(a)$ and $\gamma(b)$ with $L(c) = L(\gamma)$, then γ is **not** minimizing on $[a, b+\varepsilon]$ for any $\varepsilon > 0$.

3) If γ is minimizing on I, then also $J \subseteq I$.

Proof

1) According to Hopf-Rinow there exists a minimizing geodesic c_* between $\gamma(a)$ and $\gamma(b)$ we then have $d(\gamma(a), \gamma(b)) = L(c_*) \leq L(\gamma)$. Then by assumption we have $L(\gamma) = L(c_*) = d(\gamma(a), \gamma(b))$. This shows that γ is minimizing.

2) Let $W = B_\delta(\gamma(b))$ be a totally normal neighborhood of $\gamma(b)$.



We set

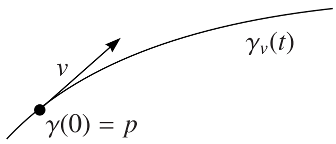
$$\alpha(t) := \begin{cases} c(t) & \text{for} \quad t \in [a,b] \\ \gamma(t) & \text{for} \quad t \in [b, b+\delta] \end{cases} . \tag{7.7}$$

α connects $\gamma(a)$ to $\gamma(b+\delta)$. Since W is totally normal there is a **unique** minimal geodesic between $\alpha(b-\delta)$ and $\alpha(b+\delta)$. Since $c \neq \gamma$ there is a corner at $\gamma(b)$ and hence α is **not** a geodesic. We conclude that the minimal geodesic segment between $\alpha(b-\delta)$ and $\alpha(b+\delta)$ is strictly shorter than the corresponding part of α. We thus have constructed a curve c_* from $\gamma(a)$ to $\gamma(b+\delta)$, which is strictly than $\alpha|_{[a,b+\delta]}$. By construction $L(\gamma|_{[a,b+\delta]}) = L(\alpha|_{[a,b+\delta]}) > L(c_*)$. Hence, γ is not the shortest connection between $\gamma(a)$ and $\gamma(b+\delta)$.

3) This is clear (use negation).

Some definitions

Be M a complete Riemannian manifold, $p \in M$, $v \in T_pM$ and $\|v\| = 1$. Let $\gamma_v(t) := \exp_p(tv)$ be the unique normal geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. We set $I := \{t \in [0,\infty) | d(\gamma_v(0), \gamma_v(t)) = t\}$ i.e. γ_v is minimizing between $\gamma(0)$ and $\gamma_v(t)$.



I_v is closed (because $d(\gamma_v(0), \bullet)$ is continuous). Hence **either** $I_v = [0,\infty)$, i.e. γ_v is a **geodesic ray**, **or** $I_v = [0, s(v)]$ is a closed interval. In the latter case $\gamma_v(s(v))$ is called **cut point of p along γ**.

Remarks

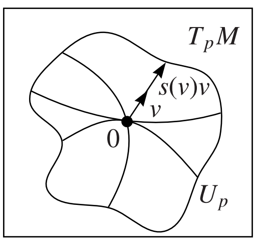
1) One can show: The map s: $\{v \in T_pM | \|v\| = 1\} \mapsto \mathbb{R} \cup \{\infty\}$ (whereas $\{v \in T_pM | \|v\| = 1\}$ is the unit tangent sphere at p, which is compact), $v \mapsto s(v)$ is continuous.

2) If M is compact then $s(v) < \infty$ for all v. (M cannot contain rays!)

Given $p \in M$ the set

$$U_p := \left\{ w \in T_pM | \|w\| < s\left(\frac{w}{\|w\|}\right) \right\} , \tag{7.8}$$

is an open neighborhood of $0 \in T_pM$ with boundary $\partial U_p = \{s(v)v \in T_pM | \|v\| = 1\}$. The **cut locus of $p \in M$** (“Schnittort”) is the set $\mathrm{Cut}(p) := \exp_p(\partial U_p) = \exp_p\{s(v)v | v \in T_pM, \|v\| \in 1\}$.

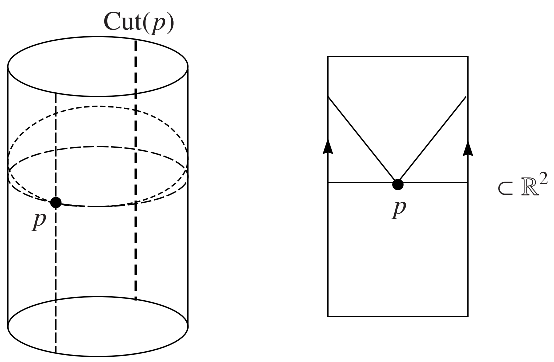


Examples

1) For$(\mathbb{R}^2, \text{can})$ or (H^2, can), geodesics are always minimizing. Hence $\text{Cut}(p) = \emptyset$ for all p.

2) For (S^n, can) geodesics are great circles and are minimizing as long as $d_p(\bullet) < \pi$, in other words for all $p \in S^n$, $U_p = \{w \in T_pM | \|w\| < \pi\}$, $\exp_p(U_p) = S^n \setminus \{-p\}$. Hence $S^n = \exp(U_p) \uplus \text{Cut}(p)$.

3) Cylinder:

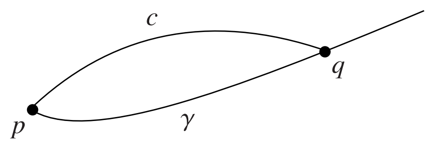


Theorem 3

Let M be a complete connected Riemannian manifold. For every point $p \in M$ one has a disjoint union $M = \exp_p(U_p) \uplus \text{Cut}(p)$.

Proof

For any $q \in M$ there is a minimizing geodesic γ_v between q and p such that $q = \gamma_v(t_0)$ with $\|v\| = 1$, $t_0 \leq s(v)$ (by Hopf-Rinow). In particular $t_0 v \in \overline{U}_p = U_p \cup \partial U_p$ and hence (as q is arbitrary) $M \subseteq \exp_p(\overline{U}_p) = \exp_p(U_p) \cup \exp_p(\partial U_p) = \exp_p(U_p) \cup \text{Cut}(p)$. There remains to show $\text{Cut}(p) \cap \exp_p(U_p) = \emptyset$. **Assume:** $q \in \exp_p(U_p) \cap \text{Cut}(p)$. Then we have on the one hand $q \in \exp_p(U_p)$ i.e. there exists a minimal geodesic $\gamma(t)$ with $\gamma(0) = p$ and $\gamma(r) = q$ with $d(p, q) = r$. As U_p is open, γ is minimizing also on $[0, r + \varepsilon]$ for ε sufficiently small. On the other hand, $q \in \text{Cut}(p)$ i.e. q is a cut point for some geodesic starting at p. This implies that there is a minimizing geodesic c with $c(0) = p$ and $c(r) = c(d(p, q)) = q$ which is **not** minimizing after r (in particular $c \neq \gamma$).

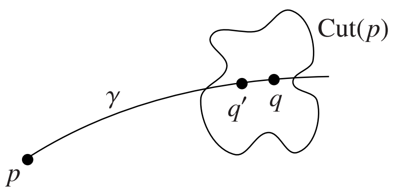


But $L(c|_{[0,r]}) = L(\gamma|_{[0,r]}) = d(p, q)$ hence by Lemma 2 (2) applied to γ we conclude that γ is not minimizing after r and this is a contradiction. Hence $\exp_p(U_p) \cap \text{Cut}(p) = \emptyset$. □

7.3.1 Further properties of the cut locus

1) $\text{Cut}(p)$ has no inner points. (For a two-dimensional manifold the cut locus is some piece of a graph.)

   Proof: Assume that there is $q \in \text{Cut}(p)$ along γ.



   Then there exists $q' \in \gamma \cup \text{Cut}(p)$ "before" q. Then by definition of $\text{Cut}(p)$ there is a minimizing geodesic c between q' and p. There are two possibilities:

   a) Case 1: $c = \gamma$. By definition of $\text{Cut}(p)$ γ is not minimizing after q', which is a contradiction.

   b) Case 2: $c \neq \gamma$. By lemma 2 (2) γ is not minimizing after q', which also is a contradiction.

   Hence there are not interior points. □

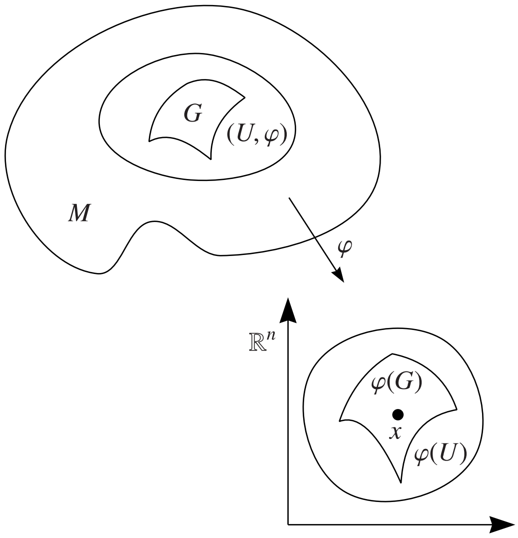
2) $\exp_p |U_p$ is injective (i.e. for any $q \in \exp_p(U_p)$ there is a **unique** geodesic from p to q).

   Proof: Let $\exp_p v_1 = q = \exp_p v_2$. If $v_1 \neq v_2$ there are two minimizing geodesics $\gamma_{v_1} \neq \gamma_{v_2}$ between p and q i.e. (by definition of $\mathrm{Cut}(p)$) $q \in \mathrm{Cut}(p)$, which is a contradiction. Hence $v_1 = v_2$. □

   Remark: One can show that $\exp_p |_{U_p}$ is an embedding $U_p \hookrightarrow M$.

7.4 An application of the decomposition theorem: the volume of Riemannian manifolds

Consider first a domain $G \subset M$ (G open, connected, relatively compact) which is completely contained in some chart (U, φ) with $\varphi(p) = (x^1(p), \ldots, x^n(p))$ as coordinates.



We define

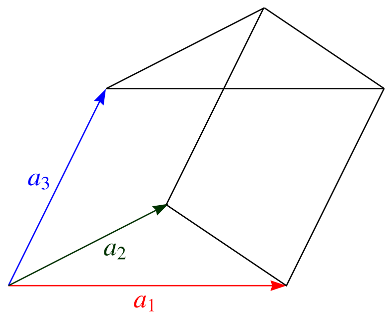
$$\mathrm{vol}(G) := \int\limits_{\varphi(G)} \sqrt{\det(g_{ij}(\varphi^{-1}(x)))}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n = \int\limits_{\varphi(G)} \mathrm{d}\,\mathrm{vol}\,, \tag{7.9}$$

where d vol is the volume element and

$$g_{ij}(p) = \left\langle \left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p \right\rangle_p . \tag{7.10}$$

Remarks

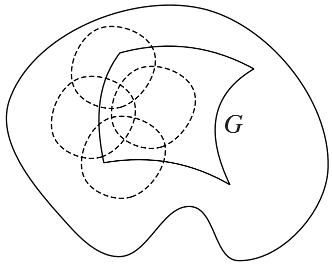
1) In linear algebra the volume of a parallel-epiped spanned by $a_1, \ldots, a_n \in \mathbb{R}^n$ is given by $V = \sqrt{\det\langle a_i, a_j\rangle}$.



2) Using the substitution rule for integrals (in $\mathbb{R}^n$) one shows that $\mathrm{vol}(G)$ is independent of the chosen chart (with $G \subset U \subset M$).

To define the volume of a compact domain G which is **not** contained in a chart one chooses a covering of G by finitely many charts $(U_i, \varphi_i)_{i=1,\ldots,m}$ with associated partition of unity $(f_i)_{i=1,\ldots,m}$ and sets

$$\mathrm{vo}(G) := \sum_{i=1}^{m} \int\limits_{\varphi_i(G\cap U_i)} f_i\, \mathrm{d}\,\mathrm{vol}_i\,. \tag{7.11}$$



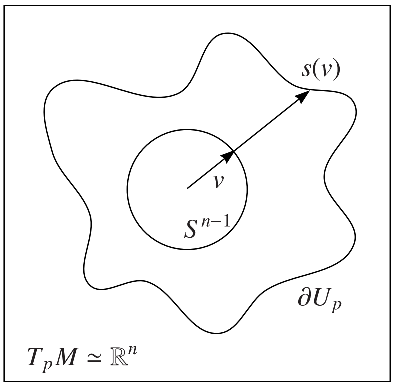
Then one shows that vol(G) does not depend on the choices of charts and the partition. (For the proof of this we refer to the books.)

Remark

For ($\mathbb{R}^n$, can) one can take $\varphi = \mathrm{id}$ and hence $\mathrm{d}\,\mathrm{vol} = \mathrm{d}x^1 \dots \mathrm{d}x^n$ (Lebesgue measure) which is equal to $t^{n-1}\,\mathrm{d}t\,\mathrm{d}\varphi$ with respect to polar coordinates (t, u) with $t \in \mathbb{R}_{>0}$ and $\mathrm{d}\sigma$ being the volume element on the unit sphere $S^{n-1} \subset \mathbb{R}^n$ (see analysis).
We know the following:

- $M = \exp_p(U_p) \uplus \mathrm{Cut}(p)$
- $\mathrm{Cut}(p)$ has no interior points. Hence, it does not contribute to the volume: $\mathrm{vol}(M) = \mathrm{vol}(\exp_p(U_p))$.

  $\mathrm{Cut}(p)$ has measure zero. One has to use the fact that the map $s\colon \{v \in T_pM \| \|v\| = 1\} \simeq S^{n-1}$, $v \mapsto s(v)$ is continuous. Hence we have a continuous copy of the sphere S^{n-1} in the tangent space.

  Hence, ∂U_p is locally a graph of a continuous function and thus by Fubini in polar coordinates

  $$\mathrm{vol}_{\mathbb{R}^n}(\partial U_p) = \int\limits_{S^{n-1}} \left[\int\limits_{s(v)}^{s(v)} t^{n-1}\,\mathrm{d}t \right] \mathrm{d}v = 0\,. \tag{7.12}$$

  The countable union of zero sets is again a zero set and as a result of that, this also holds globally. Furthermore, the map $\exp_p\colon T_pM \mapsto M$ is C^∞ and, in particular, Lipschitz on compact subsets. Hence $\mathrm{vol}_M(\mathrm{Cut}(p)) = \mathrm{vol}(\exp_p(\partial U_p)) = 0$, since Lipschitz maps $f\colon X \mapsto Y$ have the general property that $\mathrm{vol}_Y(f(A)) \leq (\mathrm{lip}_f)^n \mathrm{vol}_X(A)$ (with the Lipschitz constant lip_f).
- $\exp_p|_{U_p}$ is a diffeomorphism onto its image.

Hence we can use $\exp_p^{-1}\colon \exp_p(U_p) \mapsto \mathbb{R}^n$ ($\simeq T_pM$) as a chart. We next compute the volume element with respect to this chart using Jacobi fields: Let $c(t) = \exp_p tu$ be a normal geodesic and $\{u, e_2, \dots, e_n\}$ an orthonormal basis of T_pM. Further let $Y_i(t)$, $i = 2, \dots, n$ be the unique Jacobi fields along c with $Y_i(0) = 0$ and $Y_i'(0) = (D_{c'}Y_i)(0) = e_i$. From section (7.1) we have $(\mathrm{d}\exp_p)_{tu}(u) = c'(t)$ and $\mathrm{d}\exp_p|_{tu}(te_i) = Y_i(t)$. Now compute the corresponding tangent vectors by choosing curves in tangent space that correspond to the coordinates:

$$\left.\frac{\partial}{\partial t}\right|_{c(t)} = c'(t)\,, \quad \left.\frac{\partial}{\partial x^i}\right|_{c(t)} = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \exp_p(tu + se_i) = \mathrm{d}\exp_p|_{tu}(e_i) = \frac{1}{t} Y_i(t)\,, \tag{7.13}$$

since the differential of the exponential map is a linear map. Now we want to compute the metric. Since $c(t)$ is a normal geodesic

$$g_{11}(c(t)) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right\rangle_{c(t)} = \|c'(t)\|^2_{c(t)} = 1\,. \tag{7.14}$$

From $Y_i(t) \perp c'(t)$ we obtain

$$g_{1k}(c(t)) = \left\langle \frac{\partial}{\partial t}, \frac{\partial}{\partial x^k} \right\rangle_{c(t)} = 0\,. \tag{7.15}$$

Last, but not least

$$g_{ij}(c(t)) = \left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle_{c(t)} = \left\langle \frac{1}{t} Y_i(t), \frac{1}{t} Y_j(t) \right\rangle_{c(t)}\,, \tag{7.16}$$

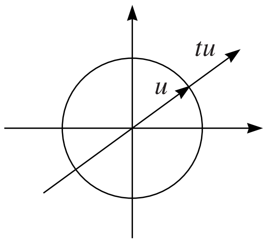
for $2 \leq i,\, j \leq n$. So and therefore

$$\sqrt{\det(g_{ij}(c(t)))} = \frac{1}{t^{n-1}} \sqrt{\det(\langle Y_i, Y_j \rangle_{c(t)})} =: J(n,t)\,, \tag{7.17}$$

whereas $g_{ij}(c(t))$ is an $(n \times n)$-matrix and $\langle Y_i(t), Y_j(t) \rangle_{c(t)}$ is an $(n-1) \times (n-1)$-matrix. Hence we obtain

$$\mathrm{d}\,\mathrm{vol} = \sqrt{\det(g_{ij})}\, \mathrm{d}x^1 \ldots \mathrm{d}x^n = J(n,t)\, \mathrm{d}x^1 \ldots \mathrm{d}x^n = J(n,t) t^{n-1}\, \mathrm{d}t\, \mathrm{d}u\,, \tag{7.18}$$

whereas (t,u) are polar coordinates and $\mathrm{d}u$ is the volume element of S^{n-1} in $\mathbb{R}^n$.



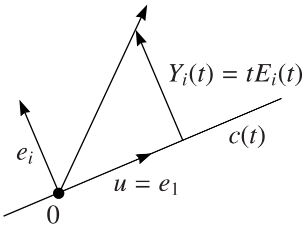
This leads us to

$$\boxed{\mathrm{vol}(M) = \mathrm{vol}(\exp_p(U_p)) = \int_{S^{n-1}} \int_0^{s(u)} J(n,t) t^{n-1}\, \mathrm{d}t\, \mathrm{d}u\,.} \tag{7.19}$$

In order to really compute this one needs the cut value $s(u)$ and one also needs the Jacobi fields, which is, in general, difficult. Hence, one often can only obtain estimates for the volume; an exact calculation is only possible in simple cases, as for example spaces of constant curvature (see section 7.2).

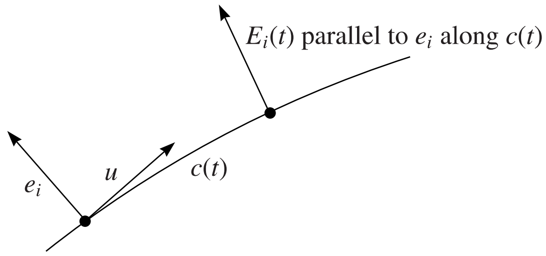
7.4.1 Some simple examples: the volume of spaces of constant curvature

1) $(\mathbb{R}^n, \mathrm{can})$:



Here, $c(t) = \exp_p(tu) = tu$. It holds that $E_i(t)$ are parallel to e_i along $c(t)$. As a result of that $J(n,t) = 1$.

2) (S^n, can):



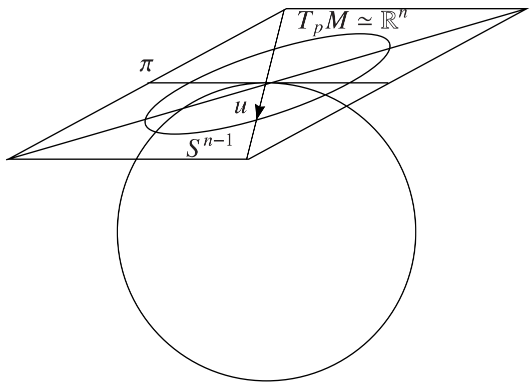
We know that $Y_i(t) = \sin(t)E_i(t)$. With $s(u) = \pi$ we obtain:

$$\mathrm{vol}(S^n) = \int^{S^{n-1}} \left[\int_0^{\pi} \left(\frac{\sin(t)}{t} \right)^{n-1} t^{n-1} \, \mathrm{d}t \right] \mathrm{d}u = \mathrm{vol}(S^{n-1}) \int_0^{\pi} (\sin(t))^{n-1} \, \mathrm{d}t \,, \tag{7.20}$$

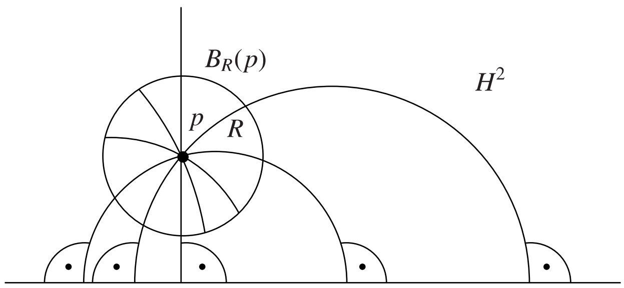
which is a recursive formula for the calculation of S^n. Especially

$$\mathrm{vol}(S^{2n}) = \frac{2(2\pi)^n}{(2n-1)\cdot(2n-3)\cdot\ldots\cdot 3\cdot 1} \,, \quad \mathrm{vol}(S^{2n+1}) = 2\frac{\pi^{n+1}}{n!} \,. \tag{7.21}$$

The observation is that $\mathrm{vol}(S^n) \mapsto 0$ for $n \mapsto \infty$: A sphere of very high dimension has an extremely small volume.



3) (H^n, can):



Since the geodesics diverge exponentially and never meet again, $s(u) = \infty$ (as in $(\mathbb{R}^n, \mathrm{can})$). The calculation can be done analogously to the sphere. We just have to use

$$J(n,t) = \left(\frac{\sinh(t)}{t} \right)^{n-1} = \mathrm{vol}(H^n) = \infty \,. \tag{7.22}$$

Since H^n is not compact, its volume is infinite. However, we can do a restriction to compact sets of H^n. For balls of radius R around p, $B_R(p) := \{q \in H^n | d(p,q) \leq R\}$, we obtain

$$\mathrm{vol}(B_R(p)) = \int_{S^{n-1}} \int_0^R (\sinh(t))^{n-1} \, \mathrm{d}t \, \mathrm{d}u = \mathrm{vol}(S^{n-1}) \int_0^R (\sinh(t))^{n-1} \, \mathrm{d}t \,. \tag{7.23}$$

For $R \gg 1$ one has

$$\sinh(R) = \frac{1}{2}(\mathrm{e}^R - \mathrm{e}^{-R}) \approx \frac{1}{2}\mathrm{e}^R \,, \tag{7.24}$$

and hence

$$\mathrm{vol}(B_R^{H^n}(p)) \approx C_1 \mathrm{e}^{(n-1)R} + C_2\,, \tag{7.25}$$

so the volume of such a ball in the hyperbolic planes grows exponentially with R. This is a phenomenon that appears in Riemannian manifolds that are noncompact. Compare to the volume of balls in $(\mathbb{R}^n, \mathrm{can})$, where the volume just grows polynomially: $\mathrm{vol}(B_R^{\mathbb{R}^n}(p)) \approx C_3 R^n$. It is common to characterize manifolds just by the volume growth.

For more information one can have a look into the book by M. Berger: "Panorama of Riemannian Geometry", in which one can find a lot of history and developments on the field of Riemannian geometry.

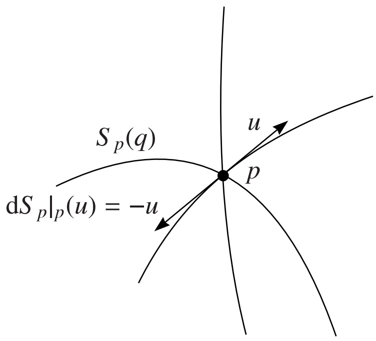
Kapitel 8

Outlook: symmetric spaces

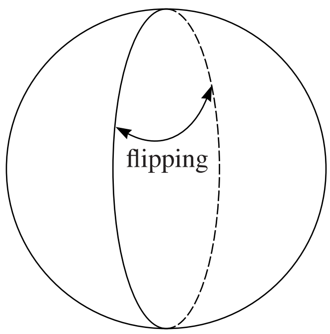
Riemannian manifolds can be classified in a hierarchy. There is the big set of general Riemannian manifolds. One subclass are the homogeneous Riemannian manifolds, which include the spaces of constant curvature. From the point of view of Riemannian geometry, these are the simplest ones, if they are additionally simply connected (this means that every loop is contractible to one point); they are $\mathbb{R}^n$, S^n, and H^n. The so-called symmetric spaces are a generalization of spaces of constant curvature.
Historically, one tried to compute g_{ij} in local coordinates by using a Taylor series, which involves (partial) derivatives of any order of components of the curvature tensor R_{ijkl}. In the case of constant curvature, these components vanish, and one obtains simpler formulas. What can be said about metrics with $DR = 0$? This question was motivated by such local computations. The conjecture was that R has to be constant. However, it turned out later that there is actually a very big class of such manifolds: the symmetric spaces. The geometric definition of a symmetric space S is as follows: A connected Riemannian manifold $(S, \langle \bullet, \bullet \rangle)$ is called symmetric "if there are a lot of symmetries", or more precisely if for all $p \in S$ there exists and isometry S_p such that

1) $S_p(p) = p$
2) $\mathrm{d}S_p|_p = -\mathrm{id}_{T_pM}$

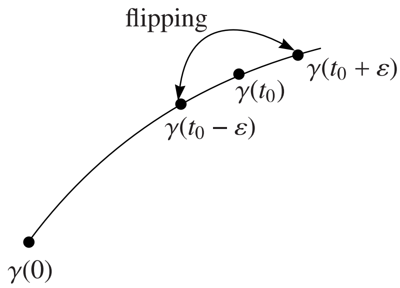
From the rigidity of isometries it follows that from (1), (2) S_p is uniquely determined, namely **geodesic reflection**.



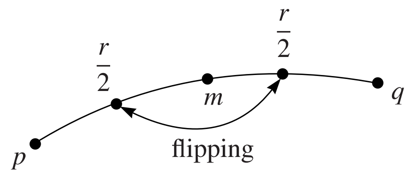
$\mathbb{R}^n$, S^n, and H^n are symmetric. This is clear in Euclidian space. Rotations in the abient space gives that also spheres are symmetric spaces.



However, there are a lot more symmetric spaces than just these. One important property of a symmetric space is that it is complete. Idea of a proof: By flipping it follows that one can define the geodesic everywhere and hence the manifold is complete.



S is homogeneous: Given p, $q \in S$ there is an isometry ϕ which maps p to q: $\phi \in \mathrm{Iso}(S)$ with $\phi(p) = q$. Idea of a proof: Use that S is complete. By Hopf-Rinow there exists a minimizing geodesic γ between two points p, q with $\gamma(0) = p$ and $\gamma(r) = p$. Set $m_{=}\gamma(r/2)$. Flip in m by using the reflection in m. As $S_m(p) = S_m(\gamma(0)) = \gamma(r) = q$.



There exists a set X that is homogeneous for group action. It holds $X \simeq G/G_X$ (bijection), whereas $G_X = \{g \in G | g \cdot x = x\}$ (with the subgroup G_X which is the stabilizer of this point). As soon as one has a homogeneous space one has an algebraic description of the space. This allows one to do computations if one understands the corresponding group. One can describe a symmetric space as a Lie group modulo a compact subgroup: $S = \mathrm{Iso}(S)/\mathrm{Sub}(p)$. For instance, in the case of a sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. There are a lot of other ones, namely $H^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Hence one will learn more about such Lie groups. The Lie groups $\mathrm{SL}(n,\bullet)$ are important in number theory, whereas $\mathrm{SU}(n)$ are important in physics.