Mitschrift zur Vorlesung: Symmetric and homogeneous spaces

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Vorlesung Sommersemester 2010

Letzte Aktualisierung und Verbesserung: 23. Mai 2010

Mitschrieb der Vorlesung Symmetric and homogeneous spaces
von Herrn Prof. Dr. Leuzinger im Sommersemester 2010
von Marco Schreck.

Dieser Mitschrieb erhebt keinen Anspruch auf Vollständigkeit und Korrektheit.
Kommentare, Fehler und Vorschläge und konstruktive Kritik bitte an Marco.Schreck@gmx.de.

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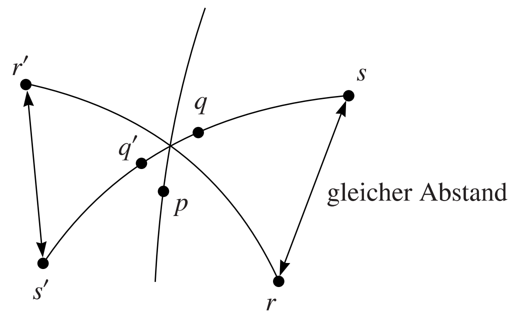
Kapitel 1

Gruppen-Aktionen

Das Ziel der Vorlesung ist die Einführung der wichtigsten Klassen von Riemannschen Mannigfaltigkeiten. Dabei handelt es sich um symmetrische und lokale symmetrische Räume. Die einfachsten Beispiele, die verallgemeinert werden, sind die klassische Geometrien, also

- die euklidische Geometrie,
- sphärische Geometrie und
- hyperbolische Geometrie.

Beispielsweise sind $\mathbb{R}^2$ und $\mathbb{R}^2/\mathbb{Z}^2 = T^2$ lokal symmetrische Räume. Zur Definition: Sei S eine zusammenhängende Riemannsche Mannigfaltigkeit, so dass für alle Punkte $p \in S$ die geodätische Spiegelung S_p eine Isometrie ist.



Diese Eigenschaft gilt in den oben genannten drei klassischen Geometrien.

1.1 Grundlegende Konzepte

Sei G eine Gruppe und X eine Menge. ***G* operiert auf *X***, falls eine Abbildung ϕ: $G \times X \mapsto X$, $(g, x) \mapsto g \cdot x$ existiert mit $e \cdot x = x$, $(gh) \cdot x = g \cdot (h \cdot x)$. Mit anderen Worten induziert ϕ einen Homomorphismus $G \mapsto$ Bijektionen(X), $g \mapsto \phi(g, \bullet) = \phi_g$ (Permutationen der Menge). Dies ist äquivalent zu $\phi_{gh} = \phi_g \circ \phi_h$, $\phi_e = \mathrm{id}$ und $\phi_{g^{-1}} = (\phi_g)^{-1}$. Man sagt auch, dass G eine **Transformationsgruppe** von X ist.
Die Menge $G \cdot x := \{\cdot x | g \in G\}$ heißt **G-Bahn von *x*** (**Orbit von *x***). Eine Gruppenoperation definiert eine Äquivalenzrelation $x \sim y \Leftrightarrow y \in G \cdot x$. Damit wird X in disjunkte Bahnen (Äquivalenzklassen) zerlegt. Der **Orbit-Raum** (**Bahnenraum**) ist die Menge der Bahnen $X/\sim \equiv X/G$ („X modulo G“). X heißt **homogen** bezüglich G, falls genau eine Bahn existiert. Man sagt dann auch, dass G **transitiv operiert**. Die Menge $G_x := \{g \in G | g \cdot x = x\}$ ist eine Gruppe, die **Isotopiegruppe oder Stabilisator** von x.

Lemma:

Die Punkte der Bahn $G \cdot x$ ($x \in X$) entsprechen bijektiv den Restklassen von G modulo Stabilisator G_x von x: $G \cdot x \simeq G/G_x$. Insbesondere gilt $X \simeq G/G_x$ (für beliebiges X), sofern G transitiv ist.

Beweis:

Betrachte $G/G_x = \{gG_x | g \in G\}$. Die Menge der Restklassen $gG_x = \{gh | h \in G_x\}$ und Abbildungen φ: $G \cdot x \mapsto G/G_x$, $g \cdot x \mapsto gG_x$.

- φ ist surjektiv nach Definition.
- φ ist injektiv, denn $\varphi(g \cdot x) = \varphi(h \cdot x) \Leftrightarrow gG_x = hG_x \Leftrightarrow g^{-1}h \in G_x \Leftrightarrow (g^{-1}h) \cdot x = x \Leftrightarrow g \cdot x = h \cdot x$.

Dies schließt den Beweis. □

Bemerkungen:

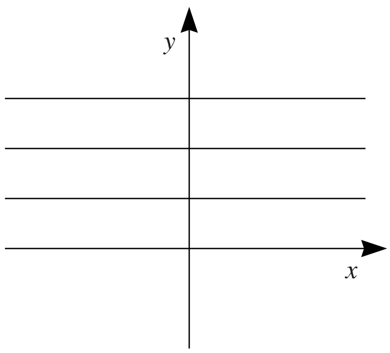
1) Interessant sind Zusatzstrukturen für X und G (und ϕ), beispielsweise [G topologische Gruppe/X topologischer Raum/ϕ stetig] oder [G Lie-Gruppe/X Mannigfaltigkeit/ϕ differenzierbar].
2) Falls G eine topologische Gruppe und X Hausdorffsch ist, so ist G_x abgeschlossen.

1.2 Einfache Beispiele

1a) $X = \mathbb{R}^2$, $G = (\mathbb{R}, +)$: Die $\mathbb{R}$-Aktion heißt auch **Fluß**. (Der Fluß kommt vor allem in der Theorie der dynamischen Systeme vor.)

$$\phi : \mathbb{R} \times \mathbb{R}^2 \mapsto {}^2; \left(t, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \begin{pmatrix} x+t \\ y \end{pmatrix} . \tag{1.1}$$

Die Bahnen sind die Parallelen zur x-Achse. Der Bahnenraum $\mathbb{R}^2/\mathbb{R}$ ist isomorph zur y-Achse, also zu $\mathbb{R}$.



1b) Sei $X = \mathbb{R}^2$, $G = \mathbb{R}^2$ und

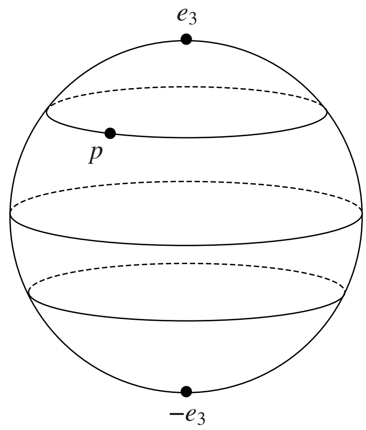
$$\phi : \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}^2; \left(\begin{pmatrix} s \\ t \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \mapsto \begin{pmatrix} s+x \\ t+y \end{pmatrix} . \tag{1.2}$$

Die Bahn von $(0,0) = \mathbb{R}^2$, womit G transitiv operiert. Der Bahnenraum $\mathbb{R}^2/\mathbb{R}^2$ besteht aus einem Punkt und damit ist $\mathbb{R}^2$ homogen bezüglich $\mathbb{R}^2$.

2a) $X = S^2 = \{x \in \mathbb{R}^3 | \|x\| = 1\}$

G seien eigentliche Drehungen um die x_3-Achse:

$$\left\{ \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \theta \in [0, 2\pi) \right\} . \tag{1.3}$$



Die komplette Spähre wird zerlegt in Bahnen nach der Äquivalenzklasseneinteilung. Es gibt Bahnen verschiedener Dimension, was ein typisches Phänomen ist. Die Bahnen von $p \neq \{e_3, -e_3\}$ sind Breitenkreise mit Stabilisator $G_p = \operatorname{diag}(1,1,1)$, also ist $G/G_p \simeq G \simeq \mathrm{SO}(2) \simeq S^1$. Mit $G_{\pm e_3} = G$ gilt $G/G_{\pm e_3} = \{\text{Punkt}\}$. Der Bahnenraum X/G ist isomorph zu $[0, \pi]$; G ist also nicht transitiv. Die Gruppe muss eine genügend große Dimension haben, damit sie überhaupt transitiv sein kann.

2b) $X = S^2$, $G = \mathrm{SO}(3)$

Der Orbit eines jeden Punktes von S^2 ist ganz S^2. Damit ist SO(3) transitiv (*) und S^2 is homogen bezüglich SO(3). Der Stabilisator G_{e_3} sind gerade Drehungen um die x_3-Achse, also

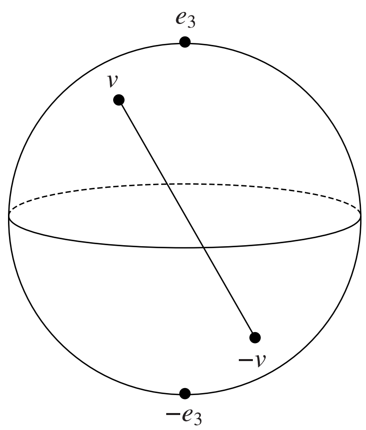
$$\left\{ \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \middle| A \in \mathrm{SO}(2) \right\} \simeq \mathrm{SO}(2)\,. \tag{1.4}$$

Nach dem Lemma gilt dann $S^2 \simeq G/G_{e_3} = \mathrm{SO}(3)/\mathrm{SO}(2)$. Für (*) ist zu zeigen, dass für $v \in S^2$ ein $g \in \mathrm{SO}(3)$ existiert, so dass $g \cdot e_3 = v$. Dazu ergänze $v := v_3$ zur Orthonormalbasis v_1, v_2 und b_3 (mit $\det(v_1|v_2|v_3) = \det(g \in \mathrm{SO}(3)) = 1$). Dann ist $ge_3 = v_3 = v$. □

Analog: $\mathrm{SO}(n+1) = \{A \in \mathrm{Gl}(n+1, \mathbb{R}) | AA^\intercal = E, \det(A) = 1\}$ operiert transitiv auf $S^n = \{x \in \mathbb{R}^{n+1} | \|x\| = 1\}$. Damit gilt $S^n \simeq \mathrm{SO}(n+1)/\mathrm{SO}(n)$.

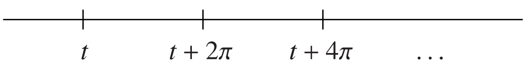
3) $X = S^n$, $G = \{\pm E\} \simeq \mathbb{Z}_2$

ϕ: $\mathbb{Z}_2 \times S^n \mapsto S^n$; $E \cdot v = v$, $-E \cdot v = -v$. Jede Bahn besteht aus zwei Punkten: einem Punkt v und dem Antipodenpunkt $-v$.



Damit ist $G \cdot v = \{v, -v\}$ und der Bahnenraum ist $S^n/\mathbb{Z}_2 = S^n/\sim = P^n\mathbb{R}$, also der n-dimensionale reelle projektive Raum.

4) $X = \mathbb{R}$, $G = \mathbb{Z}$ mit $\phi(k, t) := t + 2\pi k$



Es ist $X/\sim = \mathbb{R}/\mathbb{Z} \simeq S^1$. Analog gilt dies für $X = \mathbb{R}^n$, $G = \mathbb{Z}^n$ und $\phi((k_1, \ldots, k_n), (t_1, \ldots, t_n)) = (t_1 + 2\pi k_1, \ldots, t_n + 2\pi k_n)$. Der Bahnenraum ist dann $\mathbb{R}^n/\mathbb{Z}^n \simeq T^n$, also der n-dimensionale Torus.

5) Hopf-Faserung:

   Wir betrachten $X = S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \simeq \mathbb{R}^4 | |z_1|^2 + |z_2|^2 = 1\}$ und $G = (\mathbb{R}, +)$. Die Gruppenoperation lautet $\phi(t, (z_1, z_2)) := \exp(it); (z_1, z_2) = (\exp(it)z_1, \exp(it)z_2)$. Die Bahn von (z_1, z_2) ist isomorph zu S^1 und der Bahnenraum ist $S^3/\mathbb{R} \simeq S^2$ $(= \mathbb{C} \cup \{\infty\})$. Das Beispiel wird in den Übungen ausführlich nachvollzogen!

6) $X = \mathrm{Pos}(n)$ = Menge der $(n \times n)$-positiv definiten symmetrischen Matrizen mit Determinante 1.

   A ist positiv definit genau dann, wenn ${}^{\top}xAx = \langle Ax, x\rangle > 0$ für $x \neq 0$. Dies ist genau dann der Fall, sofern alle Eigenwerte von A positiv sind. Wir betrachten die Gruppe $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) | \det(A) = 1\}$ mit der Gruppenoperation ϕ: $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{Pos}(n) \mapsto \mathrm{Pos}(n)$; $(A, P) \mapsto {}^{\top}APA = A \cdot P$. Noch zu zeigen ist, dass ϕ wohldefiniert ist, also $A \cdot P \in \mathrm{Pos}(n)$.

   – Symmetrie: ${}^{\top}({}^{\top}APA) = {}^{\top}APA$
   – Positive Definitheit: Es existiert eine Orthonormalbasis $\{u_1, \ldots, u_n\}$ mit $Pu_i = \lambda_i u_i$ für $\lambda_i > 0$. Für $x \in \mathbb{R}^n$ und $Ax = \sum_i a_i u_i$ gilt

$$\langle {}^{\top}APAx, x\rangle = \langle PAx, Ax\rangle = \sum_{i,j} a_i u_j \langle Pu_i, u_j\rangle = \sum_{i,j} a_i u_j \lambda_i \delta_{ij} = \sum_i a_i^2 \lambda_i > 0\,. \tag{1.5}$$

   Da $\mathrm{SL}(n, \mathbb{R})$ transitiv operiert, ist $\mathrm{SL}(n, \mathbb{R}) \cdot E = \mathrm{Pos}(n)$, also $\mathrm{Pos}(n) = \mathrm{SL}(n, \mathbb{R})/\text{Stabilisator von } E$. Der Stabilisator der Einheitsmatrix ist $\{A \in \mathrm{SL}(n, \mathbb{R}) | A \cdot E = E\} = \mathrm{SO}(n)$. Somit ist $\mathrm{Pos}(n) \simeq \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)$. $\mathrm{Pos}(n)$ is homogen bezüglich $\mathrm{SL}(n, \mathbb{R})$.

   $\mathrm{SL}(n, \mathbb{R})$ operiert auf $\mathrm{Pos}(n)$: $(A, P) \mapsto A \cdot P = {}^{\top}APA$. **Behauptung:** Diese Operation ist transitiv, also ist die Bahn der Einheitsmatrix E ganz $\mathrm{Pos}(n)$. Dazu sei $B \in \mathrm{Pos}(n)$. Zu zeigen ist, dass ein $A \in \mathrm{SL}(n, \mathbb{R})$ existiert mit ${}^{\top}AEA = {}^{\top}AA = B$. B ist diagonalisierbar, also existiert $S \in \mathrm{SO}(n)$ mit $B = S^{-1}DS = {}^{\top}SDS$. Da B symmetrisch ist, ist sie auch diagonalisierbar, also existiert eine Matrix $S \in \mathrm{SO}(n)$ mit $B = S^{-1}DS = {}^{\top}SDS$. Da B zusätzlich positiv definit ist, gilt für die Matrix D: $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ mit $\lambda_i > 0$. Wir definieren $\sqrt{D} := \mathrm{diag}(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n})$. Dann ist $B = {}^{\top}S\sqrt{D} \cdot \sqrt{D}S = {}^{\top}(\sqrt{D}S)\sqrt{D}S = (\sqrt{D}S)E$ mit $\sqrt{D}S =: A$. Weiterhin gilt $\det(A) = \det(\sqrt{D}) \cdot \det(S) = 1$. **Folgerung:**

$$\mathrm{Pos}(n) = \mathrm{SL}(n, \mathbb{R}) \cdot E \simeq \mathrm{SL}(n, \mathbb{R})/\mathrm{SO}(n)\,. \tag{1.6}$$

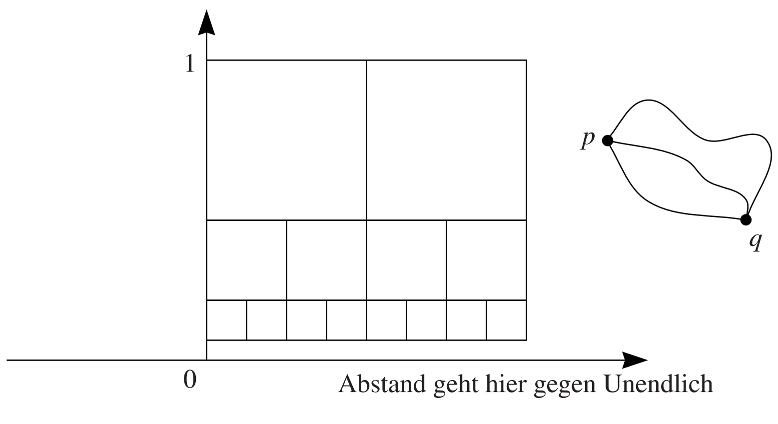
   Die Menge der positiv symmetrischen Matrizen ist also isomorph zu einem homogenen Raum.

1.3 Der Fall Pos(2) als Prototyp: die Poincaré-Halbebene

Sei $H^2\mathbb{R} := \{(x, y) \in \mathbb{R}^2 | y > 0\} = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$ mit der Riemannschen Metrik

$$g|_{(x,y)} = \begin{pmatrix} 1/y^2 & 0 \\ 0 & 1/y^2 \end{pmatrix}, \tag{1.7}$$

also ist $g|_{(x,y)} = 1/y^2 \cdot (\text{euklidische Metrik})|_{(x,y)}$. Bewegt man sich in der Nähe von $y = 1$ ist die Geometrie als näherungsweise euklidisch.



Die hyperbolische Abstandsfunktion (Längenmetrik) ist $d_h(p, q) = \inf(L_h(c))$, wobei c eine stückweise differenzierbare Kurve zwischen p und q ist (vergleiche Vorlesung zur Riemannschen Geometrie).

Lemma 2:

(H^2, d_h) ist ein metrischer Raum (siehe Vorlesung Riemannsche Geometrie).

Bemerkung:

Es existierten explizite Formeln für d_h. Zum Beispiel gilt

$$\cosh(d_h(z,w)) = 1 + \frac{|z-w|^2}{2\mathrm{Im}(z)\mathrm{Im}(w)}\,, \tag{1.8}$$

wobei es viele weitere Darstellungen für diese Metrikfunktion gibt.

Lemma 3:

$\mathrm{SL}(2,\mathbb{R})$ operiert auf $H^2\mathbb{R}$ durch Möbiustransformationen ϕ: $\mathrm{SL}(2,\mathbb{R}) \times H^2\mathbb{R} \mapsto H^2\mathbb{R}$ mit

$$\left(A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z\right) \mapsto Az = \frac{az+b}{cz+d} \equiv T_A(z)\,. \tag{1.9}$$

Beweis-Skizze:

$$\mathrm{Im}(T_A(z)) \overset{\det(A)=1}{=} \frac{\mathrm{Im}(z)}{|cz+d|^2} > 0\,. \tag{1.10}$$

Daraus folgt $T_A(H^2\mathbb{R}) \subseteq H^2\mathbb{R}$. Weiter gilt (Rechnung!):

$$T_E(z) = z \text{ für alle } z \text{ mit } E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.11}$$

und $T_{AB}(z) = T_A(T_B(z))$. Insbesondere ist ϕ eine Operation mit $(T_A)^{-1} = T_{A^{-1}}$.

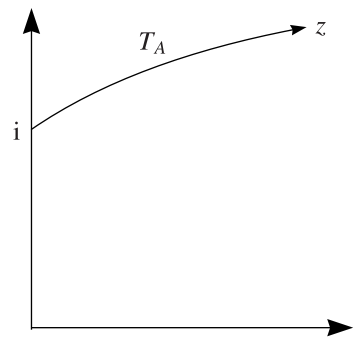
Satz 1:

a) $\mathrm{SL}(2,\mathbb{R})$ operiert transitiv auf $H^2\mathbb{R}$, also ist $H^2\mathbb{R}$ ein homogener Raum bezüglich $\mathrm{SL}(2,\mathbb{R})$ und es gilt $H^2\mathbb{R} \simeq \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \simeq \mathrm{Pos}(2)$.

b) Die Gruppe $\mathrm{SL}(2,\mathbb{R})$ operiert durch Isometrien auf $H^2\mathbb{R}$. Es gilt also $d_h(T_A(z), T_A(w)) = d_h(z,w)$ für alle $A \in \mathrm{SL}(2,\mathbb{R})$ und z, $w \in H^2\mathbb{R}$.

Beweis:

a) Sei $z = x + \mathrm{i}y \in H^2\mathbb{R}$. Zu zeigen ist, dass ein $A \in \mathrm{SL}(2,\mathbb{R})$ existiert mit $T_A(\mathrm{i}) = z$.



Setze

$$A_1 := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad A_2 := \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})\,. \tag{1.12}$$

Dann ist

$$T_A(\mathrm{i}) := T_{A_1 A_2}(\mathrm{i}) = T_{A_1}(y\mathrm{i}) = x + \mathrm{i}y = z\,. \tag{1.13}$$

Damit ist $H^2\mathbb{R} = \mathrm{SL}(2,\mathbb{R}) \cdot \mathrm{i}$. Der Stabilisator von i ist $\{A \in \mathrm{SL}(2,\mathbb{R}) | T_A(\mathrm{i}) = \mathrm{i}\}$, also

$$\frac{a\mathrm{i}+b}{c\mathrm{i}+d} = \mathrm{i} \Leftrightarrow a\mathrm{i}+b = -c+\mathrm{i}d \Leftrightarrow b = -c,\, a = d\,. \tag{1.14}$$

Da dann $a^2 + b^2 = ad - bc = 1$ ist, setze $a = \cos\theta$, $b = \sin\theta$ und somit ist

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{1.15}$$

also $A \in \mathrm{SO}(2)$.

b) Nach Definition der Metrik genügt es zu zeigen, dass für eine stückweise differenzierbare Kurve $t \mapsto z(t)$ in $H^2\mathbb{R}$ gilt: $L_h(T_A(z(t))) = L_h(z(t))$. Schreibe $z(t) = x(t) + \mathrm{i}y(t)$ und

$$w(t) := T_A(z(t)) = \frac{az(t)+b}{cz(t)+d}\,. \tag{1.16}$$

Es gilt dann

$$w'(t) = \frac{\mathrm{d}w}{\mathrm{d}t} = \frac{\mathrm{d}w}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}t} \overset{ad-bc=1}{=} \frac{z'(t)}{(cz(t)+d)^2}\,. \tag{1.17}$$

Weiterhin gilt

$$\mathrm{Im}(w(t)) = \frac{\mathrm{Im}(z(t))}{|cz+d|^2}, \tag{1.18}$$

ebenso nach Lemma 2. Somit ist

$$\|w'(t)\|_{\mathrm{hyp}} = \frac{\|w'(t)\|_{\mathrm{euk}}}{\mathrm{Im}(w(t))} = \frac{\|z'(t)\|_{\mathrm{euk}}}{\mathrm{Im}(z(t))} = \|z'(t)\|_{\mathrm{hyp}}\,. \tag{1.19}$$

Durch Integration folgt dann die Behauptung.

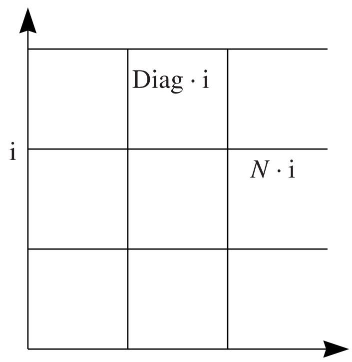
Bemerkungen:

1) Die Riemannsche Metrik auf $H^2\mathbb{R}$ ist gerade so gewählt, dass die Möbius-Transformationen T_A Isometrien sind.

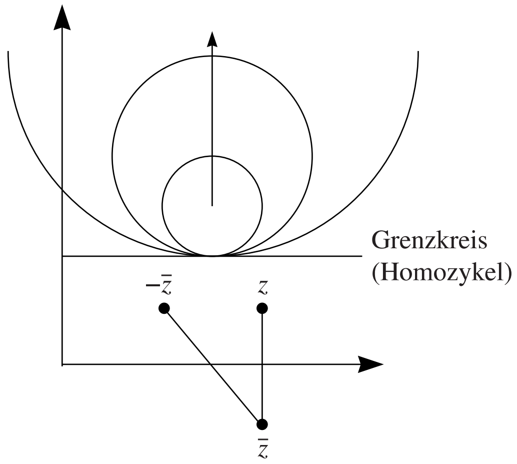
2) Im Beweis von (a) wurde die Iwasawa-Zerlegung von $\mathrm{SL}(2,\mathbb{R})$ verwendet: $\mathrm{SL}(2,\mathbb{R}) = N \cdot \mathrm{Diag} \cdot \mathrm{SO}(2)$. Das heißt $A \in \mathrm{SL}(2,\mathbb{R})$ kann eindeutig geschrieben werden als ein Produkt $A = n \cdot d \cdot k$, wobei n eine obere Dreiecksmatrix, d eine positive Diagonalmatrix mit $\det(d) = 1$ und k eine orthogonale Matrix mit $\det(k) = 1$ ($\in \mathrm{SO}(2)$) ist:

$$n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda > 0\,. \tag{1.20}$$

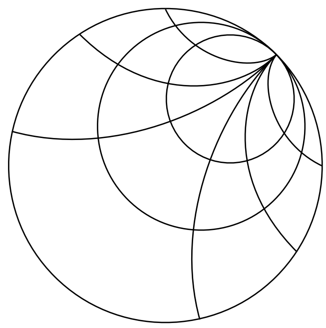
Das „n“ steht für nilpotent. Hinter der Beziehung steckt das Gram-Schmidtsche Orthogonalisierungsvefahren. Die Bahnen bilden das Netz der kartesischen Koordinaten. N bildet die y-Achse als Geodätische auf parallele Geodätische ab. Man bezeichnet diese Koordinaten als homozyklisch.



Die horizontalen Kurven bekommt man so: Man betrachtet einen hyperbolischen Kreis um einen bestimmten Punkt. Anschließend verschiebt man den Mittelpunkt des Kreises in Richtung der positiven imaginären Achse. Infolgedessen werden die Kreise immer größer, was im Grenzfall einer Parallelen zur x-Achse (also einer horizontalen Geraden) entspricht. Diesen „Grenzkreis“ nennt man Homozykel.



Durch eine geeignete Möbiustransformation lässt sich H^2 auf das Innere des Einheitskreies abbilden. Dann sind die Grenzkreise Bouquets und Geodätische stehen orthogonal auf diesen Kreisen. Außerdem laufen die Geodätischen zusammen. Auf diesem Objekt operiert auch eine Gruppe, aber nicht die SL(2,ℝ).



3) **Beachte:** $H^2\mathbb{R} \simeq \text{Pos}(2)$, aber $\text{Pos}(n) \neq H^n\mathbb{R}$ (also der n-dimensionale hyperbolische Raum). Pos(2) besitzt zwar eine konstante negative Krümmung, jedoch handelt es sich bei $\text{Pos}(n)$ für $n \geq 3$ um eine Mannigfaltigkeit mit anderen Krümmungseigenschaften. Beispielsweise gibt es viele Untermannigfaltigkeiten von $\text{Pos}(n)$ für $n \geq 3$, die flach sind.

4) Es ist $T_A = T_{-A}$. Mit $A \in \text{SL}(2,\mathbb{R})$ ist ebenso $-A \in \text{SL}(2,\mathbb{R})$; außerdem ist die Wirkung von T_{-A} auf $H^2\mathbb{R}$ analog zur der Wirkung von T_A (Kürzung des Nenners). Man definiert $\text{PSL}(2,\mathbb{R}) := \text{SL}(2,\mathbb{R})/\{\pm E\}$, die sogenannte projektive spezielle lineare Gruppe. Diese operiert **effektiv** auf $H^2\mathbb{R}$. Dies bedeutet, dass nur die Identität trivial operiert: Aus $\phi_g(x) = x$ für alle x folgt $g = e$.

5) Nach Satz 1 ist $\text{PSL}(2,\mathbb{R}) \subset \text{Iso}(H^2\mathbb{R})$. Die volle Isometriegruppe von $H^2\mathbb{R}$ ist $\text{Iso}(H^2\mathbb{R}) = \text{PSL}(2,\mathbb{R}) \cup \sigma \cdot \text{PSL}(2,\mathbb{R})$, wobei σ die Spiegelung an der imaginären Achse ist: $\sigma(z) := -\bar{z}$. $\text{PSL}(2,\mathbb{R})$ sind die orientierungserhaltenden Isometren und $\sigma \cdot \text{PSL}(2,\mathbb{R})$ die orientierungsumkehrenden Isometrien.

6) Historisch: Es war von der projektiven Geometrie her bekannt, dass Möbiustransformationen auf der oberen Halbebene operieren und suchte deshalb nach einer entsprechenden Metrik in Gl. (1.7), um den Vorfaktor $1/(cz(t)+d)$ loszuwerden.

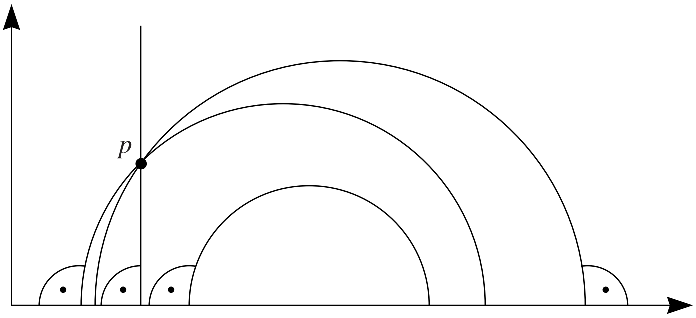
Bemerkung: Da die Vorlesung nach der ersten Woche internationalen Zuwachs bekam, geht die Vorlesung im Folgenden auf Englisch weiter ;-)

1.4 Geodesics

Now we are coming to geodesics: A smooth curve (parameterized by arc length c: $I \mapsto H^2\mathbb{R}$ is a **geodesic** if and only if c is locally a shortest curve between any two of its points, i.e. for $t \in I$ there exists a neighborhood $U(t)$ such that for all t_1, $t_2 \in U$ one has $L_{\text{hyp}}(c|_{[t_1,t_2]}) = |t_1 - t_2| = d_{\text{hyp}}(c(t_1), c(t_2))$.

Theorem 2:

The geodesics in $(H^2\mathbb{R}, d_{\text{hyp}})$ are straight lines parallel to the imaginary axis (parameterized by hyerbolic arc-length) and semi-circles with centre on the real axis (parameterized by hyperbolic arc-length).



This is a model for non-Euclidian geometry, in which the parallel axiom does not hold. The parallel axiom in Euclidian space means that to every straight line and a given point p, which does not lie on that straight line, there exists a unique parallel line through the point.

Idea of proof:

1) Show that the imaginary axis is a geodesic: $c(t) = x(t) + \mathrm{i}y(t)$

$$L_{\text{hyp}}(c) = \int_{t_1}^{t_2} \frac{\sqrt{x'^2 + y'^2}}{y(t)} \,\mathrm{d}t \geq \int_{t_1}^{t_2} \frac{|y'|}{y} \,\mathrm{d}t \geq \int_{t_1}^{t_2} \frac{y'}{y} \,\mathrm{d}t = \ln(t_2) - \ln(t_1) = L_{\text{hyp}}([\mathrm{i}t_1, \mathrm{i}t_2])\,. \tag{1.21}$$

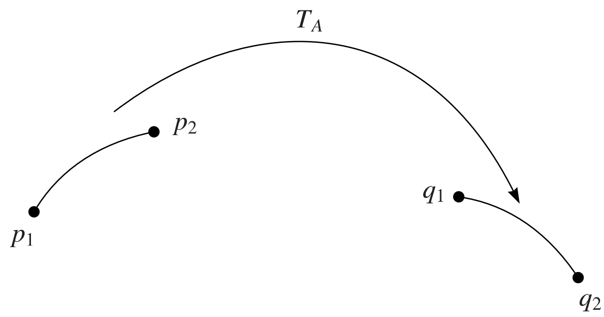
2) Möbius transformations map circles/straight lines to circles/straigh lines (see e.g. Aklfor: "Complex Analysis").

3) Möbius transformations are isometries of $H^2\mathbb{R}$, in particular $T_A(\text{geodesic}) = \text{geodesic}$.

This completes the proof. □

Remark:

The set of geodesics of $H^2\mathbb{R}$ is a homogeneous space for $\mathrm{SL}(2,\mathbb{R})$, i.e. $\mathrm{SL}(2,\mathbb{R})$ acts transitively on a set of geodesics. This in turn implies that $H^2\mathbb{R}$ is **2-point homogeneous**: then the pairs (p_1, p_2), $(q_1, q_2) \in H^2\mathbb{R}$ with $d_{\text{hyp}}(p_1, p_2) = d_{\text{hyp}}(q_1, q_2)$ then there is a Möbius transformation T_A such that $T_A(p_i) = q_i$ for $i = 1$, 2.



A map consists of a translation and a rotation around a fixed point, since these are isometries. (Note: S^2 and $\mathbb{R}^2$ are also 2-point-homogeneous.)

Kapitel 2

Lie groups

2.1 Definition and examples

A Lie group is an (abstract) group with the additional structure of a differentiable manifold, such that multiplication and formation of the inverse are differentiable maps:

$$m : G \times G \mapsto G,\ (g,h) \mapsto gh\,, \quad i : G \mapsto G,\ g \mapsto g^{-1}\,. \tag{2.1}$$

Examples:

0) Finite groups (0-dimensional manifold)
1) Multiplicative group of real numbers $(\mathbb{R} \setminus \{0\}, \cdot)$, additive group of real numbers $(\mathbb{R}, +)$, additive group of $\mathbb{R}$-vectorspace $(V, +)$
2) $S^1 \simeq \mathrm{SO}(2)$, tori $T^n = S^1 \times \ldots \times S^1$, flat tori $T^n \simeq \mathbb{R}^n/\mathbb{Z}^n$

2.2 Matrix groups

- Consider:
  - Real **general linear** group $\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n} | \det(A) \neq 0\}$
  - Complex **general linear** group $\mathrm{GL}(n, \mathbb{C}) = \{A \in \mathbb{C}^{n\times n} | \det(A) \neq 0\}$

  $\mathrm{GL}(n, \mathbb{R})$ is an open submanifold of $\mathbb{R}^{n^2}$. The rule multiplication is given by

  $$(A \cdot B)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}\,, \tag{2.2}$$

  hence m is smooth. Furthermore there is a formula for the inverse:

  $$(A^{-1})_{ij} = (-1)^{i+j} \frac{\det(A(i,j))}{\det(A)}\,, \tag{2.3}$$

  whereas $A(i, j)$ is the $\mathbb{R}^{(n-1)\times(n-1)}$-dimensional submatrix that follows from A by erasing the i-th line and the j-th column. From (2.3) follows the smoothness of i.

- A **fact** is (see later) that closed subgroups of Lie groups are again Lie groups. For example, the **special linear** group $\mathrm{SL}(n, \mathbb{R}) := \{A \in \mathrm{GL}(n, \mathbb{R}) | \det(A) = 1\}$ is closed in $\mathrm{GL}(n, \mathbb{R})$, hence it is a Lie group. The same holds for $\mathrm{SL}(n, \mathbb{C}) = \{A \in \mathrm{GL}(n, \mathbb{C}) | \det(A) = 1\}$.

- The next example is the **orthogonal group** $\mathrm{O}(n) := \{A \in \mathrm{GL}(n, \mathbb{R}) | {}^{\intercal}AA = I_n\}$, where I_n is the n-dimensional identity matrix. If $\langle \bullet, \bullet \rangle$ is the standard scalar product on $\mathbb{R}^n$, hence

  $$\langle a, b\rangle = \sum_{i=1}^{n} a_i b_i\,, \quad a = (a_1, \ldots, a_n) \in \mathbb{R}^n\,, \tag{2.4}$$

  then $\mathrm{O}(n) = \{A \in \mathrm{GL}(n, \mathbb{R}) | \langle Ax, Ay\rangle = \langle x, y\rangle\ \forall\ x,\ y \in \mathbb{R}^n\}$. The **special orthogonal** group is defined by $\mathrm{SO}(n) = \mathrm{O}(n) \cup \mathrm{SL}(n, \mathbb{R})$.

- Define **unitary groups** by $\mathrm{U}(n) := \{A \in \mathrm{GL}(n, \mathbb{C}) | {}^{\intercal}\overline{A}A = I_n\}$. If $\langle \bullet, \bullet \rangle$ is the standard hermitian scalar product on $\mathbb{C}^n$

$$\langle a, b \rangle = \sum_{i=1}^{n} a_i \bar{b}_i \,, \quad a = (a_1, \ldots, a_n) \in \mathbb{C}^n \,, \tag{2.5}$$

  then $\mathrm{U}(n) = \{A \in \mathrm{GL}(n, \mathbb{C}) | \langle Ax, Ay \rangle = \langle x, y \rangle \ \forall\ x,\ y \in \mathbb{C}^n\}$. Analogously, the **special unitary** group is defined by $\mathrm{SU}(n) = \mathrm{U}(n) \cap \mathrm{SL}(n, \mathbb{C})$.

- The **pseudo-orthogonal** groups are defined by $\mathrm{O}(p, q) = \{A \in \mathrm{GL}(p+q, \mathbb{R}) | A$ leaves the quadratic form $(*)$ invariant$\}$, whereas $(*)$ is given by

$$-x_1^2 - x_2^2 - \ldots - x_p^2 + x_{p+1}^2 + \ldots + x_{p+q}^2 \,. \tag{2.6}$$

  Another possibility to define that is

$$\begin{aligned} \mathrm{O}(p, q) = \{A \in \mathrm{GL}(p+q, \mathbb{R}) | {}^{\intercal}A\mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)A = \\ = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)\} \,. \end{aligned} \tag{2.7}$$

- The **Lorentz group** is defined by $\mathrm{SO}(p, q) := \mathrm{O}(p, q) \cap \mathrm{SL}(p+q, \mathbb{R})$, whereas $\mathrm{O}(1, 3)$ plays a crucial role in special relativity. Analogously, $U(p, q)$ are matrices in $\mathrm{GL}(p+q, \mathbb{C})$, which leave

$$-z_1\overline{z}_1 - \ldots - z_p\overline{z}_p + z_{p+1}\overline{z}_{p+1} + \ldots + z_{p+q}\overline{z}_{p+q} \,, \tag{2.8}$$

  invariant, hence

$$\mathrm{U}(p, q) = \{A \in \mathrm{GL}(p+q, \mathbb{C}) | {}^{\intercal}\overline{A}I_{p,q}A = I_{p,q}\} \,. \tag{2.9}$$

- **Symplectic groups**:

  Use

$$\mathcal{I}_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \,, \tag{2.10}$$

  and define

$$\mathrm{Sp}(n, \mathbb{R}) = \{A \in \mathrm{GL}(2n, \mathbb{R}) | {}^{\intercal}A\mathcal{I}_nA = \mathcal{I}_n\} \,, \tag{2.11}$$

$$\mathrm{Sp}(n, \mathbb{C}) = \{A \in \mathrm{GL}(2n, \mathbb{C}) | {}^{\intercal}A\mathcal{I}_nA = \mathcal{I}_n\} \,, \tag{2.12}$$

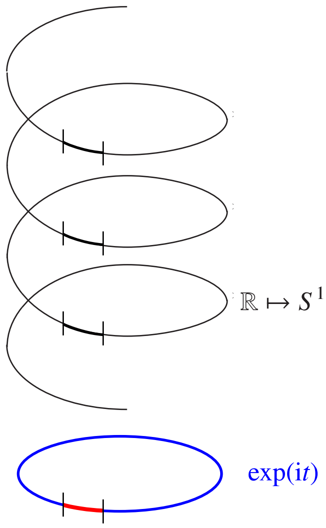
  and

$$\mathrm{Sp}(n) = \mathrm{Sp}(n, \mathbb{C}) \cap \mathrm{U}(2n) \,, \tag{2.13}$$

  whereas the latter is the compact symmetric group. These groups play a role in classical Hamiltonian mechanics.

2.3 Constructions of new Lie groups out of given ones

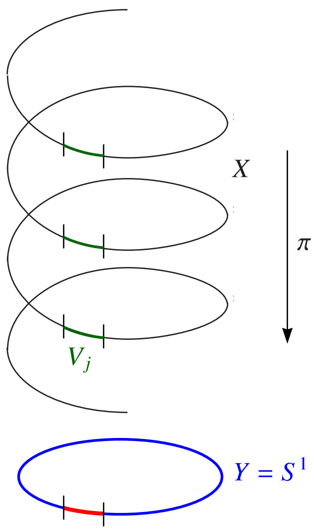
- Product of Lie groups is again a Lie group: examples are $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$, $S^1 \times S^1 = T^2$ or $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$, which is the 12-dimensional gauge group of the standard model of particle physics.
- Closed subgroups of Lie groups are Lie groups. For example $\mathrm{O}(n) \subset \mathrm{GL}(n, \mathbb{R})$.
- The universal covering of a connected Lie group is a Lie group: $R \xrightarrow{\pi} S^1$, $t \mapsto \exp(\mathrm{i}t)$ covering



$$\mathrm{U}(1) \simeq S^1 \simeq \mathrm{SO}(2)\,, \quad S^3 \simeq \mathrm{SU}(2)\,. \tag{2.14}$$

Definition:

We would like to define the **covering space** for Hausdorff topological spaces X and Y: π: $X \mapsto Y$ (surjective and continuous) is a covering, if and only if for every point $y \in Y$ there is a neighborhood $U = U(y)$ open in Y such that $\pi^{-1}(U)$ is a disjoint union $\bigcup_{j \in I} U_j$ with $\pi|_{V_j} \simeq U$ (homeomorphic to U).



Remarks:

- There exists a theorem from topology: Given a topological Hausdorff space Y, there exists a simply connected covering space $\widetilde{Y}$ (i.e. every loop in $\widetilde{Y}$ is homotopic to a point).
- The universal covering space of a connected Lie group is again a Lie group (without proof). Examples are
  - $(\mathbb{R}, +)$ is the universal covering group of $S^1 = \mathbb{R}/\mathbb{Z} \simeq \mathrm{U}(1) \simeq \mathrm{SO}(2)$.
  - Spin group $\mathrm{Sp}(n) = \mathrm{SO}(2n)$
    The pre-image of the point exists of two copies, hence this is a two-fold covering $\pi^{-1}(A) = \{A_+, A_-\}$.
  - The identification of antipodal points on the sphere is a two-fold covering.

2.4 Some isomorphisms between low-dimensional Lie groups

- $\mathrm{SO}(2) \simeq \mathrm{U}(1) \simeq S^1$

This is an isomorphism which follows from the addition theorem for trigonometric functions and for multiplication of complex exponentials.

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto \exp(\mathrm{i}\theta)\,. \tag{2.15}$$

- $\mathrm{SU}(2) \simeq S^3 \simeq \mathrm{Sp}(3)$

  The three-sphere S^3 is a group, which is simply connected. Furthermore it is the group of unit quaternions. The Hamiltonian quaternions are defined on the four-dimensional real vectorspace

  $$\mathbb{H} = \{q = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} | t, x, y, z \in \mathbb{R}\}\,, \tag{2.16}$$

  with basis 1, i, j, k and the multiplication rules $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1$, $\mathrm{ij} = \mathrm{k} = -\mathrm{ji}$. There exists a bijection to a matrix group as follows:

  $$q = t + x\mathrm{i} + y\mathrm{j} + z\mathrm{k} \mapsto A = \begin{pmatrix} v_1 & v_2 \\ -\overline{v}_2 & \overline{v}_1 \end{pmatrix} \in \mathbb{C}^{2\times 2}\,, \tag{2.17}$$

  where $v_1 := t + x\mathrm{i} \in \mathbb{C}$ and $v_2 := y + z\mathrm{i} \in \mathbb{C}$. The standard scalarproduct on $\mathbb{R}^4$ (or $\mathbb{H}$) corresponds (under this bijection) to $\langle A_1, A_2\rangle := \mathrm{Tr}(A_1\overline{A}_2^{\mathsf{T}})/2$. In particular $\|A\|^2 = \langle A, A\rangle = \det(A)$. One defines **unit quaternions** by $\{q \in \mathbb{H} | |q| = 1\}$. Because of $\overline{q} := t - \mathrm{i}x - \mathrm{j}y - \mathrm{k}z$ one obtains $q\overline{q} = |q|^2 = t^2 + x^2 + y^2 + z^2$ and hence the group of unit quaternions is isomorphic to S^3. As a result of that, S^3 is a group with $\mathbb{H}$-multiplication and therefore a Lie group. Under the bijection unit-quaternions correspond to

  $$\left\{ A = \begin{pmatrix} v_1 & v_2 \\ -\overline{v}_2 & \overline{v}_1 \end{pmatrix} \in \mathbb{C}^{2\times 2} \middle| \det(A) = 1 \right\} = \left\{ A \in \mathrm{GL}(2,\mathbb{C}) | A\overline{A}^{\mathsf{T}} = I_2, \det(A) = 1 \right\} = \mathrm{SU}(2)\,. \tag{2.18}$$

  Furthermore, define the zero-sphere by $S^0 = \{x \in \mathbb{R}^1 | |x| = 1\} = \{\pm 1\} = \mathbb{Z}_2$. Note then that there exists a theorem by Hopf which tells us that only S^0, S^1, and S^3 are equipped with Lie group structure.

2.5 A non-linear Lie group

The observation here is that not all Lie groups are groups of matrices. An example for that is the groups

$$N := \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x,\, y,\, z \in \mathbb{R} \right\}\,, \tag{2.19}$$

$$H := \left\{ \begin{pmatrix} 1 & 0 & m \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| m \in \mathbb{Z} \right\}\,, \tag{2.20}$$

whereas the latter is a subgroup of the center of N, which is given by

$$\mathrm{Center}(N) = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbb{R} \right\}\,. \tag{2.21}$$

H is a normal subgroup and the factor group N/H is a Lie group (the fact that N/H is a manifold will be shown later), but is is not a matrix group (i.e. N/H is not isomorphic to a subgroup of some $\mathrm{GL}(n,\mathbb{C})$). The key to this is the following lemma:

<u>**Lemma:**</u>

Let G be a subgroup of $\mathrm{GL}(n,\mathbb{C})$ and $p > 1$ a prime number. **Assume** that G contains special elements S and T, whose commutator $R := S^{-1}T^{-1}ST$ is of order p (i.e. $R^p = I$, $R^k \neq I$ if $1 < k < p$) such that $SR = RS$ and $TR = RT$. **Then** $n \geq p$.

We would like to apply the lemma to N/H. Assume that N/H is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{C})$ for some n. Let π: $N \mapsto N/H$, $n \mapsto nH = \pi(n)$ be the canonical projection. Now let p be an arbitrary prime number. Let S (and T, respectively) be the image in $\mathrm{GL}(n,\mathbb{C})$ under the above isomorphism of

$$\pi\left\{ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}\,, \quad \pi\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/p \\ 0 & 0 & 1 \end{pmatrix} \right\} \in N/H\,. \tag{2.22}$$

Then $R = S^{-1}T^{-1}ST$ is the image in $\mathrm{GL}(n, \mathbb{C})$ of

$$\pi \left\{ \begin{pmatrix} 1 & 0 & 1/p \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} . \tag{2.23}$$

R is of order p. Then by the Lemma we have $n \geq p$, which is a contradiction to the fact that p is arbitrary and n fixed. □

2.6 Lie algebra and exponential maps

Be G a Lie group with $g \in G$. The **left-multiplication** by g is the map L_g: $G \mapsto G$, $h \mapsto gh = L_g(h)$. L_g is a diffeomorphism, since it is differentiable and there exists a differentiable inverse $(L_g)^{-1} = L_{g^{-1}}$. Since $L_{gh} = L_g \circ L_h$, $L_e = \mathrm{id}$ left-multiplication defines a transitiv action of G on itself. Analogously one defines **right-multiplication** by R_g: $G \mapsto G$, $h \mapsto hg$. A vectorfield $X \in \mathcal{V}G$ is **left-invariant**, if fore g, $h \in G$ one has

$$X(gh) = X(L_g(h)) = \mathrm{d}L_g|_h(X(h)) , \quad \mathrm{d}L_g|_h : T_hG \mapsto T_{gh}G . \tag{2.24}$$

Let $\mathfrak{g} \equiv \mathrm{Lie}(G)$ be the set of all left-invariant vector fields on G ($\subset \mathcal{V}G$). We have seen that $(\mathcal{V}G, [\bullet, \bullet])$ is a Lie algebra, whereas $[X, Y](\varphi) := XY\varphi - YX\varphi$ with $\varphi \in C^\infty G$. $[\bullet, \bullet]$ is bilinear, $[X, Y] = -[Y, X]$ and it fulfills the Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. In general, the Lie algebra $(\mathcal{V}G, [\bullet, \bullet])$ is of infinite dimension.

Lemma:

The set $\mathfrak{g}$ of all left-invariant vector fields on a Lie group G is an ***n*-dimensional** ($n = \dim(G)$) Lie subalgebra of $\mathcal{V}G$, i.e.

1) $\mathfrak{g}$ is a vector subspace of $\mathcal{V}G$.

2) $\mathfrak{g}$ is closed under bracketing: for X, $Y \in \mathfrak{g}$ follows $[X, Y] \in \mathfrak{g}$.

Proof:

1) $\mathrm{d}L_g(\alpha X + \beta Y) = \alpha \mathrm{d}L_g X + \beta \mathrm{d}L_g Y = \alpha X \circ L_g + \beta Y \circ L_g = (\alpha X + \beta Y) \circ L_g$

2) This can be performed as an exercise.

Lemma:

Let τ: $T_eG \mapsto g$, $v \mapsto X_v$, where X_v is defined by $X_v(g) := \mathrm{d}L_g|_e v$. Then τ is an isomorphism of vector spaces.

Proof:

- X_v is left-invariant: for g, $h \in G$ holds

$$X_v(gh) = \mathrm{d}L_{gh}|_e v = \mathrm{d}L_{Lgh}|_e v \overset{\substack{\text{definition} \\ \text{of differential}}}{=} \mathrm{d}L_g|_h \circ \mathrm{d}L_h|_e = \mathrm{d}L_g|_h(X_v(h)) . \tag{2.25}$$

- τ is surjective: for $X \in g$ let $v := X(e)$. Then $X(g) = \mathrm{d}L_g|_e v = X_v(g)$. Every left-invariant vector field can be written as such an X_v.

- τ is injective: if $v \neq w$ in T_eG, then $X_v(e) \neq X_w(e)$. So $X_v \neq X_w$.

This closes the proof. □

Corollary:

Directly from the lemma follows $\dim(G) = \dim(T_eG) = \dim(g)$, whereas $\dim(g)$ is the dimension of the left-invariant vector space.
Via τ, we can transport the Lie algebra structure from g to the tangent space T_eG. For u, $v \in T_eG$ we define $[u, v] := [\tau(u), \tau(v)](e)$.

Definition:

Let $(V, [\bullet, \bullet]_v)$, $(W, [\bullet, \bullet]_w)$ Lie algebras. A **homomorphism [isomorphism] of Lie algebras** is a vector space homomorphism [isomorphism] $\varphi\colon V \mapsto W$, such that $\varphi([X,Y]_v) = [\varphi(x), \varphi(y)]_w$. This definition makes τ into a Lie algebra isomorphism. So we shall identify g and T_eG and call them the **Lie algebra of *G***.

Theorem:

Let X be left-invariant on the Lie group G. Further, let c be a locally defined integral curve through e. Then

1) there exists an open interval $I \subset \mathbb{R}$, $0 \in I$ such that $c(0) = e$ and $c(s+t) = c(s)c(t)$ for all s, t, $s+t \in I$.
2) If $c\colon I \mapsto G$ is the maximally defined integral curve of X with $c(0) = e$, then $I = \mathbb{R}$.

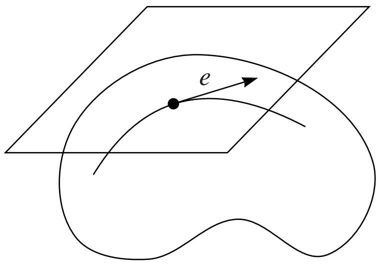
Proof:

1) For an integral curve holds $c'(t) = X(c(t))$. $L_h \circ c$ is also an integral curve of X, since for any $h \in G$:
$$(L_hc)'(t) \overset{\substack{\text{chain}\\ \text{rule}}}{=} \mathrm{d}L_h|_{c(t)}c'(t) \overset{\substack{\text{integral}\\ \text{curve}}}{=} \mathrm{d}L_h|_{c(t)}X(c(t)) \overset{\substack{\text{left-}\\ \text{invariance}}}{=} X(hc(t)) = X((L_h \circ c)(t))\,. \tag{2.26}$$
Now, let $c\colon I \mapsto G$ be the locally defined integral curve of X through e. Let $I \subseteq J$ such that $I + I \subseteq J$. For s, $t \in I$ the curves $t \mapsto c(s+t)$ and $t \mapsto c(s)c(t) = (L_{c(s)} \circ c)(t)$ are integral curves, coinciding at $t = 0$. By uniqueness on I, $c(s+t) = c(s)c(t)$ for all s, $t \in I$.
2) Let $I = (\alpha, \beta)$ be maximal and assume $\beta < \infty$. Then for all $0 < \eta < \beta$ the curve $c_1\colon (\alpha + \eta, \beta + \eta) \mapsto G$, $t \mapsto c_1(t) := c(\eta)c(t-\eta) = L_{c(\eta)}(c(t-\eta))$ with $c_1(0) = c(\eta)c(-\eta) = c(0) = e$ is an integral curve (by (1)) of X, such that $c_1(\eta) = c(\eta)$. So we can extend the curve c beyond β by c_1, which is a contradiction to the fact that I is maximal.

A maximally defined integral curve $c\colon \mathbb{R} \mapsto G$ such that $c(s+t) = c(s)c(t)$ is called a **one-parameter subgroup**.

Definition:

Let $\exp\colon T_eG \simeq g \mapsto G$, $X \mapsto c_X(1)$, where $c_X\colon \mathbb{R} \mapsto G$ is the unique integral curve of X with $c_X(0) = e$. This map is called the **exponential map**.



Lemma:

exp is defined on all of T_eG and is a homomorphism on any one-dimensional subspace, that is

i) $\exp((s+t)X) = \exp(sX)\exp(tX)$
ii) $\exp(0) = e$, $\exp(X)^{-1} = \exp(-X)$

for all s, $t \in \mathbb{R}$.

Proof:

We have $c_{\lambda X}(1) = c_X(\lambda)$ fir $\lambda \in \mathbb{R}$ fixed. If c is an integral curve of X, then $\widetilde{c}(s) := c(\lambda s)$ is an integral curve of λX:
$$\widetilde{c}'(s) = \lambda c'(\lambda s) = \lambda X(c(\lambda s)) = (\lambda X)(\widetilde{c}(s))\,, \tag{2.27}$$
hence
$$c_{\lambda X}(1) = \widetilde{c}(1) = c(\lambda) = c_X(\lambda)\,. \tag{2.28}$$
Now the lemma follows from the theorem before:
$$\exp((t+s)X) = c_{(t+s)X}(1) = c_X(t+s) = c_X(t)c_X(s) = c_{tX}(1)c_{sX}(1) = \exp(tX)\exp(sX)\,. \qquad \square \tag{2.29}$$

Example:

Let $G = (\mathbb{R}_{>0}, \bullet)$. Then $T_1 G = (\mathbb{R}, +)$ and $\exp(t) = \mathrm{e}^t$.

Lemma:

exp is a local diffeomorphism at $0 \in g = T_e G$.

Proof:

Consider $\mathrm{d}\exp_0$: $T_0(T_e G) \simeq T_e G \mapsto T_{\exp(0)} G = T_e G$. Let $X \in T_e G$ ($\alpha(t) := tX$ (such that $\alpha(0) = 0$, $\alpha'(0) = X$)). Then

$$\mathrm{d}\exp_0(X) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp(tX) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c_X(t) = X\,, \tag{2.30}$$

hence $\mathrm{d}\exp_0 = \mathrm{id}_{T_e G}$. By the inverse function theorem (for manifolds), exp is a local diffeomorphism $U_0 \mapsto \exp(U_0)$ for some open neighborhood of 0 in $T_e G$. □

Remark:

In general, exp is neither injective nor surjective. A small example is the vector space of skew-symmetric 2×2 matrices $\mathrm{Lie}(\mathrm{O}(2)) = \mathrm{o}(2)$. This set is generated by matrices $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$:

$$\exp\left\{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right\} = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}\,, \tag{2.31}$$

which is not injective because of the periodicity of the involved trigonometric functions. Furthermore, a matrix with $\mathrm{Tr}(0)$ gets mapped to a matrix with $\det = 1$, hence the image is not $\mathrm{O}(2)$, but $\mathrm{SO}(2)$ and therefore it is not surjective.

2.7 Matrix (linear) Lie algebras

We will study the exponential map for $\mathrm{GL}(n,\mathbb{R})$ (or $\mathrm{GL}(n,\mathbb{C})$) and subgroups G of $\mathrm{GL}(n,\mathbb{R})$. Let $M(n,\mathbb{R})$ be the associative algebra of $n \times n$-matrices, which is $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ as a vector space. Then $\mathrm{GL}(n,\mathbb{R}) = \mathbb{R}^{n\times n} \setminus \{A|\det(A) = 0\}$ is an open subset of $\mathbb{R}^{n^2}$. The tangent space of $\mathrm{GL}(n,\mathbb{R})$ at E is $T_E\mathrm{GL}(n,\mathbb{R}) = \mathrm{gl}(n,\mathbb{R}) \simeq T_I\mathbb{R}^{n^2} = M(n,\mathbb{R})$. As $M(n,\mathbb{R})$ is associative with respect to the ordinary matrix multiplication, we can define a Lie bracket on $M(n,\mathbb{R})$ via $[A,B]^* = AB - BA$.

Lemma:

The Lie algebra $(\mathrm{gl}(n,\mathbb{R}), [\bullet,\bullet])$ and $(M(n,\mathbb{R}), [\bullet,\bullet]^*)$ are isomorphic.

Proof:

For $B = (b_{ij}) \in M(n,\mathbb{R})$ we have n^2 coordinate functions, if one uses the identity as a chart: $x_{ij}(B) = b_{ij}$. A vector field X is written as

$$X = \sum_{i,j=1}^{n} v_{ij} \frac{\partial}{\partial x_{ij}}\,, \tag{2.32}$$

whereas $v_{ij} \in C^\infty(G)$ and its action on the coordinate functions is

$$X(x_{kl}) = \sum_{i,j=1}^{n} v_{ij} \frac{\partial}{\partial x_{ij}}(x_{kl}) = \sum_{i,j=1}^{n} v_{ij}\delta_{ik}\delta_{jl} = v_{kl}\,. \tag{2.33}$$

We prove that φ: $\mathrm{gl}(n,\mathbb{R}) \mapsto M(n,\mathbb{R})$, $X \mapsto (X(x_{ij})(E))_{i,j} =: (a_{ij}(X))_{i,j}$ is an isomorphism of Lie algebras.

- Clearly, φ is linear.

- φ is injective: Assume $\varphi(X) = 0$, that is, all $a_{ij}(X) = 0$. Then, by (2.33)

$$X(E) = \sum_{i,j=1} a_{ij}(X) \left. \frac{\partial}{\partial x_{ij}} \right|_{E=0} = 0\,. \tag{2.34}$$

  Because of left-invariance parallel transportation gives $X = 0$.

- φ is surjective: This follows by a dimensional argument. $\sim (M(n,\mathbb{R})) = n^2 = \dim T_E\mathrm{GL}(n,\mathbb{R})$, since they are isomorphic as vector spaces.

- φ is a Lie algebra homomorphism: By definition of the differential one obtains $X(x_{ij})(g) = X(g)x_{ij} = (\mathrm{d}L_g X(e))x_{ij} = X(e)(x_{ij} \circ L_g)$. Now,

$$(x_{ij} \circ L_g)(h) = x_{ij}(gh) \overset{\substack{\text{matrix}\\ \text{product}}}{=} \sum_k x_{ik}(g) \cdot x_{kj}(h)\,. \tag{2.35}$$

  Hence

$$X(x_{ij})(g) = \sum_k x_{ik}(g) X(x_{kj})(E) = \sum_k x_{ik}(g) a_{kj}(X)\,. \tag{2.36}$$

  Finally, application of this leads to

$$(\varphi[X,Y])_{ij} = (XY - YX)(x_{ij})(E) = \sum_k a_{ik}(X)a_{kj}(Y) - a_{ik}(Y)a_{kj}(X) = ([\varphi(X), \varphi(Y)])_{ij}\,, \tag{2.37}$$

  which holds for all i, j and all X, $Y \in \mathrm{gl}(n,\mathbb{R})$. □

2.7.1 The exponential map

Let $\|A\|$ denote the **operator norm**:

$$\|A\| = \sup_{x \in \mathbb{R}^n, \|x\| \le 1} \|Ax\|\,. \tag{2.38}$$

Then $\|A + B\| \le \|A\| + \|B\|$, $\|AB\| \le \|A\| \cdot \|B\|$. Define the **matrix exponential series** as

$$\mathrm{e}:\ M(n,\mathbb{R}) \mapsto \mathrm{GL}(n,\mathbb{R}),\ , \quad A \mapsto \mathrm{e}^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k\,. \tag{2.39}$$

Let us have a look at the properties of this matrix exponential.

- e is a C^∞-map (and even analytic).
- It is well-defined: The series absolutely converges with respect to the norm defined in Eq. (2.38). (Note that $\|A^k\| \le \|A\|^k$, so the partial sums form a Cauchy sequence. In finite-dimensional vector spaces all norms are equivalent, hence one could take any norm.)
- $\mathrm{e}^0 = E$
- If A, $B \in M(n,\mathbb{R})$ commute ($AB = BA$), then $\mathrm{e}^{A+B} = \mathrm{e}^A \mathrm{e}^B$. From this follows in particular $\mathrm{e}^A \mathrm{e}^{-A} = \mathrm{e}^0 = E$. This proves that the image is indeed $\mathrm{GL}(n,\mathbb{R})$, since for every matrix e^A there exists and inverse e^{-A}. Also for s, $t \in \mathbb{R}$, $\mathrm{e}^{(s+t)A} = \mathrm{e}^{sA}\mathrm{e}^{tA}$. This property should remind us on the definition of one-parameter groups and enables us to prove that e coincides with exp. We have a differentiable map $T_E\mathrm{GL}(n,\mathbb{R}) \mapsto \mathrm{GL}(n,\mathbb{R})$ satisfying the same functional equation as the exponential map does. I.e., for all $A \in T_E\mathrm{GL}(n,\mathbb{R})$ both e^{tA} and $\exp(tA)$ are one-parameter subgroups of $\mathrm{GL}(n,\mathbb{R})$ subject to the same initial condition:

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \exp(tA) = A = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left(E + tA + \frac{t^2}{2} A^2 + \dots \right)\,. \tag{2.40}$$

  By the following lemma, the two maps coincide:

$$\boxed{\exp(A) = \mathrm{e}^A\,, \quad \forall\, A \in M(n,\mathbb{R})\,.} \tag{2.41}$$

Lemma:

A one-parameter subgroup c: $\mathbb{R} \mapsto G$ is uniquely determined by $c'(0)$.

Proof:

We have the property of one-parameter subgroups: $c(s+t) = c(s)c(t) = L_{c(s)}c(t)$. By taking the differential of this expression with respect to t, one obtains

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c(s+t) = c'(s) = \mathrm{d}L_{c(s)}c'(0)\,. \tag{2.42}$$

Hence, $c'(s)$ is left-invariant along c and can be extended to a left-invariant vector field $X(g) = \mathrm{d}L_g c'(0)$. Then c is an initial curve for X through e and as such, it is unique (Picard-Lindelöf). □

Remark:

For a matrix Lie group G holds $G \subseteq \mathrm{GL}(n,\mathbb{R})$, $g = \{X \in M(n,\mathbb{R}| \exp(tX) \in G \;\forall\; t \in \mathbb{R}\}$.

Examples:

1) $G = \mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R})| \det(A) = 1\}$

The Lie algebra of this group is given by $\mathrm{sl}(n,\mathbb{R}) = T_E\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{gl}(n,\mathbb{R})|\mathrm{Tr}(A) = 0\}$. This follows from $\det(\mathrm{e}^A) = \mathrm{e}^{\mathrm{Tr}(A)}$. This is obvious for diagonal matrices $A = \mathrm{diag}(\lambda_1,\ldots,\lambda_n)$. Then

$$\mathrm{e}^{tA} = \mathrm{diag}\left(\mathrm{e}^{t\lambda_1,\ldots,\mathrm{e}^{t\lambda_n}}\right) = \prod_{i=1}^{n}\mathrm{e}^{t\lambda_i} = \mathrm{e}^{t\sum_{i=1}^{n}\lambda_i} = \mathrm{e}^{\mathrm{Tr}(A)}\,. \tag{2.43}$$

To sketch a possible proof for non-diagonal matrices, recall that every quadratic matrix A can be transformed into block-diagonal form (Jordan canonical form) with each block being an upper triangle matrix $\mathcal{M}$:

$$\mathcal{M} = \begin{pmatrix} \lambda_1 & 1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{n-1} & 1 \\ 0 & 0 & \ldots & 0 & \lambda_n \end{pmatrix} \equiv \mathcal{A} + \mathcal{B}\,, \tag{2.44a}$$

with

$$\mathcal{A} = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\,, \quad \mathcal{B} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}\,, \tag{2.44b}$$

whereas λ_1, …, λ_n are the eigenvalues of each block matrix. The determinant is invariant with respect to base change. Hence a general quadratic matrix can be brought into the Jordan canonical form without changing the value of the determinant. One single block matrix $\mathcal{M}$ can be written as a sum over a diagonal matrix $\mathcal{A}$ (with only eigenvalues in the diagonal) and a nilpotent upper triangle matrix $\mathcal{B}$, whose coefficients vanish except for the ones in the first upper diagonal. It is sufficient to consider just one of the block matrices. By direkt calculation one obtains

$$\mathrm{e}^{\mathcal{M}} = \det\left\{\begin{pmatrix} \mathrm{e}^{\lambda_1} & 0 & 0 & \ldots & 0 \\ 0 & \mathrm{e}^{\lambda_2} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & \mathrm{e}^{\lambda_{n-1}} & 1 \\ 0 & 0 & \ldots & 0 & \mathrm{e}^{\lambda_n} \end{pmatrix} + \mathcal{B}'\right\}\,, \tag{2.45}$$

whereas $\mathcal{B}'$ is an upper triangle matrix with vanishing diagonal elements, whose explicit form is complicated, but not important, since it does not contribute to the determinant. Hence

$$\det(\mathrm{e}^{\mathcal{M}}) = \prod_{i=1}^{n}\mathrm{e}^{\lambda_i} = \mathrm{e}^{\sum_{i=1}^{n}\lambda_i} = \mathrm{e}^{\mathrm{Tr}(\mathcal{M})}\,, \tag{2.46}$$

which completes the sketch of the proof. □

So $\dim(\mathrm{SL}(n,\mathbb{R})) = \dim(\mathrm{sl}(n,\mathbb{R})) = n^2-1$, since due to the trace condition one element is fixed.

2) $G = \mathrm{O}(n) = \{A | A^\intercal A = E\}$

Here, the Lie algebra is given by $\mathrm{o}(n) = T_I\mathrm{O}(n) = \{A \in \mathrm{gl}(n,\mathbb{R}) | A^\intercal = -A\}$, which is the space of skew-symmetric matrices. To see this, look at the defining curve for the one-parameter subgroup of $\mathrm{O}(n)$, which is given by $A(t)A^\intercal(t) = I$ with $A(0) = I_n$. By differentiating that one with respect to t one obtains the Lie algebra. Then

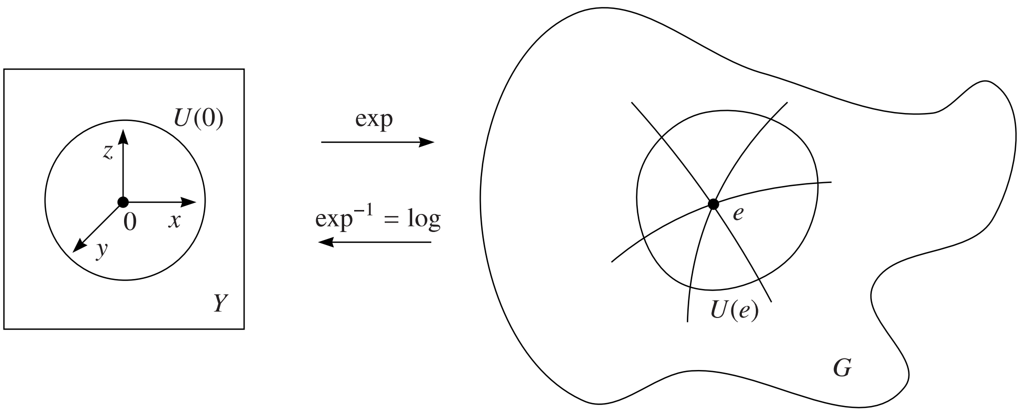
$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} c(t)^\intercal c(t) = (c'(t)^\intercal c(t) + c(t)^\intercal c'(t))|_{t=0} = c'(0)^\intercal + c'(0)\,. \tag{2.47}$$

Hence the tangent vector $c'(0)$ in the Lie algebra has this skew-symmetry and every skew-symmetric matrix corresponds to such a tangent vector. Conversely, let A be skew-symmetric. Define a curve by $c(s) = \mathrm{e}^{sA}$. Then

$$(\mathrm{e}^{sA})^\intercal = \mathrm{e}^{sA^\intercal} \overset{\substack{\text{skew-}\\ \text{symmetry}}}{=} \mathrm{e}^{-sA} = (\mathrm{e}^{sA})^{-1}\,. \tag{2.48}$$

So $A = c'(0) \in T_E\mathrm{O}(n)$. It follows that $\sim (\mathrm{O}(n)) = \dim(\mathrm{o}(n)) = n/2(n-1)$, because a skew-symmetric matrix has only zeros in the diagonal and is determined by the entries in the upper triangle part.

- Similarly by taking the derivative of the one-parameter subgroup of $\mathrm{U}(n)$, hence $A(t)\overline{A}^\intercal(t) = I_n$, $A(0) = I_n$ one obtains the Lie algebra of $\mathrm{U}(n)$: $\mathrm{u}(n) = \{A \in \mathbb{C}^{n\times n} | \overline{A}^\intercal = -A\}$. Furthermore, $\dim(\mathrm{U}(n)) = \dim(\mathrm{u}(n)) = n^2$.
- $\mathrm{su}(n) = \{A \in \mathbb{C}^{n\times n} | \overline{A}^\intercal = -A,\ \mathrm{Tr}(A) = 0\}$, which implies $\dim(\mathrm{su}(n)) = \dim(\mathrm{SU}(n)) = n^2-1$. Here one has to use the one-parameter subgroup $A(t)\overline{A}^\intercal(t) = I_n$ with $A(0) = I_n$ and the additional condition $\det(A(t)) = 1$. For this use $\det(\mathrm{e}^{tA}) = \mathrm{e}^{t\cdot\mathrm{Tr}(A)}$ (exercise).
- $\mathrm{sl}(n,\mathbb{R}) = \{A \in \mathbb{R}^{n\times n} | \mathrm{Tr}(A) = 0\}$ is the Lie algebra of $\mathrm{SL}(n,\mathbb{R})$ with $\dim(\mathrm{sl}(n,\mathbb{R})) = \dim(\mathrm{SL}(n,\mathbb{R})) = n^2-1$.



2.8 The formula of Campbell-Baker-Hausdorff

"In a neighborhood of e the structure of the group G is completely determined by the Lie algebra $\mathfrak{g}$."

Theorem 2:

Let G be a Lie group. Put a norm on the Lie algebra $\mathfrak{g}$ of G (this makes $\mathfrak{g}$ to a Banach space). Let B be the corresponding unit ball $B := \{X \in \mathfrak{g} | \|X\| \leq 1\}$. Then there is some $\varepsilon > 0$ such that for any X, $Y \in B$ there exists a map Z: $(-\varepsilon,\varepsilon) \mapsto \mathfrak{g}$ such that $(\exp(tX))(\exp(tY)) = \exp(Z(t))$ for all $t \in (-\varepsilon,\varepsilon)$, where $Z(t)$ can be written as (an absolutely convergent) power series

$$Z(t) = \sum_{k=1}^{\infty} t^k Z_k(X,Y)\,, \tag{2.49}$$

whereas $Z_k(X,Y)$ is a finite $\mathbb{R}$-linear combination of iterated brackets of X and Y. In particular, $Z_1(X,Y) = X+Y$, $Z_2(X,Y) = [X,Y]/2$, ... In particular, for $\|X\|$, $\|Y\| < \varepsilon$

$$\exp(X)\cdot\exp(Y) = \exp\left(X + Y + \frac{1}{2}[X,Y] + \dots\right)\,. \tag{2.50}$$

Idea of the proof:

The very foundation of the proof is the comparison of Taylor series. Be $\widetilde{X}$ a left-invariant vector field corresponding to $X \in T_eG = \mathfrak{g}$ and $f \in C^\infty G$ a test function. Apply f to the one-parameter subgroup $\exp(tX)$ and write this as a Taylor series:

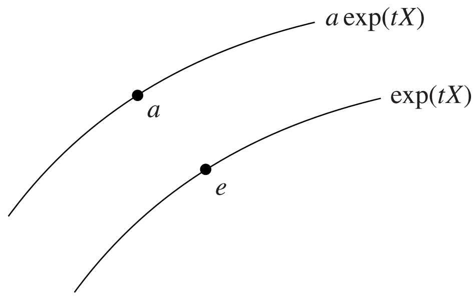
$$f(\exp(tX)) = \sum_{k=0}^{\infty} \frac{t^k}{k!} (\widetilde{X}^k f)(e) \left(= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^k \bigg|_{t=0} f(\exp(tX)) \right) . \tag{2.51}$$

This follows by applying

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_0 f(\exp(tX)) = (\widetilde{X} f)(e) , \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}\bigg|_0 f(\exp(tX)) = (\widetilde{X}(\widetilde{X}(e))(e)) = (\widetilde{X}^2 f)(e) , \tag{2.52}$$

hence more generally for $a \in G$:

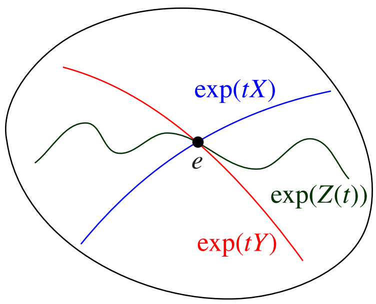
$$f(a \cdot \exp(tX)) = \sum_{k \geq 0} \frac{t^k}{k!} (\widetilde{X}^k f)(a) . \tag{2.53}$$



This implies

$$\begin{aligned} f(\exp(sX) \cdot \exp(tY)) &= \sum_{k \geq 0} \frac{t^k}{k!} (\widetilde{Y}^k f)(\exp(sX)) = \sum_{k \geq 0} \frac{t^k}{k!} \sum_{m \geq 0} \frac{s^m}{m!} \widetilde{X}^m (\widetilde{Y}^k f)(e) = \\ &= \sum_{k,m \geq 0}^{\infty} \frac{t^k}{k!} \frac{s^m}{m!} (\widetilde{X}^m \widetilde{Y}^k f)(e) . \end{aligned} \tag{2.54}$$

Note that the correct order of X and Y, when applied to the test function f, has to be kept in each term.



Since the exponential map is a local diffeomorphism, there exists a $Z(t)$ such that $\exp(tX) \cdot \exp(tY) = \exp(Z(t))$. Write $Z(t) = Z_0 + tZ_1 + tZ_2 + \ldots$ (formal series *Ansatz*). We have $Z_0 = 0$. (To see this, set $t = 0$ and hence $\exp(Z(0)) = \exp(0) \cdot \exp(0) = e \cdot e = e = \exp(Z_0)$. Since exp is a diffeomorphism $Z_0 = 0$.) With the previous formulas one gets

$$f(\exp(Z(t))) = \sum^{\infty} \frac{1}{l!} \left((t\widetilde{Z}_1 + t^2 \widetilde{Z}_2 + \ldots)^l f \right) (e) , \tag{2.55}$$

and $f(\exp(tX)\exp(tY))$ is given by Eq. (2.54) with $s = t$. Compare coefficients of powers of t:

- first power of t: t^1

$$\widetilde{X} f(e) + \widetilde{Y} f(e) = \widetilde{Z}_1 f(e) \overset{\text{definition}}{\Leftrightarrow} Xf + Yf = Z_1 f . \tag{2.56}$$

As f is arbitrary, we have $Z_1 = X + Y$.

- second power of t: t^2

  From Eq. (2.54) follows for $m+k=2$:

$$\frac{1}{2}(\widetilde{X}^2 f)(e) + (\widetilde{X}\widetilde{Y} f)(e) + \frac{1}{2}(\widetilde{Y}^2 f)(e) = \left(\frac{1}{2}\widetilde{Z}_1^2 f + \widetilde{Z}_2 f\right)(e) = \\ = \left(\frac{1}{2}(\widetilde{X}+\widetilde{Y})^2 f\right)(e) + \widetilde{Z}_2 f(e) = \left(\frac{1}{2}(\widetilde{X}^2 + \widetilde{X}\widetilde{Y} + \widetilde{Y}\widetilde{X} + \widetilde{Y}^2) f\right)(e) + \widetilde{Z}_2 f(e)\,. \tag{2.57}$$

  As a result of that

$$\left(\widetilde{X}\widetilde{Y} f - \frac{1}{2}\widetilde{X}\widetilde{Y} f - \frac{1}{2}\widetilde{Y}\widetilde{X} f\right)(e) = (\widetilde{Z}_2 f)(e)\,, \tag{2.58}$$

  and as before one obtain $Z_2 = [X,Y]/2$. □

For a complete proof, see for instance, Price: "Lie groups and compact groups" on page (65).

Corollary 1:

Be X, Y, and t as in theorem 2.

1) conjugation (ghg^{-1}): $\exp(tX)\cdot\exp(tY)\cdot\exp(-tX) = \exp(tY + t^2[X,Y] + \mathcal{O}(t^3))$
2) commutator ($ghg^{-1}h^{-1}$): $\exp(tX)\cdot\exp(tY)\cdot\exp(-tX)\cdot\exp(-tY) = \exp(t^2[X,Y] + \mathcal{O}(t^3))$

   Hence, the commutator of elements in the group corresponds to the commutator of elements in the Lie algebra.

Proof:

1) Application of the Campbell-Baker-Hausdorff formula leads to

$$\exp(tX)\cdot\exp(tY)\cdot\exp(-tX) = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + \mathcal{O}(t^3)\right)\exp(-tX) = \\ = \exp\left(t(X+Y) + \frac{t^2}{2}[X,Y] + \mathcal{O}(t^3) - tX + \frac{t^2}{2}\left[X+Y+\frac{t}{2}[X,Y] + \mathcal{O}(t^2), -X\right]\right) = \\ = \exp\left(tY + \frac{t^2}{2}[X,Y] + \frac{t^2}{2}[X,Y] + \mathcal{O}(t^3)\right)\,. \tag{2.59}$$

2) This works analogously. □

A Lie algebra $\mathfrak{g}$ is called Abelian (commutative), if $[X,Y]=0$ for all X and $Y\in\mathfrak{g}$. An example for that is the set $\mathfrak{a}$ of diagonal matrices in $\mathbb{R}^{n\times n}$ (with $[A,B]^* := AB - BA$). For matrices D_1, $D_2\in\mathfrak{a}$ holds $D_1D_2 = D_2D_1$, hence $[D_1,D_2]^* = 0$.

Corollary 2:

Let G be a **connected** Lie group with Lie algebra $\mathfrak{g}$. Then G is Abelian, if and only if $\mathfrak{g}$ is Abelian.

Proof:

- „⇒“: For X, $Y\in\mathfrak{g}$, $\exp(tX)$ and $\exp(tY)$ are corresponding one-parameter subgroups. Then we can calculate the group commutator. Since G is Abelian, it holds

$$e = \exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y) \overset{\text{CBH}}{=} \exp(t[X,Y] + \mathcal{O}(t^{\frac{3}{2}}))\,, \tag{2.60}$$

  and from this follows $[X,Y]=0$.

- „⇒“: If $\mathfrak{g}$ is Abelian, the Campbell-Baker-Hausdorff-formula implies $xyx^{-1}y^{-1} = e$ for all x, $y\in U(e)$ (for a sufficient small neighborhood of e). Hence $xy = yx$. As G is connected, $U(e)$ generates G (see exercise!) and as a result of that, G is Abelian. □

Remark:

The Lie algebra $\mathfrak{g}$ of G "determines" G only locally (see theorem 4). An example is $G = (\mathbb{R}_{>0,\bullet})$ with $\mathfrak{g} = (\mathbb{R}, +)$ or $G = \mathrm{SO}(2)$ with $\mathfrak{g} = (\mathbb{R}, +)$. Hence, as Lie groups these two are not isomorphic; topologically they differ, since the first one is not compact, whereas the second one is compact.

2.9 Locally isomorphic Lie groups

Theorem 3:

Let ϕ: $G \mapsto H$ be a Lie group homomorphism (i.e. ϕ is differentiable and is an algebra homomorphism, $\phi(g_1, g_2) = \phi(g_1)\phi(g_2)$ for all g_1, $g_2 \in G$). Then

1) $\phi \circ \exp^{(G)} = \exp^{(H)} \circ \mathrm{d}\phi_e$

   In other words, there exists the commutative diagram

2) $\mathrm{d}\phi_e \cdot \mathfrak{g} \mapsto f$ is a Lie algebra homomorphism (i.e. $[\mathrm{d}\phi_e X, \mathrm{d}\phi_e Y]_{\mathfrak{f}} = \mathrm{d}\phi_{\mathfrak{f}}[X, Y]$

Proof:

1) For $X \in \mathfrak{g}$, $c_X(s) = \exp^{(G)} sX$ is a one-parameter subgroup. Since ϕ is a homomorphism, $\phi \circ c_X(s)$ is a one-parameter subgroup in H. Moreover,

$$\mathrm{d}\phi_e X = \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} \phi \circ c_X(s) \Rightarrow \phi \circ c_X(s) \overset{\substack{\text{definition}\\ \text{of exp}}}{=} \exp^{(H)}(s\,\mathrm{d}\phi_e X)\,, \tag{2.61}$$

and with $s = 1$ this leads to

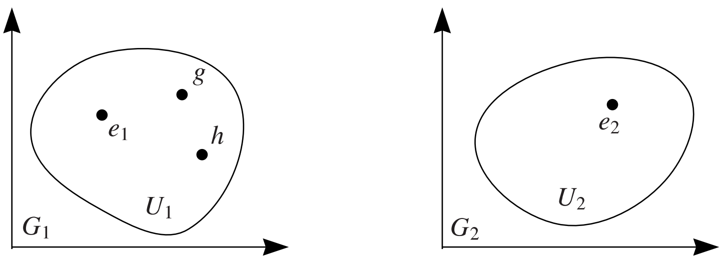
$$\phi(c_X(1)) = \phi(\exp^{(G)} X) = \exp^{(H)}(\mathrm{d}\phi_e X)\,. \tag{2.62}$$

2) $[X, Y]_{\mathfrak{g}}$ corresponds (via the Campbell-Baker-Hausdorff formula) to the tangent vector at 0 of commutator curve $\exp(\sqrt{t}X)\exp(\sqrt{t}Y)\exp(-\sqrt{t}X)\exp(-\sqrt{t}Y)$. ϕ maps this commutator curve to a commutator curve in H, since $\phi(xyx^{-1}y^{-1}) = \phi(x)\phi(y)\phi(x)^{-1}\phi(y)^{-1}$. $\mathrm{d}\phi_e[X, Y]$ is the tangent vector of the image curve, hence

$$\begin{aligned}
&\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi(\exp^{(G)}(\sqrt{t}X))\phi(\exp^{(G)}(\sqrt{t}Y))\phi(\exp^{(G)}(-\sqrt{t}X))\phi(\exp^{(G)}(-\sqrt{t}Y)) \overset{(1)}{=} \\
&\quad = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp^{(H)}(\mathrm{d}\phi_e\sqrt{t}X)\exp^{(H)}(\mathrm{d}\phi_e\sqrt{t}Y)\exp^{(H)}(-\mathrm{d}\phi_e\sqrt{t}X)\exp^{(H)}(-\mathrm{d}\phi_e\sqrt{t}Y) = \\
&\quad \overset{\text{CBH}}{=} \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \exp^{(H)}(t[\mathrm{d}\phi_e X, \mathrm{d}\phi_e Y]_{\mathfrak{f}} + \mathcal{O}(t^{\frac{2}{3}})) = [\mathrm{d}\phi_e X, \mathrm{d}\phi_e Y]_{\mathfrak{f}}\,.
\end{aligned} \tag{2.63}$$

This completes the proof. □

The main result of that is the following: Two Lie groups G_1 and G_2 are **locally isomorphic**, if there are neighborhoods $U_1(e_1)$ und $U_2(e_2)$ and a local diffeomorphism ϕ: $(U_1 \subset G_1) \mapsto (U_2 \subset G_2)$ such that $\phi(gh) = \phi(g)\phi(h)$ for all g, h with $g \cdot h \in U_1$.



Theorem 4:

Two Lie groups G_1, G_2 are locally isomorphic, if and only if their Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ are (Lie algebra) isomorphic.

Proof:

- „⇒“: If ϕ: $(U_1 \subset G_1) \mapsto (U_2 \subset G_2)$ is a local isomorphism. Then by theorem 3 $\mathrm{d}\phi_e$: $\mathfrak{g}_1 \mapsto \mathfrak{g}_2$ is a Lie algebra **isomorphism**.
- „⇐“: Let h: $\mathfrak{g}_1 \mapsto \mathfrak{g}_2$ be a Lie algebra isomorphism. Set ϕ: $\exp_2 \circ h \circ \exp_1^{-1}$. Then ϕ is a local diffeomorphism and $\phi(e_1) = e_2$. To show that ϕ is a homomorphism let x, $y \in U$ such that $xy \in U_1$. Write $x = \exp_1(X)$, $y = \exp_2(Y)$. Then by Campbell-Baker-Hausdorff

$$xy = \exp_1\left(X + Y + \frac{1}{2}[X,Y] + \dots\right) . \tag{2.64}$$

Hence

$$\begin{aligned}\phi(xy) &= \exp_2 \circ h \circ \exp_1^{-1}\left\{\exp_1\left(X + Y + \frac{1}{2}[X,Y] + \dots\right)\right\} = \\ &= \exp_2\left\{h\left(X + Y + \frac{1}{2}[X,Y] + \dots\right)\right\} .\end{aligned} \tag{2.65}$$

Using that h is a Lie algebra homomorphism, leads to

$$\exp_2\left(h(X) + h(Y) + \frac{1}{2}[h(X), h(Y)] + \dots\right) , \tag{2.66}$$

and again with Baker-Campbell-Hausdorff we obtain:

$$\exp_2(h(X))\exp_2(h(Y)) = (\exp_2 \circ h \circ \exp_1^{-1}(x))(\exp_2 \circ h \circ \exp_1^{-1}(y)) = \phi(x)\phi(y) . \qquad \square \tag{2.67}$$

Example:

The Lie algebras $\mathrm{sl}(2,\mathbb{R})$, $\mathrm{so}(2,1)$, and $\mathrm{su}(1,1)$ are isomorphic (exercise!). Hence, by theorem 4 one can conclude that the groups $\mathrm{SL}(2,\mathbb{R})$, $\mathrm{SO}(2,1)$, and $\mathrm{SU}(1,1)$ are locally isomorphic. This is the reason for the existence of different models of the plane hyperbolic geometry.

- SO(2,1) leaves the quadratic form $x^2 + y^2 - z^2$ invariant. Hence, the set $x^2 + y^2 - z^2 = 0$, which describes a cone in $\mathbb{R}^3$ and additionally the two sheets of the hyperboloid $x^2 + y^2 - z^2 = 1$ are left invariant. (If one restricts to the connected component, each part of the hyperboloid is left invariant. For the full group there exists a map which interchanges the two parts.) SO(2,1) models the hyperbolic geometry.
- $\mathrm{SL}(2,\mathbb{R})$ is called Poincaré model or upper half-plane model.
- The Lie group SU(1,1) acts on the unit disk and leaves it invariant (see later chapter). Hence it is called the unit disk model.

2.10 The adjoint representation

In general, a representation is a homomorphism F: $G \mapsto \mathrm{Aut}(V) \simeq \mathrm{GL}(n,\mathbb{C})$, whereas V is an n-dimensional $\mathbb{C}$-vector space. The idea is to get a picture of the group as a group of matrices. This group of matrices is much simpler to understand, since one can use methods from linear algebra. The homomorphism has a kernel, so elements can be mapped on the identity, which leads to some information of the group being lost.
Be G a Lie group, $\mathfrak{g}$ its Lie algebra and $g \in G$. Define the map i_g: $G \mapsto G$, $h \mapsto ghg^{-1} =: i_g(h)$ (conjugation), which is a group isomorphism, hence

$$i_g(h_1, h_2) = i_g(h_1)i_g(h_2) , \quad i_g(e) = e , \quad (i_g)^{-1} = i_{g^{-1}} . \tag{2.68}$$

Since $i_g = L_g \circ R_{g^{-1}}$ is a diffeomorphism, i_g is a **Lie group automorphism**. By theorem 3 the differential of i_g is a Lie algebra automorphism $\mathrm{Ad}(g) := \mathrm{d}i_g|_e : \mathfrak{g} \mapsto \mathfrak{g}(\simeq T_eG)$. We have

$$\mathrm{Ad}(g_1g_2) = \mathrm{d}i_{(g_1g_2)}|_e = \mathrm{d}(i_{g_1} \circ i_{g_2})|_e = \mathrm{d}i_{g_1}|_e \circ \mathrm{d}i_{g_2}|_e = \mathrm{Ad}(g_1) \circ \mathrm{Ad}(g_2) . \tag{2.69}$$

So we have a representation (a homomorphism) Ad: $G \mapsto \mathrm{Aut}(\mathfrak{g})$, $g \mapsto \mathrm{d}i_g|_e$, called the **adjoint representation** of G. With fixed basis, $\mathrm{Aut}(\mathfrak{g}) \subseteq \mathrm{GL}(n,\mathbb{R})$ $(\dim G = n)$. This is a closed subgroup, hence a Lie group with Lie algebra $\mathrm{End}(\mathfrak{g}) \subseteq \mathbb{R}^{n\times n}$. One can again differentiate: $\mathrm{ad} := \mathrm{d}(\mathrm{Ad})_e$: $\mathfrak{g} \simeq T_eG \mapsto \mathrm{End}(\mathfrak{g})$. So take some element of the Lie algebra and associate $\mathrm{ad}(X)$ to it. What is $\mathrm{ad}(X)$ for fixed X, hence $\mathrm{ad}\, X(Y)$? The idea is to use (1) of theorem 3:

$$i_g \circ \exp = \exp \circ \mathrm{Ad}(g)\,, \tag{2.70}$$

for g fixed. By Campbell-Baker-Hausdorff one obtains

$$\begin{aligned}\exp(\mathrm{Ad}(\exp(tX))(tY)) = i_{\exp(tX)}(\exp(tY)) = \exp(tX)\exp(tY)\exp(-tX) = \\ = \exp(tY + t^2[X,Y] + \mathcal{O}(t^3))\,.\end{aligned} \tag{2.71}$$

Since exp is a local diffeomorphism, it follows

$$\mathrm{Ad}(\exp(tX))(tY) = tY + t^2[X,Y] + \mathcal{O}(t^3)\,, \tag{2.72}$$

which implies

$$\mathrm{Ad}(\exp(tX))Y = Y + t[X,Y] + \mathcal{O}(t^2)\,, \tag{2.73}$$

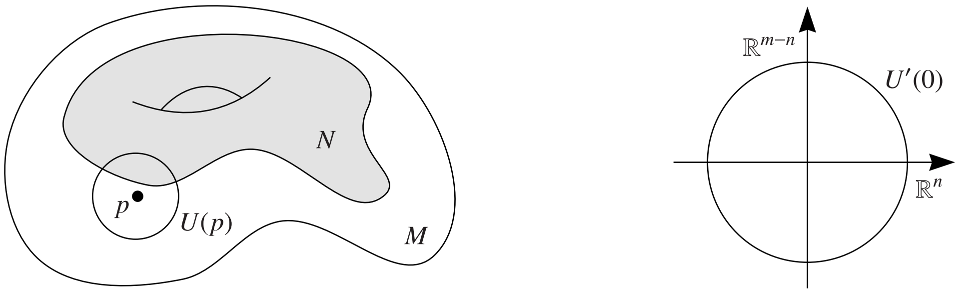
and as a result of this

$$(\mathrm{ad}\, X)(Y) = (\mathrm{d}(\mathrm{Ad})_e(X))(Y) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(tX))(Y) = [X,Y]\,. \tag{2.74}$$

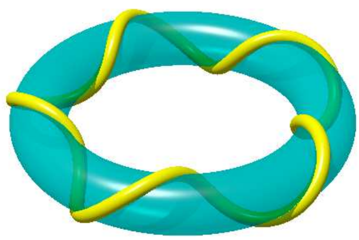
Hence, the second derivative of conjugation gives the bracket.

2.11 Lie subgroups

A n-dimensional subset N of a m-dimensional smooth manifold M is a **n-dimensional submanifold**, if for every point $p \in N$ there is a chart $\varphi\colon U(p) \mapsto U'(0) \subset \mathbb{R}^n \oplus \mathbb{R}^{m-n}$ (of M) such that $\varphi(U(p) \cap N) = U'(0) \cap \mathbb{R}^n \oplus \{0\}$.



Let G be a Lie group. Then an (abstract) subgroup $H \subset G$ is a **Lie subgroup** of G, if H is a submanifold of G. (H is a Lie group, hence multiplication/inverse are smooth restrictions of smooth maps in G). **Note**: In literature a Lie subgroup is sometimes defined as a homomorphism $\phi\colon H \mapsto G$, whereas ϕ is smooth and G and H are Lie groups. ($\phi(H)$ is an (abstract) subgroup but not restricted to a submanifold.) For instance, the image of a one-parameter subgroup is in general not a Lie subgroup in our sense. An example for this is $H := (\mathbb{R}, +)$ and $G = \mathrm{U}(1) \times \mathrm{U}(1)$ ($\simeq \mathrm{SO}(2) \times \mathrm{SO}(2)$) (as a manifold a 2-torus). For $\alpha \in \mathbb{R}$, $\phi_\alpha\colon H \mapsto G$, $t \mapsto (\mathrm{e}^{\mathrm{i}t}, \mathrm{e}^{\mathrm{i}\alpha t})$. For $\alpha \notin \mathbb{Q}$, ϕ_α is injective and $\phi_\alpha(\mathbb{R}) = T^2$. For $\alpha \in \mathbb{Q}$, $\phi_\alpha(\mathbb{R})$ is a submanifold (see Amol'd: Differential equations 3.12), since it is a closed curve on the torus.

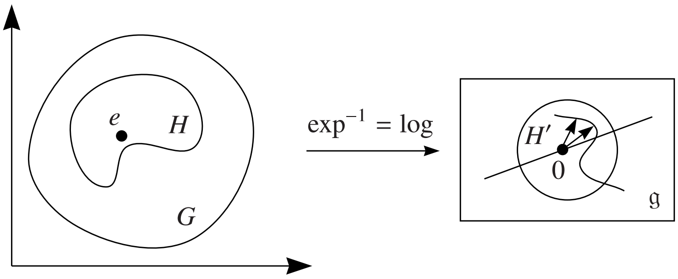


In this case the one-parameter subgroup is a subgroup in our sense, since its image is a closed subgroup. However, for $\alpha \in \mathbb{R}$ the one-parameter subgroup is an injective curve on the torus G and because of that reason it is not closed. In a neighborhood of some point on the torus one has an infinite number of orbits and the winding of the curve gets dense and therefore the image of the curve forms the whole torus G (see also last lecture on Riemannian geometry).

Theorem 5 (E. Cartan):

Let $H \subset G$ be an abstract subgroup of a Lie group G. If H is closed in G, then H is a submanifold of G and in particular a **Lie subgroup**.
Hence, a weak topological condition results in a strong algebraic condition.

Proof:



We equip $\mathfrak{g} = T_eG$ with an (auxiliary) scalar product and consider the chart $\exp^{-1}|_U$ for a sufficiently small neighborhood of e in G.

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\exp} & G \subset H \\ \downarrow \cup & & \downarrow \cup \\ U(0) & \xrightarrow{\exp|_U} & U(e) \end{array}$$

Set $H' := \exp^{-1}|_U(U \cap H) \subset \mathfrak{g}$ and define

$$W := \left\{ sX \middle| X = \lim_{n \mapsto \infty} \frac{h_n}{\|h_n\|}, h_n \in H', \|h_n\| \mapsto 0\,(n \mapsto \infty), s \in \mathbb{R} \right\} , \tag{2.75}$$

which is the “tangent cone of H' at 0”.

1) We want to show that $\exp(W) \subseteq H$. Let $t \in \mathbb{R}$. Then we have

$$tX = \lim_{n \mapsto \infty} \frac{th_n}{\|h_n\|} , \tag{2.76}$$

for $X \in W$. Since $\|h_n\| \mapsto 0$ $(n \mapsto \infty)$ there are numbers $m_n \in \mathbb{Z}$ such that $|\|h_n\| - t/m_n| \mapsto 0$ for $n \mapsto \infty$. This is equivalent to $m_n\|h_n\| \mapsto t$ for $n \mapsto \infty$. Hence

$$\lim_{n \mapsto \infty} \exp(m_n h_n) = \lim_{n \mapsto \infty} \exp\left(m_n\|h_n\| \cdot \frac{h_n}{\|h_n\|} \right) = \exp(tX) , \tag{2.77}$$

but

$$\exp(m_n h_n) = \exp\Big(\underbrace{h_n + \ldots + h_n}_{m_n \text{ summands}}\Big) = \exp(h_n)^{m_n} \in H , \tag{2.78}$$

since this is a one-parameter subgroup. From that and due to the closeness of H follows $\exp(tX) \in H$.

2) Furthermore, we claim that W is a vector subspace of $\mathfrak{g}$. Let X, $Y \in W$. It remains to show that $X + Y \in W$. For doing so, set $h(t) := \log(\exp(tX) \cdot \exp(tY))$ for $\exp(tX) \cdot \exp(tY) \in H$ (by (1)). By Campbell-Baker-Hausdorff

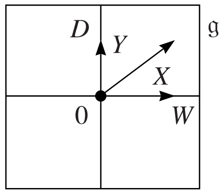
$$\lim_{t \mapsto 0} \frac{1}{2} h(t) = X + Y . \tag{2.79}$$

As a result of that

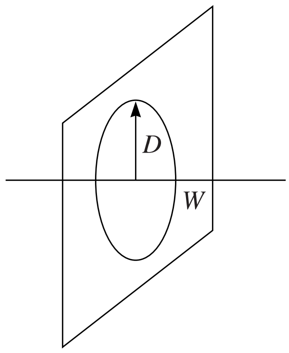
$$\lim_{t \mapsto 0} \frac{h(t)}{\|h(t)\|} = \lim_{t \mapsto 0} \frac{\frac{h(t)}{t}}{\left\| \frac{h(t)}{t} \right\|} = \frac{X+Y}{\|X+Y\|} , \tag{2.80}$$

i.e. $X + Y \in W$. By definition follows then $\lambda X \in W$ for $X \in W$.

3) Third, we claim that $\exp(W)$ is a neighborhood of e in H (submanifold property). A priori, $\exp(W)$ could be much smaller than $H \cap U$.

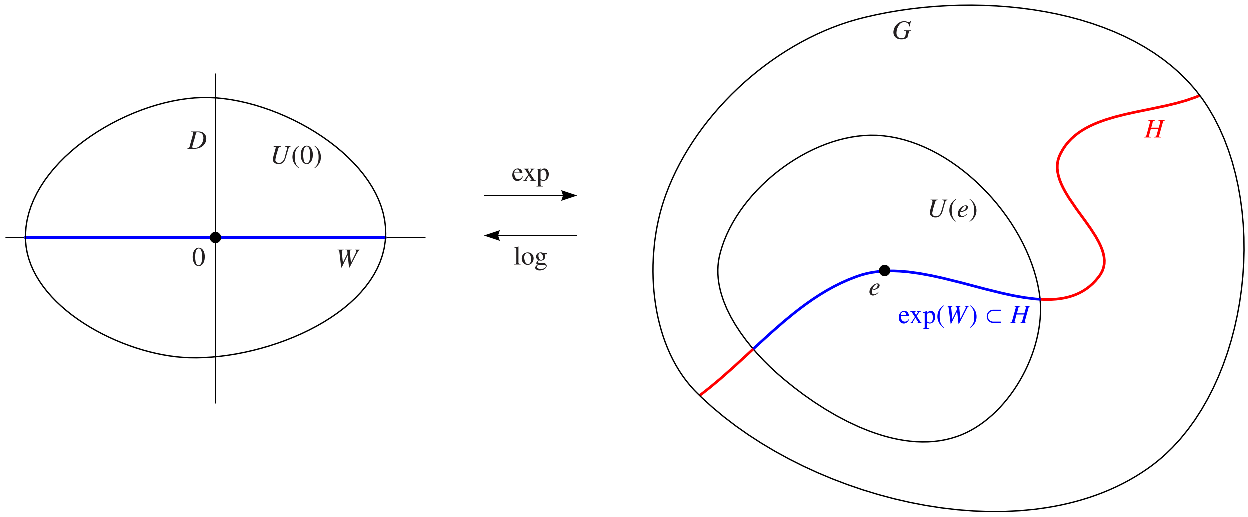


Let D be the orthogonal complement of W in $\mathfrak{g}$. Then the map $W \oplus D(= \mathfrak{g}) \mapsto G$, $(X, Y) \mapsto \exp(X) \cdot \exp(Y)$ is a local diffeomorphism. Assume, (3) is wrong. Then there are pairs $(X_n, Y_n) \in W \oplus D$ with $\exp(X_n) \cdot \exp(Y_n) \in H$ $(*)$, $Y_n \neq 0$, and $(X_n, Y_n) \mapsto (0, 0)$ for $n \mapsto \infty$. As D is a (closed) linear subspace of $\mathfrak{g}$ we can find a subsequence of $Y_n / \|Y_n\|$ converging to some $Y \in D$.



As $\|Y\| = 1$, $Y \neq 0$ holds. Passing to a subsequence again, such that $X_n \mapsto X \in W$. By $(*)$ we have $\exp(Y_n) \in H$ (for members of a subsequence since $\exp(X_n) \in H$ by (1)). By definition of W it follows that $Y \in W$. As $W \cup D = \{0\}$ we have $Y = 0$, which is a contradiction to the construction of Y.

Hence, the construction of a submanifold chart.



H is a submanifold near e. Since left-translations are diffeomorphisms, H is a submanifold. □

2.11.1 Examples of closed subgroups

1) $\mathrm{SO}(n) \subset \mathrm{SL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$

   $\mathrm{SL}(n, \mathbb{R}) \subset \mathrm{GL}(n, \mathbb{R})$ since $\mathrm{SL}(n, \mathbb{R})$ is the restriction of $\mathrm{GL}(n, \mathbb{R})$ under the equation $\det(A) = 1$ and therefore it is closed. $\mathrm{SL}(n, \mathbb{R})$ is a restriction of $\mathrm{SL}(n, \mathbb{R})$ by the matrix equations $AA^\intercal = A^\intercal A = E_n$ and as a result of this it is a closed subgroup.

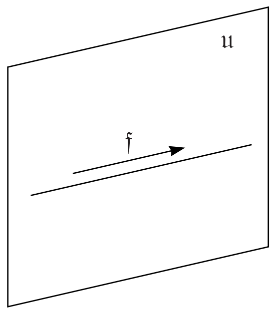
2) Be $\phi \colon H \mapsto G$ be a differentiable homomorphism. Then $\mathrm{Kern}(\phi) := \{h \in H | \phi(n) = e_G\}$ is a closed subgroup of H. A special case if $\det \colon \mathrm{GL}(n, \mathbb{R}) \mapsto \mathbb{R}$. Since $\mathrm{Kern}(\det) = \mathrm{SL}(n, \mathbb{R})$, $\mathrm{SL}(n, \mathbb{R})$ is a closed subgroup of $\mathrm{GL}(n, \mathbb{R})$.

3) Stabilizors are closed subgroups: in particular, $G_x = \{g \in G | g \cdot x = x\}$. If G is a Lie group acting on a manifold then G_x is a Lie subgroup of G (e.g. G is the isometry group of a Riemannian manifold, which are always closed subgroups).

Remarks:

- There is a relationship between Lie subgroups and Lie subalgebras. (A **Lie subalgebra** $\mathfrak{n}$ of a Lie algebra $\mathfrak{g}$ is a subvectorspace with $[\mathfrak{n}, \mathfrak{u}] \subseteq \mathfrak{n}$.
- Theorem 6 (without proof):
  - If H is a Lie subalgebra of $\mathfrak{g}$, then $\mathfrak{f} = \text{Lie}(H)$ is a Lie subalgebra of $\mathfrak{g} = \text{Lie}(G)$.
  - Every Lie subalgebra of $\mathfrak{g} = \text{Lie}(G)$ is the Lie algebra of a connected Lie subgroup of G (i.e. there exists a bijection of Lie subalgebras and connected Lie subgroups).

  The proof of this can be found in Helgason II.2. Theorem 2.1). Idea of the proof: An inclusion map $H \hookrightarrow_i G$, $h \mapsto i(j) = h$ is a Lie groups homomorphism, hence (by theorem 3) $\mathrm{d}i_e \colon T_eH(= \mathfrak{f}) \mapsto T_eG(= \mathfrak{g})$ is a Lie algebra homomorphism. This implies that $\mathrm{d}i_e(\mathfrak{f})$ is a subalgebra of $\mathfrak{g}$. Furthermore, $\mathfrak{f} \subseteq \mathfrak{g}$ subalgebra. Define H as the unique connected subgroup of G generated by the one-parameter subgroup $\exp(tX)$, $t \in \mathbb{R}$, $X \in \mathfrak{f}$.



Kapitel 3

Homogeneous spaces

3.1 Homogeneous spaces of Lie groups

Consider the action of a Lie group as a differentiable manifold. ϕ: $G \times M \mapsto M$, $(g,p) \mapsto g \cdot p$. If ψ is differentiable, G is called **Lie transformation group**.

Examples:

1) SL(2, $\mathbb{R}$ acts on H^2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \mapsto \frac{az+b}{cz+d}. \tag{3.1}$$

2) SL($n, \mathbb{R}$) acts on positive definite matrices SL($n, \mathbb{R}$) $\times$ Pos(n) $\to$ Pos(n), $(A, P) \mapsto A^\intercal \cdot P \cdot A$.

The **goal** is to show that orbits of $G \cdot p \simeq G/G_p$ are differentiable manifolds. Recall: G_p is closed, hence a Lie subgroup. More generally, consider any closed subgroup $H \subseteq G$. The **coset space** of G modulo H is

$$G/H = \{gH | g \in G\} = G/\sim, \tag{3.2}$$

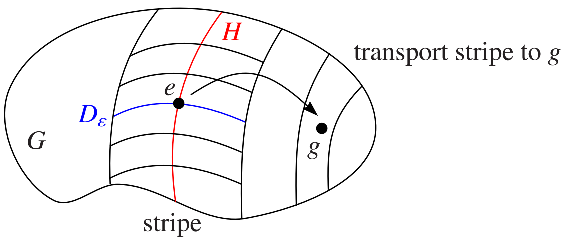
where $g_1 \sim g_2$ if and only if $g_1^{-1} g_2 \in H$. Via canonical projection π: $G \mapsto G/H$, $g \mapsto gH$, endow G/H with the quotient topology. $U \subseteq G/H$ is open, if and only if $\pi^{-1}(U)$ is open in G. G has an action on G/H. ϕ: $G \times (G/H) \mapsto G/H$, $(g, g'H) \mapsto g \cdot g'H$. The action is transitive: $\phi(g, eH) = gH$. The stabilizer of $eH(= H)$ is $G_{eH} = \{g \in G | geH = gH = H\} = H$.

Theorem:

Let G be a Lie group, $H \subseteq G$ a closed subgroup. Then G/H is a differentiable manifold of dimension $\dim(G/H) = \dim(G) - \dim(H)$. The canonical projection π is differentiable. The maps τ_a: $G/H \mapsto G/H$, $gH \mapsto agH$ (for all $a \in G$) are differentiable.

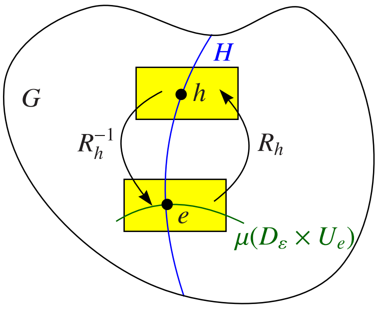
Proof:

1) First, we prove that G/H is a differentiable manifold. We choose an inner product on the Lie algebra $T_eG = g$. The Lie algebra h of H is a vector subspace of g and we can perform the decomposition $g = h \oplus h^\perp$, whereas $h^\perp$ is, in general, not a Lie algebra. Set $V := h^\perp$ and choose a ball $V_\varepsilon := \{v \in V | \|v\| < \varepsilon\}$ for $\varepsilon > 0$. Furthermore define $D_\varepsilon := \exp(V_\varepsilon)$, which is a subset of G. **Claim:** For a sufficiently small $\varepsilon > 0$, the map μ: $D_\varepsilon \times H \mapsto G$, $(g, h) \mapsto gh$ is a diffeomorphism onto its image in G.



In other words, μ provides us with a local product structure around e. The claim can be proven as follows:

i) The differential of μ in (e, e) is the identity on each subspace $V = T_e D_\varepsilon$ and $h = T_e H$ of g. From the inverse function theorem we get the existence (for some $\varepsilon > 0$) of a neighborhood U_e in H such that $\mu|_{D_\varepsilon \times U_e}\colon D_\varepsilon \times U_e \mapsto D_\varepsilon U_e \subseteq G$, which is a local diffeomorphism. For all $h \in H$, right-multiplication R_h is a diffeomorphism. Hence $\mu|_{D_\varepsilon \times U_e h} = R_h \circ \mu|_{D_\varepsilon \times U_e} \circ (\mathrm{id} \times R_h^{-1})$ is a local diffeomorphism in a neighborhood of $h \in H$.



ii) We can choose ε small enough such that μ is injective on all of H. To see this, consider d_1, $d_2 \in D_\varepsilon$ and h_1, $h_2 \in H$. Assume $\mu(d_1, h_1) = \mu(d_2, h_2)$, hence $d_1 h_1 = d_2 h_2$ or $d_1 h_1 h_2^{-2} = d_2$ with $h_1 h_2^{-1} =: h \in H$ (*). Choose ε small enough such that $d_1^{-1} d_2 = h \in U_e$. Now, by (i), $\mu|_{D_\varepsilon \times U_e}$ is injective, so write $(*)$ as $d_1 h = d_2 e \in D_\varepsilon U_e = \mu(D_\varepsilon \times U_e)$, which implies $d_1 = d_2$, $h = e$ and so $h_1 = h_2$. Hence, μ is injective on all of H. This proves the claim.

Now we can define charts for G/H. For $g \in G$, define $U_g := gD_\varepsilon H = \{gdh | d \in D_\varepsilon, h \in H\}$. U_g is invariant under right-multiplication by H and $\pi(U_g) = U_g/H$ constitute a cover of G/H by open sets (note that $D_\varepsilon H = \bigcup_{h \in H} D_\varepsilon U_e h$ is open). Define charts $\varphi_g\colon U_g/H \mapsto D_\varepsilon \mapsto V_\varepsilon$, $gdH \mapsto \log(d)$. **Exercise:** $\varphi_{g_1} \circ \varphi_{g_2}^{-1}$ is differentiable, since it can be written as a decomposition of differentiable maps as follows:

$$v \in V_\varepsilon \xrightarrow{\exp} d \mapsto g_2 dH = g_1 d' H \mapsto d' \xrightarrow{\log} v' \,. \tag{3.3}$$

2) Now prove that π and the τ_a are differentiable maps. For ε as above and $\widetilde{U}_e \subset G$ given by $D_\varepsilon U_e \simeq D_\varepsilon \times U_e$ we have $\pi(\widetilde{U}_e) = D_\varepsilon U_e / H$. We have charts φ_e (as above) and $\log \times \log\colon \widetilde{U}_e \mapsto V_\varepsilon \times h$, $du \mapsto (\log(d), \log(u))$. Then $\varphi_e \circ \pi \circ (\log \times \log)^{-1}\colon V_\varepsilon \times \log(U_e) \mapsto V_\varepsilon$ is the projection on the first factor, hence differentiable. It follows that π is differentiable. The map $\tau_a\colon G/H \mapsto G/H$, $gH \mapsto agH$, makes the following diagram commute:

$$\begin{array}{ccc} G & \xrightarrow{L_a} & G \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ G/H & \xrightarrow{\tau_a} & G/H \end{array}$$

As L_a and π are differentiable, so is τ_a.

3) Now we have

$$\dim(G/H) = \dim(D_\varepsilon) = \dim(V_\varepsilon) = \dim(h^\perp) = \dim(g) - \dim(h) = \dim(G) - \dim(H) \,. \tag{3.4}$$

This completes the proof of the theorem. □

Remark:

$\pi\colon G \mapsto G/H$ is continuous with respect to the quotient topology. So, if G is compact then G/H is compact as well. For example take $G = \mathrm{SO}(n)$, so $\mathrm{SO}(n)/H$ is compact. In the following lectures we will see that $\mathrm{Gr}(k, n) = \mathrm{SO}(n)/(\mathrm{SO}(k) \times \mathrm{SO}(n-k))$, which is an example for a compact homogeneous space. Think of $\mathrm{SO}(k) \times \mathrm{SO}(n-k)$ as a block diagonal matrix with a $(k \times k)$-submatrix $\in \mathrm{SO}(k)$ and $(n-k) \times (n-k)$-submatrix $\in \mathrm{SO}(n-k)$.

3.1.1 Examples

We consider a Lie Transformation group which acts transitively on a manifold M with the diffeomorphism $M \simeq G/G_{x_0}$. Now we want to look at some examples of this situation.

1) Grassmann-manifolds: $\mathrm{Gr}_p(\mathbb{R}^n) := \{p\text{-dimensional vector subspaces of } \mathbb{R}^n\}$ with $1 \leq p \leq n-1$

The claim is $\mathrm{Gr}_p(\mathbb{R}^n) \simeq \mathrm{O}(n)/\mathrm{O}(p) \times \mathrm{O}(n-p)$, whereas $\mathcal{O}(n) = \{A \in \mathrm{GL}(n,\mathbb{R}|A^\intercal A = I_n)$ is the orthogonal group.

i) To show this, one first has to probe the $\mathrm{O}(n)$ acts transitively on $\mathrm{Gr}_p(\mathbb{R}^n)$. Let $e_1, \ldots, e_p, e_{p+1}, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$ and $V_0 := [e_1, \ldots, e_p]$. Let V be in $\mathrm{Gr}_p(\mathbb{R}^n)$. Choose an orthonormal basis $f_1, \ldots, f_p$ of V and extend to an orthonormal basis of $\mathbb{R}^n$: $f_1, \ldots, f_p, f_{p+1}, \ldots, f_n$. Consider the orthogonal matrix $A := (f_1|f_2|\ldots|f_p|f_{p+1}|\ldots|f_n) \in \mathrm{O}(n)$. Hence, $A \cdot V_0 = V$.

ii) Now one has to find the stabilizer of $V_0 = \{B \in \mathrm{O}(n)|B \cdot V_0 = V_0\}$. As $\mathbb{R}^n = V_0 \overset{\perp}{\oplus} V_0^\perp$ we see that

$$B = \begin{pmatrix} * & 0 \\ 0 & ** \end{pmatrix} \in \mathrm{O}(n)\,, \tag{3.5}$$

whereas the first block is a $(p \times p)$-submatrix and the second one is a $(n-p) \times (n-p)$-submatrix. As a result of that

$$\mathrm{O}(n)_{V_0} = \left\{ \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \middle| B_1 \in \mathrm{O}(p), B_2 \in \mathrm{O}(n-p) \right\}\,. \tag{3.6}$$

Analogously, consider

$$\widetilde{\mathrm{Gr}}_p(\mathbb{R}^n) := \{p\text{-dimensional } \textbf{oriented} \text{ subspaces of } \mathbb{R}^n\} = \mathrm{SO}(n)/\mathrm{SO}(p) \times \mathrm{SO}(n-p)\,. \tag{3.7}$$

Orientation on a vector space means that there are two equivalence classes of bases, those of positive and those of negative orientation: $B_1 \sim B_2 \Leftrightarrow \det(B_1) = \det(B_2)$, whereas B_1 is the matrix of columnvectors of B_1.

Special cases are the real projective space $\mathrm{Gr}_1(\mathbb{R}^n) = P^{n-1}\mathbb{R} = \mathrm{O}(n)/\mathrm{O}(1) \times \mathrm{O}(n-1)$ or $\widetilde{\mathrm{Gr}}_1(\mathbb{R}^n) = S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(1) \times \mathrm{SO}(n-1) \simeq \mathrm{SO}(n)/\mathrm{SO}(n-1)$ (for $p=1$).

Remarks:

a) $\mathrm{O}(n)$ and $\mathrm{SO}(n)$, respectively, are compact Lie groups. Because of $\sum_j a_{ij}a_{jk} = \delta_{ik}$ and hence $|a_{ij}| \leq 1$, $\mathrm{O}(n)$ is a closed bounded subset of $\mathbb{R}^n$ and hence compact. $G \mapsto G/H$ is continuous. That means, if G is compact so is G/H. This leads to the fact that $\mathrm{Gr}_p(\mathbb{R}^n)$ and $\widetilde{\mathrm{Gr}}_p(\mathbb{R}^n)$ are compact manifolds.

b) From $\dim(G/H) = \dim(G) - \dim(H)$ follows:

$$\begin{aligned}\dim(\mathrm{Gr}_p(\mathbb{R}^n)) &= \dim(\mathrm{O}(n)) - \dim(\mathrm{O}(p)) - \dim(\mathrm{O}(n-p)) = \\ &= \frac{n(n-1)}{2} - \frac{p(p-1)}{2} - \frac{(n-p)(n-p-1)}{2} = p(n-p)\,.\end{aligned} \tag{3.8}$$

c) Similarly, one can define **complex Grassmann manifolds** as follows:

$$\mathrm{Gr}_p(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(p) \times \mathrm{U}(n-p)\,. \tag{3.9}$$

d) $\mathrm{Gr}_p(\mathbb{R}^n) \simeq \mathrm{Gr}_{n-p}(\mathbb{R}^n)$

2) Generalization: Flag-manifolds ("Fahnen-manifold")

A **flag** in V (K-vectorspace) is a chain of nested subspaces, hence $V_0 = 0 \subset V_1 \subset V_2 \ldots \subset V_k = V$, such that $\dim(V_1) = d_i$ (whereas the notation is $F(d_1, d_2, \ldots, d_k)$). The set of flags is defined to be the flag manifold $\mathscr{F}(d_1, d_2, \ldots, d_n; \mathbb{R}^n) = \{F(d_1, d_2, \ldots, d_k)\}$. $\mathrm{GL}(n,\mathbb{R})$ acts on $F(d_1, \ldots, d_k)$ transitively (use basis for $V_1 \subset V_2 \subset \ldots \subset V_k$ and map standard flag $[e_1, \ldots, e_{d_1}] \subset [e_1, \ldots, e_{d_1}, e_{d_1+1}, \ldots, e_{d_2}] \subset \ldots$). The stabilizer of $[e_1, \ldots, e_{d_1}] \subset [e_1, \ldots, e_{d_2}] \subset \ldots \subset [e_1, \ldots, e_{d_k}]$ consists of all matrices in $\mathrm{GL}(n,\mathbb{R})$ of the form As a result of this

$$\begin{aligned}F(d_1, \ldots, d_k) &\simeq \mathrm{GL}(n,\mathbb{R})/\text{"block matrices"} = \mathrm{SO}(n)/\text{"block-diagonal matrices"} = \\ &= \mathrm{SO}(n)/\mathrm{SO}(d_1) \times \mathrm{SO}(d_2 - d_1) \times \mathrm{SO}(d_3 - (d_1 + d_2)) \times \ldots\,.\end{aligned} \tag{3.10}$$

Remark:

a) Different groups can act on the same manifold.

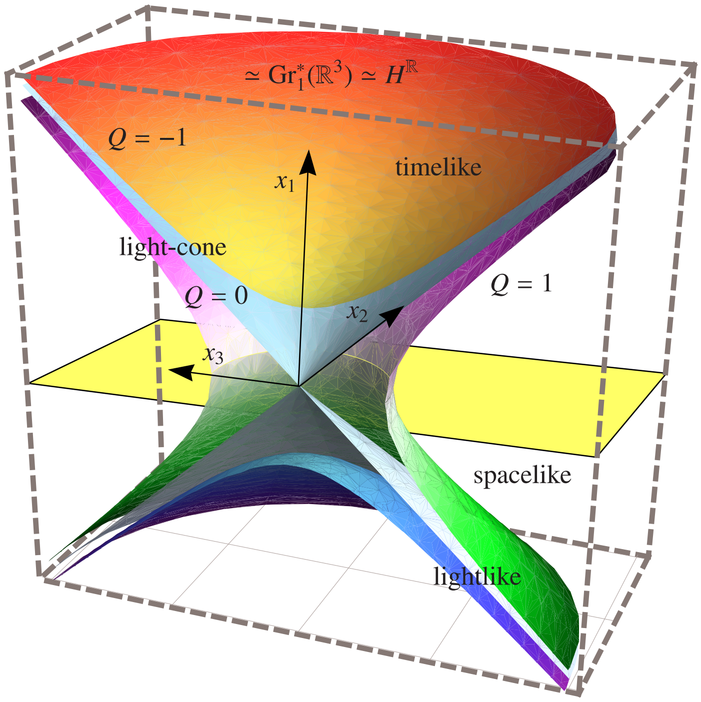
b) Flag manifolds are compact.

3) Non-compact Grassmann manifolds:

Consider a symmetric bilinear form of signature $(p, n-p)$ on $\mathbb{R}^n$:

$$\langle x, y\rangle := -x_1y_1 - \ldots - x_py_p + x_{p+1}y_{p+1} + \ldots + x_ny_n\,. \tag{3.11}$$

Be $\mathrm{Gr}_p^*(\mathbb{R}^n) := \{p\text{-dimensional subspaces } U \text{ of } \mathbb{R}^n \text{ such that } \langle\bullet,\bullet\rangle|_U \text{ negative definite}\} \in [e_1, \ldots, e_p]$. So $\mathrm{Gr}_p^*(\mathbb{R}^n) = \mathrm{SO}(p, n-p)/\mathrm{S}(\mathrm{O}(p) \times \mathrm{O}(n-p))$. A special case for $p = 1$ is $\mathrm{Gr}_1^*(\mathbb{R}^n)$, which is the $(n-1)$-dimensional real hyperbolic space. Furthermore, the case with $n = 3$, $p = 1$ is a model for the hyperbolic plane. So consider $Q(x, y) = \langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$. $Q(x, x) = 0$ is a three-dimensional cone:



The whole family of hyperboloids form a foliation of the whole space. This is especially important for special relativity, since it defines timlike, spacelike and lightlike vectors. **Remark:**

a) $\mathrm{Gr}_p^*(\mathbb{R}^n)$ is not compact.

b) $\mathrm{O}(1, n)$ is not compact, since they can be written in the following way (Lorentz boosts):

$$\begin{pmatrix} \cosh(\varphi) & \mathbf{0}^\intercal & \sinh(\varphi) \\ \mathbf{0} & I_{n-2} & \mathbf{0} \\ \sinh(\varphi) & \mathbf{0}^\intercal & \cosh(\varphi) \end{pmatrix}. \tag{3.12}$$

These groups are called pseudo-orthogonal. Unlike in rotations, where one-parameter subgroups consist of trigonometric functions, they are here made up of hyperbolic functions. This comes from the fact that the defining equation with this pseudo-orthogonal group is $x^2 - y^2 = 1$ which is satisfied by hyperbolic functions. The defining equation of rotations is $x^2 + y^2 = 1$, which is fulfilled by trigonometric functions.

Kapitel 4

Symmetric spaces

Question: Is there a Riemannian metric on $G/H = M$ such that $G \subseteq \mathrm{Isom}(M)$? We will proceed in the opposite direction.

4.1 Definition

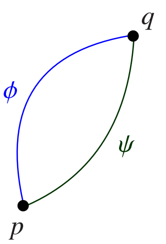
A connected Riemmannian manifold $(M, \langle \bullet, \bullet \rangle)$ is called **symmetric**, if for all $p \in M$ there exists an isometry $S_p \colon M \mapsto M$ with

1) $S_p(p) = p$ (fixpoint) and
2) $\mathrm{d}S_p|_p = -\mathrm{id}_{T_pM}$.

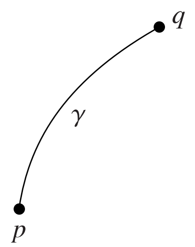
$\phi \in \mathrm{Isom}(M)$ is uniquely determined by $\phi(p)$, $\mathrm{d}\phi_p$ for some $p \in M$ ("rigidity property of isometries").

Proof:

Assume $\phi(p) = \psi(p)$, $\mathrm{d}\phi|_p = \mathrm{d}\psi|_p$ for ϕ, $\psi \in \mathrm{Isom}(M)$. Claim: $\phi = \psi$. This is equivalent to $\psi_0^{-1}\phi = \mathrm{id}_M$. It suffices to show $\phi(p) = p$ and $\mathrm{d}\phi|_p = \mathrm{id}_{T_pM}$, then $\phi = \mathrm{id}_M$.



To simplify assume that M is **complete** so that every geodesic is defined in $\mathbb{R}$. Let $q \neq p \in M$. By Hopf-Rinow there exists a geodesic γ joining p to q, hence $p = \gamma(0)$ and $q = \gamma(d)$.



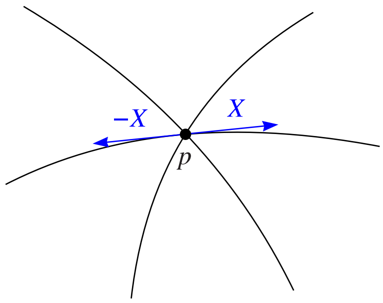
Consider $\phi \circ \gamma := \widetilde{\gamma}$, which is a geodesic (since γ is a geodesic and ϕ an isometry). From $\widetilde{\gamma}(0) = \phi(\gamma(0)) = \phi(p) = p$ follows

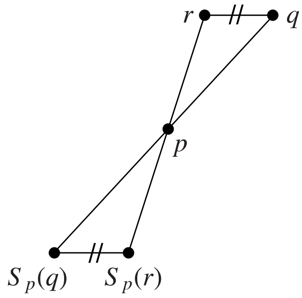
$$\dot{\widetilde{\gamma}}(0) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \phi \circ \gamma(t) = \mathrm{d}\phi_p(\dot{\gamma}(0)) \overset{!}{=} \dot{\gamma}(0)\,, \tag{4.1}$$

hence $\widetilde{\gamma}(0) = \gamma(0)$. So $q = \gamma(d) = \widetilde{\gamma}(d) = \phi(\gamma(d)) = \phi(q)$. □

4.1.1 Geometric interpretation

S_p is a geodesic reflection in p.





(Simple) examples of symmetric spaces are the Euclidian spaces $\mathbb{E}^n = (\mathbb{R}^n, \langle \bullet, \bullet \rangle)$, the n-dimensional spheres S^n or the hyperbolic plane $H^2\mathbb{R}$.

Lemma 1:

- Symmetric spaces are complete.
- Symmetric spaces are homogeneous.